

Four-loop results on anomalous dimensions and splitting functions in QCD

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- Introduction, (methods of) diagram computations, structural properties
- Non-singlet: analytic (large- n_c) and numerical results for P_{ns} and γ_{cusp}
- Outlook: flavour-singlet cases, 5-loop non-singlet at low spin N

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References

Prerequisites: FORCER & FORM (latest version)

B. Ruijl, T. Ueda and J.A.M Vermaseren, arXiv:1704.06650 & arXiv:1707.06453

Results, up to now

**B. Ruijl, T. Ueda, J.A.M. Vermaseren, J. Davies and A. Vogt,
First Forcer results on deep-inelastic scattering and related quantities,
arXiv:1605.08408, PoS LL 2016 (2016) 071**

**J. Davies, B. Ruijl, T. Ueda, J.A.M. Vermaseren and A. Vogt,
Large- n_f contributions to the four-loop splitting functions in QCD,
arXiv:1610.07477, Nucl. Phys. B915 (2017) 335**

**S. Moch, B. Ruijl, T. Ueda, J.A.M. Vermaseren and A. Vogt,
Four-loop non-singlet splitting functions in the planar limit and beyond,
arXiv:1707.08315 (JHEP, to appear)**

Hard processes in perturbative QCD

Observables in ep and pp scattering, schematically, up to power corrections

$$O^{ep} = f_i \otimes c_i^o, \quad O^{pp} = f_i \otimes f_k \otimes c_{ik}^o$$

c^o : coefficient functions, scale $\mu \leftrightarrow$ physical hard scale, e.g., M_{Higgs}

Parton densities f_i : renormalization-group evolution (\otimes : Mellin convolution)

$$\frac{\partial}{\partial \ln \mu^2} f_i(\xi, \mu^2) = [P_{ik}(\alpha_s(\mu^2)) \otimes f_k(\mu^2)](\xi)$$

Splitting functions (\Leftrightarrow twist-2 anomalous dimensions) & coefficient funct's:

$$\begin{aligned} P &= \alpha_s P^{(0)} + \alpha_s^2 P^{(1)} + \alpha_s^3 P^{(2)} + \alpha_s^4 P^{(3)} + \dots \\ c_a &= \underbrace{\alpha_s^{n_a} [c_a^{(0)} + \alpha_s c_a^{(1)} + \alpha_s^2 c_a^{(2)} + \alpha_s^3 c_a^{(3)} + \dots]}_{\text{NNLO: now the standard approximation for many processes}} \end{aligned}$$

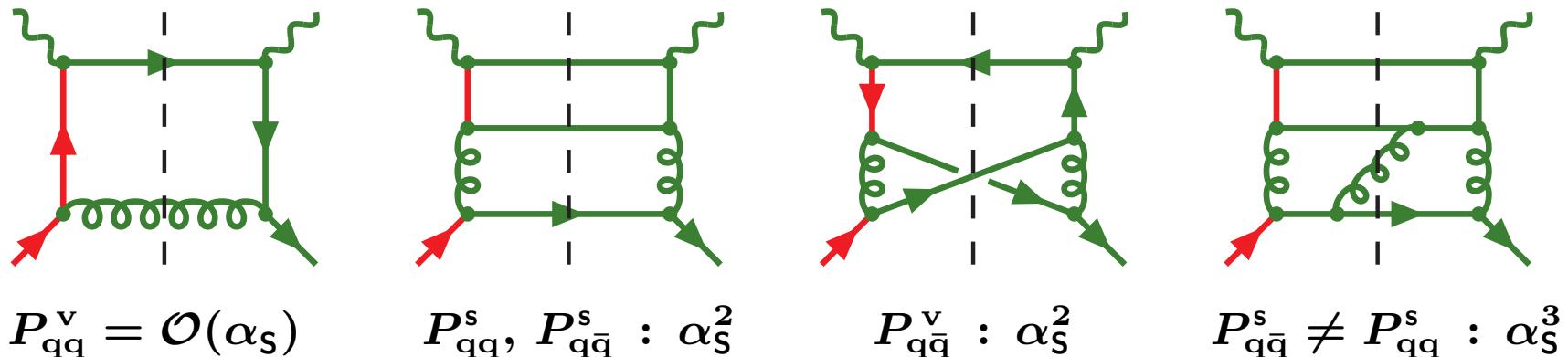
$N^{n>2}\text{LO}$: for high precision (α_s from DIS); slow convergence (Higgs in pp)

..., Anzai et al ('15); Anastasiou, Duhr, Dulat, Herzog, Mistlberger ('15); Dreyer, Karlberg ('16); ...

Quark-quark splitting functions

General structure of the (anti-)quark (anti-)quark splitting functions

$$P_{q_i q_k} = P_{\bar{q}_i \bar{q}_k} = \delta_{ik} P_{qq}^v + P_{qq}^s , \quad P_{q_i \bar{q}_k} = P_{\bar{q}_i q_k} = \delta_{ik} P_{q\bar{q}}^v + P_{q\bar{q}}^s$$



⇒ three types of independent difference (non-singlet, ns) combinations

$2(n_f - 1)$ flavour asymmetries of $q_i \pm \bar{q}_i$ + one total valence distribution

with

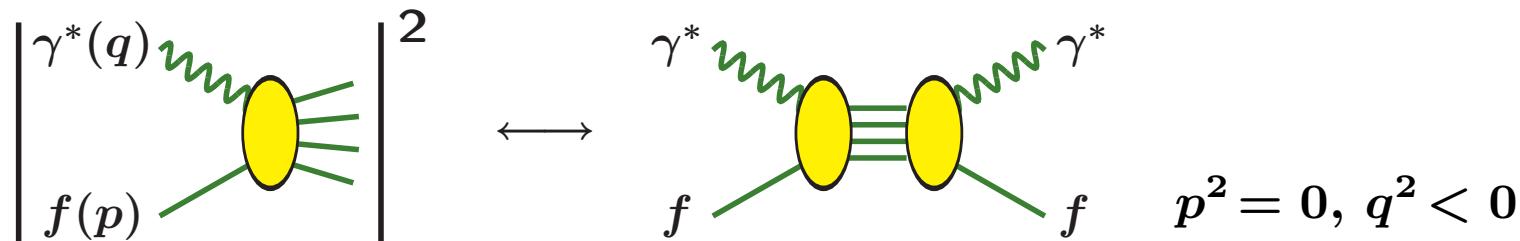
$$q_{ns,ik}^\pm = q_i \pm \bar{q}_i - (q_k \pm \bar{q}_k) , \quad q_{ns}^v = \sum_{r=1}^{n_f} (q_r - \bar{q}_r)$$

$$P_{ns}^\pm = P_{qq}^v \pm P_{q\bar{q}}^v , \quad P_{ns}^v = P_{ns}^- + P_{ns}^s . \quad \text{Large } n_c: \quad P_{ns}^+(x) = P_{ns}^-(x)$$

N -space calculations (OPE, ‘NIKHEF method’): even (+) or odd (−, v) N

Splitting functions via DIS calculations

Inclusive DIS: probe-parton total cross sections \leftrightarrow forward amplitudes



Dispersion relation in x : coefficient of $(2p \cdot q)^N \leftrightarrow N\text{-th Mellin moment}$

$$A(N) = \int_0^1 dx x^{N-1} A(x)$$

Fixed N : harmonic projection \rightarrow self-energy integrals \rightarrow MINCER/ FORCER

Projection on structure functions, $D=4-2\epsilon$ dimensions, mass factorization

ϵ^{-1} : splitting functions $P_{ik}^{(n)}(N) = -\gamma_{ik}^{(n)}(N)$, ϵ^0 : n -loop coefficient fct's

2 and 3 loops: Larin, Vermaseren ('91); Larin, van Ritbergen, Vermaseren [, Nogueira] ('93, '96); ...

[D]RUVV ('16): 4 loops to $N \leq 6$ in general; up to $N > 40$ for high- n_f parts

Splitting functions via the OPE (non-singlet)

Spin- N twist-two flavour non-singlet quark operators (symmetric, traceless)

$$O_{\{\mu_1, \dots, \mu_N\}}^{\text{ns}} = \bar{\psi} \lambda^\alpha \gamma_{\{\mu_1} D_{\mu_2} \dots D_{\mu_N\}} \psi$$

ψ : quark field, $D_\mu = \partial_\mu - igA_\mu$: covariant derivative, A_μ : gluon field,
 $\lambda^\alpha, \alpha = 3, 8, \dots, (n_f^2 - 1)$: diagonal generators of flavour group $SU(n_f)$



Here: operator matrix elements with off-shell quarks, $p_1 = p_2 = p$, $p^2 < 0$

$$A^{\text{ns}}(N) = \Delta^{\mu_1} \dots \Delta^{\mu_N} \langle p | O_{\{\mu_1, \dots, \mu_N\}}^{\text{ns}} | p \rangle, \quad \Delta^2 = 0$$

Renormalization constants $Z_{\text{ns}}(N) \rightarrow$ anomalous dimensions $\gamma_{\text{ns}}(N)$

2 and 3 loops: Floratos, Ross, Sachrajda ('77); Blümlein et al. [heavy quarks, $p^2 = 0$] ('09), ...

MRUVV ('17): 4 loops to $N = 16$ in general; to $N = 20$ for large- n_c parts

Towards all- N expressions (I)

If analogous to lower orders: harmonic sums S_w and simple denominators

$$\gamma_{\text{ns}}^{(n)}(N) = \sum_{w=0}^{2n+1} c_{00w} S_w(N) + \sum_a \sum_{k=1}^{2n+1} \sum_{w=0}^{2n+1-k} c_{akw} D_a^k S_w(N),$$

$D_a^k \equiv (N+a)^{-k}$. Denominators at calculated N values: $a = 0, 1$ for γ_{ns}^\pm

Coefficients c_{00w}, c_{akw} : integer modulo low powers of $1/2$ and $1/3$

$\gamma_{\text{ns}}(N)$: constrained by ‘self-tuning’ (conjecture, conformal symmetry)

$$\gamma_{\text{ns}}(N) = \gamma_u (N + \sigma \gamma_{\text{ns}}(N) - \beta(a_s)/a_s))$$

Space-like/time-like anom. dimensions: $\sigma = -1/+1$; universal kernel γ_u : reciprocity-respecting (RR), i.e., invariant under replacement $N \rightarrow (1-N)$

Dokshitzer, Marchesini ('06); Bass, Korchemsky ('06); ...

Non-RR parts, spacelike/timelike difference: ‘inherited’ from lower orders, need to find ‘only’ γ_u ; weight w : 2^{w-1} RR (combinations of) harmonic sums

Towards all- N expressions (II)

Limit of a large number of colours n_c : $\gamma_{\text{ns}}^+ = \gamma_{\text{ns}}^-$, no alternating sums

\Rightarrow remaining sums at $w = 1, \dots, 7$: $1, 1, 2, 3, 5, 8, 13 = \text{Fibonacci}(w)$

Velizhanin [website] ('10)

Together with powers of $\eta = [N(N + 1)]^{-1}$ (RR): 87 basis funct's at $w = 7$

Large- N and small- x limits: more than 40 constraints. Large- N :

$$\gamma_{\text{ns}}^{(n-1)}(N) = A_n \ln \widetilde{N} - B_n + N^{-1} \{C_n \ln \widetilde{N} - \widetilde{D}_n + \frac{1}{2} A_n\} + O(N^{-2})$$

C_n, \widetilde{D}_n : fixed by lower-order information Dokshitzer, Marchesini, Salam ('05), ...

Small- x resummation: 4-loop coeff's of $x^a \ln^b x$ known for all $a, 4 \leq b \leq 6$

Kirschner, Lipatov ('83), Blümlein, A.V. (95); [Davies,] Kom, A.V. ('12, '16)

$a = 0$ large- n_c single logs: $\gamma_{\text{ns}}(N) \cdot (\gamma_{\text{ns}}(N) + N - \beta(a_s)/a_s) = O(1)$

Velizhanin ('14)

$N \leq 18$ Diophantine eqs. \Rightarrow remaining large- n_c coeff's. Check: $N = 19, 20$

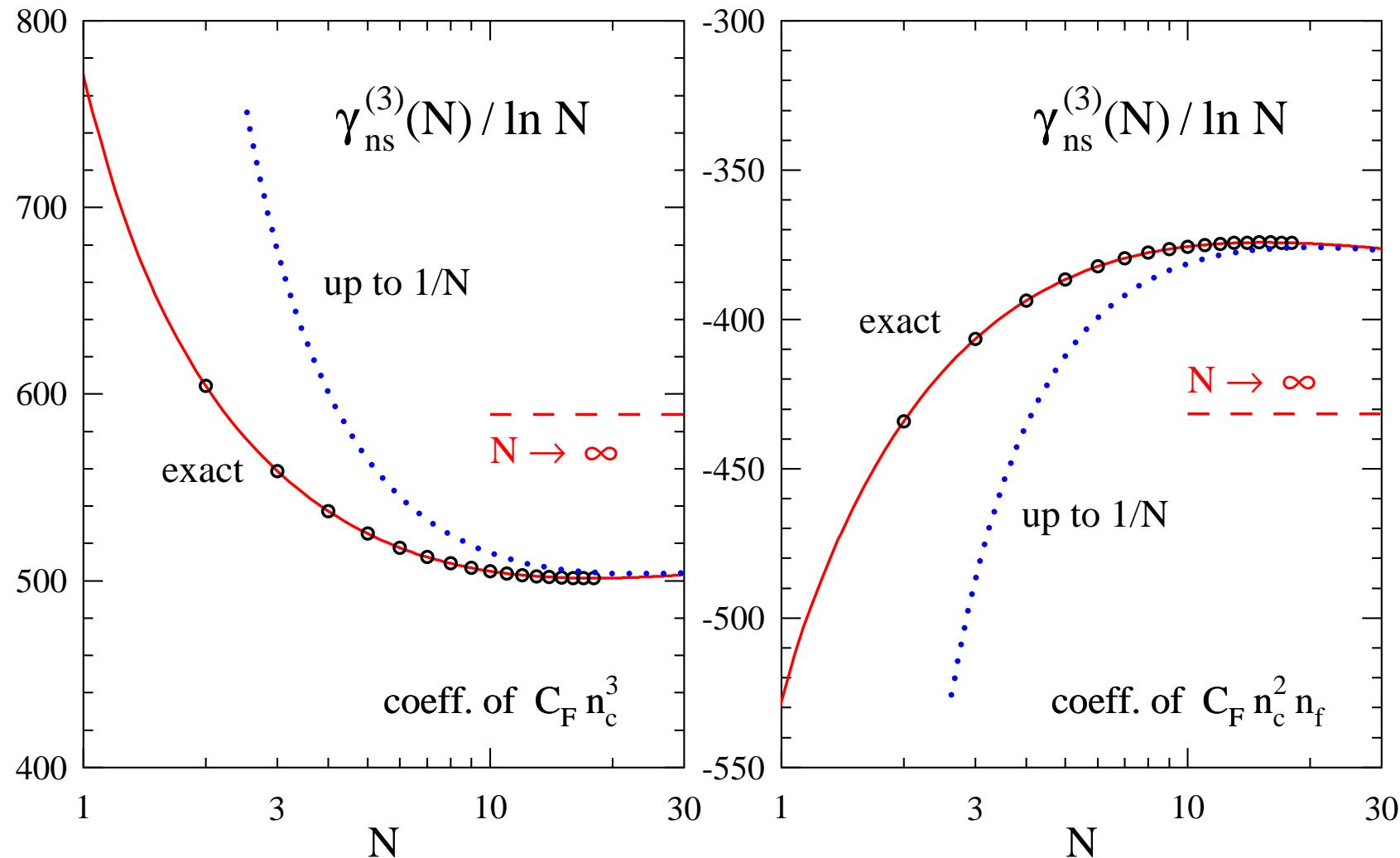
All- N anomalous dimension in the large- n_c limit (I)

$$\gamma_{\text{ns}}^{(3)\pm}(N) =$$

$$\begin{aligned}
 & 16 C_F n_c^3 (\dots - 21 S_{3,1} \eta^2 - 4 S_{3,1} \eta^3 - 4 S_{3,1} \zeta_3 - 5581/72 S_{3,2} + 22 S_{3,2} D_1^2 - 53/3 S_{3,2} \eta \\
 & - 16 S_{3,2} \eta^2 + 143/6 S_{3,3} - 11 S_{3,3} \eta - 14 S_{3,4} - 6899/72 S_{4,1} + 24 S_{4,1} D_1^2 - 74/3 S_{4,1} \eta \\
 & - 11 S_{4,1} \eta^2 + 57/2 S_{4,2} - 25 S_{4,2} \eta - 26 S_{4,3} + 63/2 S_{5,1} - 23 S_{5,1} \eta - 36 S_{5,2} - 28 S_{6,1} \\
 & - 12 S_{1,1,1} D_1^4 + 12 S_{1,1,1} \eta^2 + 24 S_{1,1,1} \eta^3 + 6 S_{1,1,1} \eta^4 + 18 S_{1,1,2} \eta^2 + 6 S_{1,1,2} \eta^3 - 20 S_{1,1,3} \eta \\
 & + 8 S_{1,1,3} \eta^2 + 20/3 S_{1,1,4} - 20 S_{1,1,4} \eta + 8 S_{1,1,5} + 18 S_{1,2,1} \eta^2 + 6 S_{1,2,1} \eta^3 + 134/3 S_{1,2,2} \\
 & - 12 S_{1,2,2} D_1^2 + 12 S_{1,2,2} \eta + 6 S_{1,2,2} \eta^2 - 22/3 S_{1,2,3} - 6 S_{1,2,4} + 1447/18 S_{1,3,1} - 16 S_{1,3,1} D_1^2 \\
 & + 104/3 S_{1,3,1} \eta - 6 S_{1,3,1} \eta^2 - 38/3 S_{1,3,2} + 16 S_{1,3,2} \eta + 22 S_{1,3,3} - 56/3 S_{1,4,1} + 12 S_{1,4,1} \eta \\
 & + 50 S_{1,4,2} + 46 S_{1,5,1} + 18 S_{2,1,1} \eta^2 + 6 S_{2,1,1} \eta^3 + 134/3 S_{2,1,2} - 12 S_{2,1,2} D_1^2 + 12 S_{2,1,2} \eta \\
 & + 6 S_{2,1,2} \eta^2 - 22/3 S_{2,1,3} - 6 S_{2,1,4} + 134/3 S_{2,2,1} - 12 S_{2,2,1} D_1^2 + 12 S_{2,2,1} \eta + 6 S_{2,2,1} \eta^2 \\
 & - 13 S_{2,2,2} + 6 S_{2,2,2} \eta + 12 S_{2,2,3} - 44/3 S_{2,3,1} + 38 S_{2,3,2} + 36 S_{2,4,1} + 307/6 S_{3,1,1} \\
 & - 20 S_{3,1,1} D_1^2 + 86/3 S_{3,1,1} \eta + 16 S_{3,1,1} \eta^2 - 43/3 S_{3,1,2} + 10 S_{3,1,2} \eta + 14 S_{3,1,3} - 43/3 S_{3,2,1} \\
 & + 10 S_{3,2,1} \eta + 24 S_{3,2,2} + 22 S_{3,3,1} - 37/3 S_{4,1,1} + 26 S_{4,1,1} \eta + 28 S_{4,1,2} + 28 S_{4,2,1} + 44 S_{5,1,1} \\
 & + 40 S_{1,1,1,4} - 16/3 S_{1,1,3,1} + 16 S_{1,1,3,1} \eta - 32 S_{1,1,3,2} - 24 S_{1,1,4,1} - 12 S_{1,2,2,2} - 28/3 S_{1,3,1,1} \\
 & - 16 S_{1,3,1,1} \eta - 20 S_{1,3,1,2} - 20 S_{1,3,2,1} - 52 S_{1,4,1,1} - 12 S_{2,1,2,2} - 12 S_{2,2,1,2} - 12 S_{2,2,2,1} \\
 & - 36 S_{2,3,1,1} - 12 S_{3,1,1,2} - 12 S_{3,1,2,1} - 12 S_{3,2,1,1} - 12 S_{4,1,1,1} - 32 S_{1,1,1,3,1} + 32 S_{1,1,3,1,1})
 \end{aligned}$$

+ large- n_c terms with n_f (known) + terms suppressed at large n_c (see below)

All- N anomalous dimension in the large- n_c limit (II)



Limit $N \rightarrow \infty$: large- n_c cusp anomalous dimension A_4 Korchemsky ('89), ...

Large- N coefficients in the large- n_c limit

Cusp anomalous dimension, expansion in $a_s = \alpha_s/(4\pi)$

$$\begin{aligned} A_{L,4} &= C_F n_c^3 \left(\frac{84278}{81} - \frac{88832}{81} \zeta_2 + \frac{20992}{27} \zeta_3 + 1804 \zeta_4 - \frac{352}{3} \zeta_2 \zeta_3 - 352 \zeta_5 \right. \\ &\quad \left. - 32 \zeta_3^2 - 876 \zeta_6 \right) \\ &- C_F n_c^2 n_f \left(\frac{39883}{81} - \frac{26692}{81} \zeta_2 + \frac{16252}{27} \zeta_3 + \frac{440}{3} \zeta_4 - \frac{256}{3} \zeta_2 \zeta_3 - 224 \zeta_5 \right) \\ &+ C_F n_c n_f^2 \left(\frac{2119}{81} - \frac{608}{81} \zeta_2 + \frac{1280}{27} \zeta_3 - \frac{64}{3} \zeta_4 \right) - C_F n_f^3 \left(\frac{32}{81} - \frac{64}{27} \zeta_3 \right) \end{aligned}$$

B. Ruijl [Seminar, Zurich, 6 Dec '16]

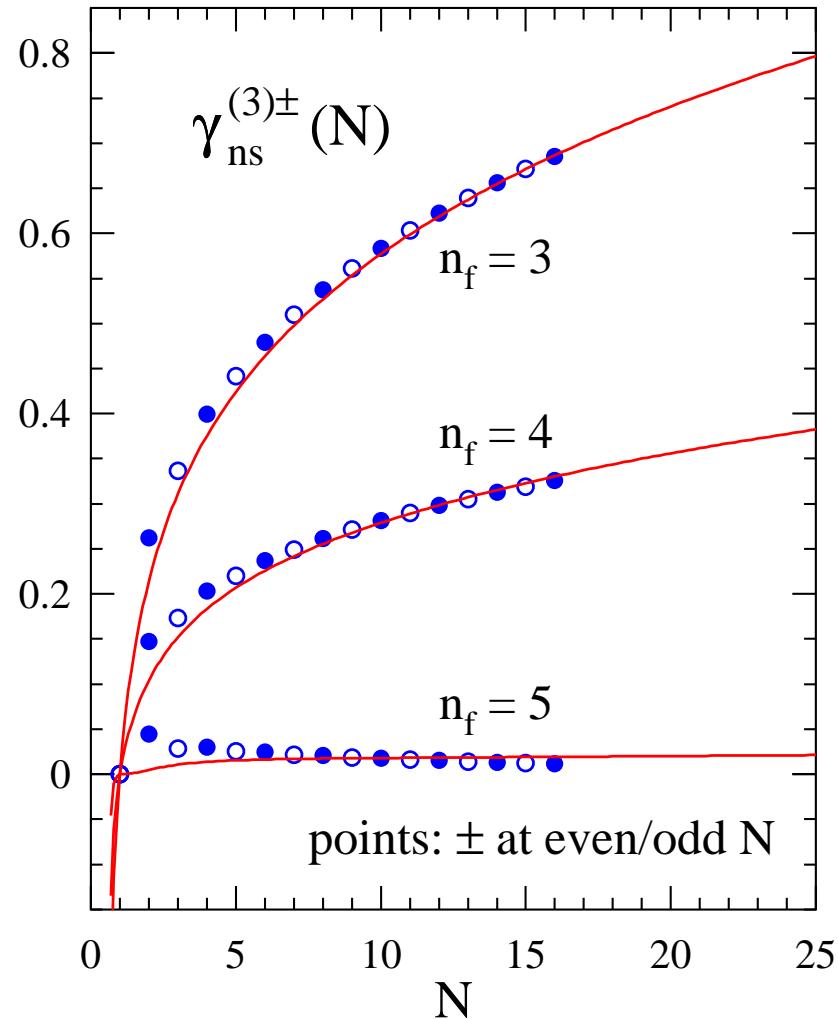
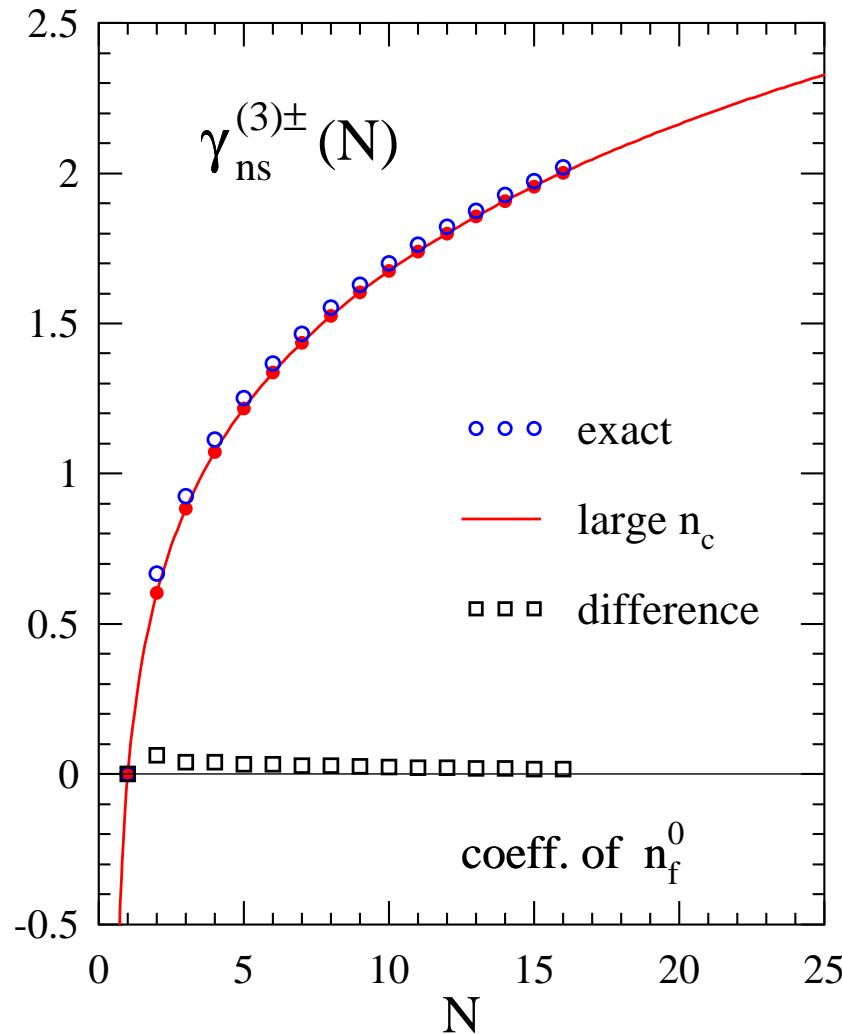
Agreement with result obtained from the large- n_c photon-quark form factor

Henn, Smirnov, Smirnov, Steinhauser [, Lee] (n_f : Apr '16, n_f^0 : 13 Dec '16)
 ζ_3^2, ζ_6 ($\mathcal{N} = 4$ SYM), Bern, Czakon, Dixon, Kosower, Smirnov ('06); ...

Further non-trivial check of the determination of the all- N form of $\gamma_{ns}(N)$

Also relevant beyond evolution of parton densities: $\delta(1-x)$ coefficient $B_{L,4}$

$\gamma_{\text{ns}}^{(3)\pm}(N)$: large- n_c limit vs QCD



Cancellations between powers of n_f : non large- n_c terms relevant at low N

Approximations of large- n_c suppressed parts

Analogous, but more accurate than those used before 2004 at 3 loops

van Neerven, A.V. ('99, '00); MVV [photon structure] ('01), ...

n_f^0 and n_f^1 parts $P_{N,0/1}^{(3)+}$ of $P_{\text{ns}}^{(3)+}(x)$: ansatz consisting of

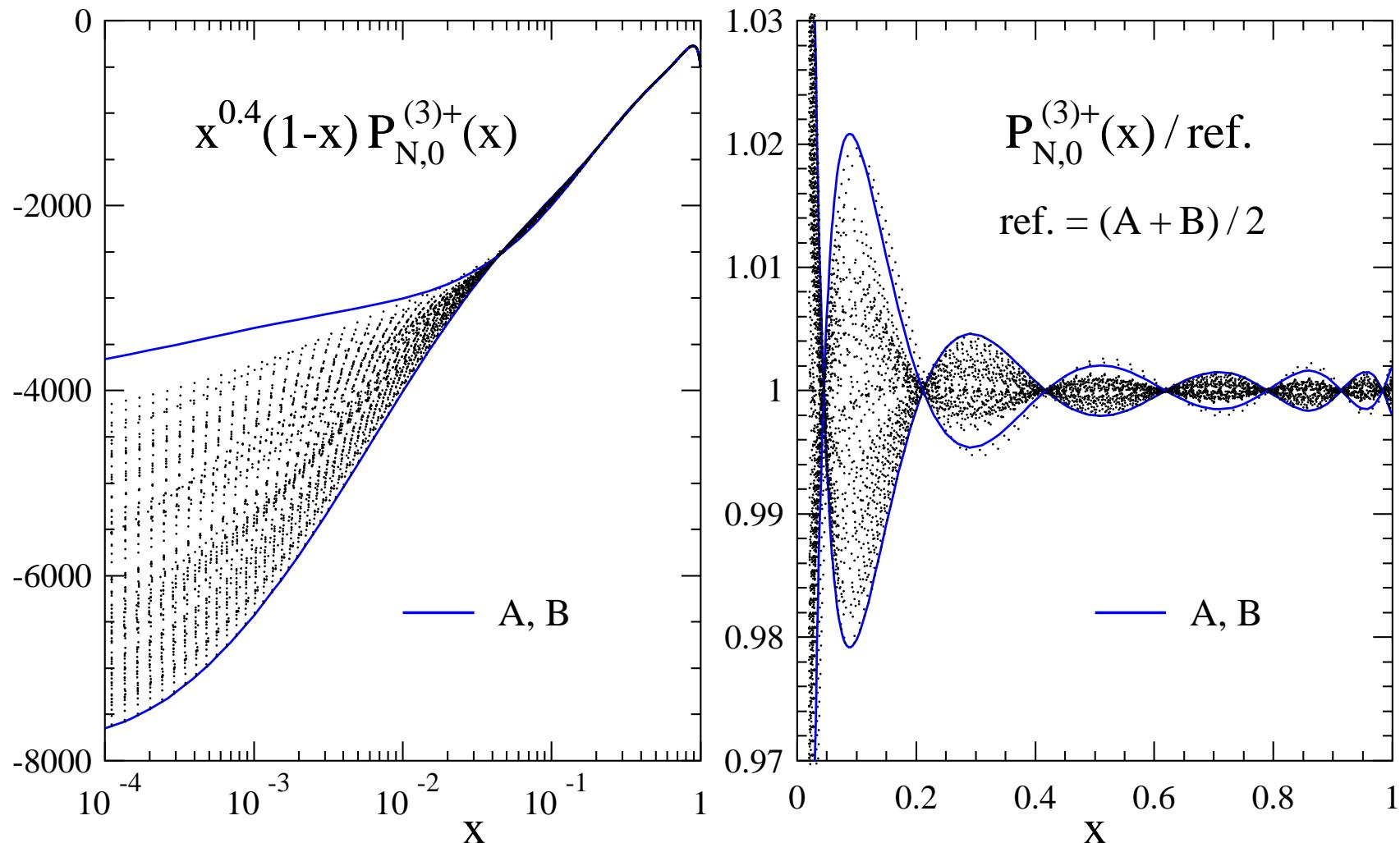
- the 2 large- x parameters A_4 and B_4
- 2 of 3 suppressed large- x logs $(1-x) \ln^k (1-x)$, $k = 1, 2, 3$
- one of ten 2-parameter polynomials in x vanishing for $x \rightarrow 1$
- 2 of the 3 unknown small- x logarithms $\ln^k x$, $k = 1, 2, 3$

90 resulting trial functions, parameters fixed from the 8 available moments,
two representatives chosen that indicate the remaining uncertainty

Checks: compare same treatment for the large- n_c parts to exact results;
compare $N=18$ prediction for n_f^1 part due exact result, ...

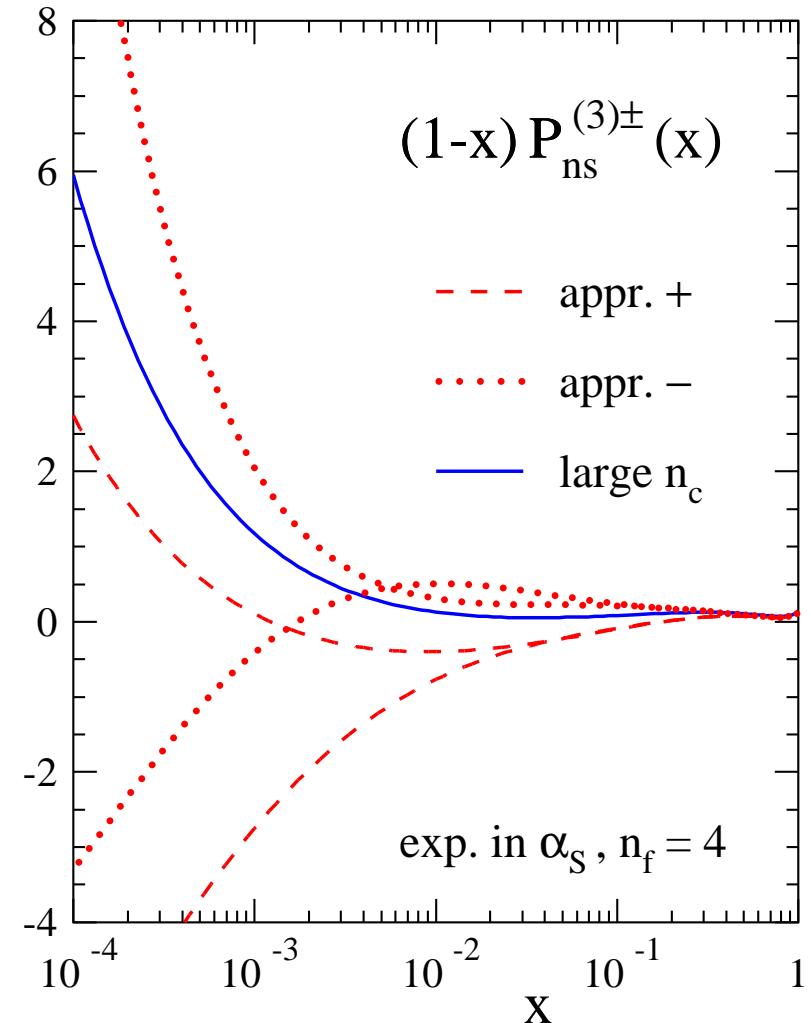
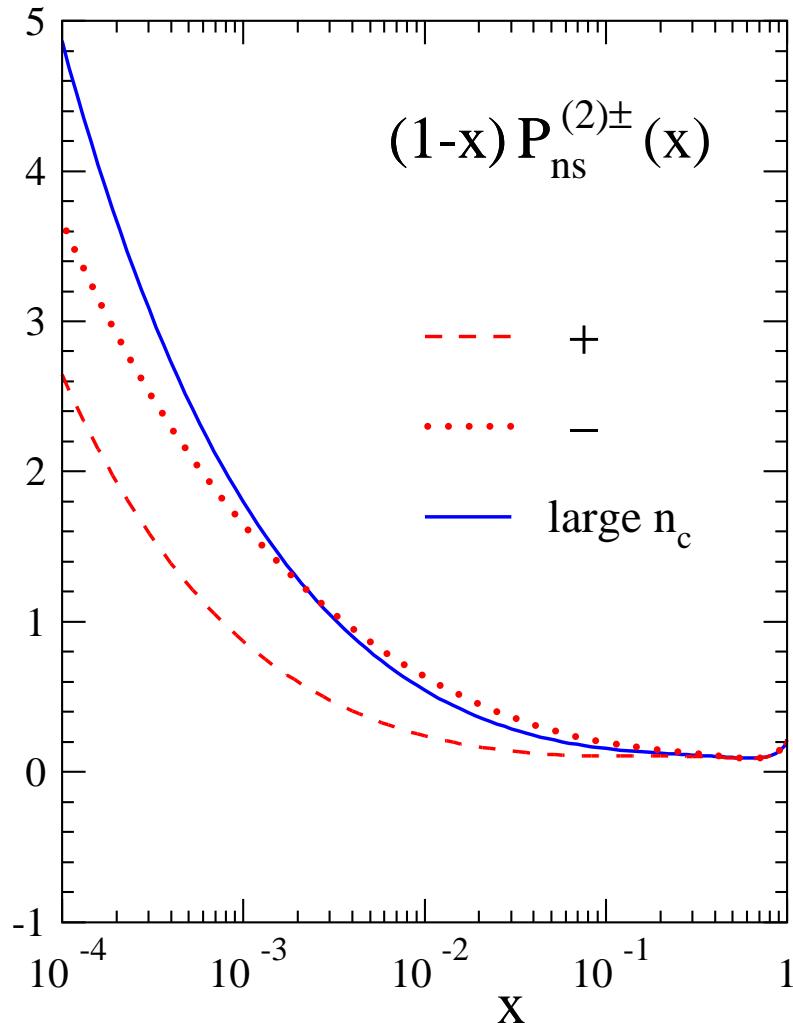
$P_{\text{ns}}^{(3)-}(x)$ and $P_{\text{ns}}^{(3)s}(x)$: similar, but with less small- x information

Example: approximations of the n_f^0 component



Factor $(1-x) \rightarrow$ value at $x=1$: contribution to cusp anom. dimension A_4

3- and 4-loop large- n_c limits vs $P_{\text{ns}}^{\pm}(x)$ in QCD



Weight-6 harmonic polylogarithms, numerically: Gehrmann, Remiddi (ext. by T.G.); Ablinger, Blümlein, Round, Schneider → compact high-accuracy parametrizations

Numerical applications: soft-gluon coefficients

4-loop cusp anom. dim. for QCD with n_f quark flavours, expansion in $\alpha_s/4\pi$

$$A_4 = 20702(2) - 5171.9(2) n_f + 195.5772 n_f^2 + 3.272344 n_f^3$$

In brackets: uncertainty of the preceding digit, conservative estimate

$$A_q(\alpha_s, n_f=3) = 0.42441 \alpha_s (1 + 0.72657 \alpha_s + 0.73405 \alpha_s^2 + 0.6647(2) \alpha_s^3 + \dots)$$

$$A_q(\alpha_s, n_f=4) = 0.42441 \alpha_s (1 + 0.63815 \alpha_s + 0.50998 \alpha_s^2 + 0.3168(2) \alpha_s^3 + \dots)$$

$$A_q(\alpha_s, n_f=5) = 0.42441 \alpha_s (1 + 0.54973 \alpha_s + 0.28403 \alpha_s^2 + 0.0133(2) \alpha_s^3 + \dots)$$

$n_f = 5$: much smaller than prev. Padé estimate (\Leftarrow quartic Casimirs, cf. β_3)

Corresponding coefficient of $\delta(1-x)$: similarly benign expansion

$$B_4 = 23393(10) - 5551(1) n_f + 193.8554 n_f^2 + 3.014982 n_f^3$$

$$B_q(\alpha_s, n_f=3) = 0.31831 \alpha_s (1 + 0.99712 \alpha_s + 1.24116 \alpha_s^2 + 1.0791(13) \alpha_s^3 + \dots)$$

$$B_q(\alpha_s, n_f=4) = 0.31831 \alpha_s (1 + 0.87192 \alpha_s + 0.97833 \alpha_s^2 + 0.5649(13) \alpha_s^3 + \dots)$$

$$B_q(\alpha_s, n_f=5) = 0.31831 \alpha_s (1 + 0.74672 \alpha_s + 0.71907 \alpha_s^2 + 0.1085(13) \alpha_s^3 + \dots)$$

A_4 , B_4 : individual colour factors

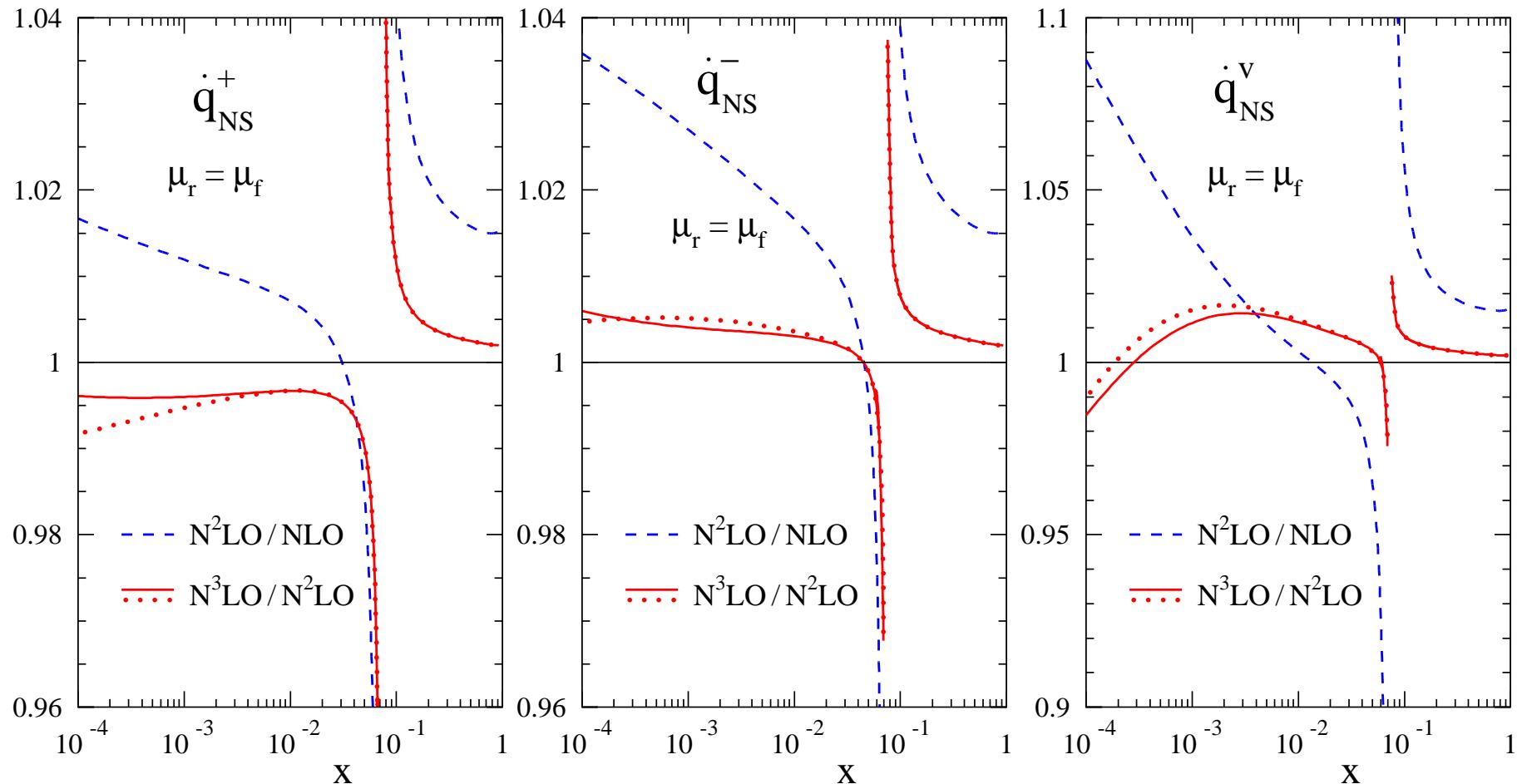
	A_4	B_4
C_F^4	0	$197. \pm 3.$
$C_F^3 C_A$	0	$-687. \pm 10.$
$C_F^2 C_A^2$	0	$1219. \pm 12.$
$C_F C_A^3$	610.3 ± 0.3	295.6 ± 2.4
$d_R^{abcd} d_A^{abcd} / N_R$	-507.5 ± 6.0	$-996. \pm 45.$
$n_f C_F^3$	-31.00 ± 0.4	81.4 ± 2.2
$n_f C_F^2 C_A$	38.75 ± 0.2	-455.7 ± 1.1
$n_f C_F C_A^2$	-440.65 ± 0.2	-274.4 ± 1.1
$n_f d_R^{abcd} d_R^{abcd} / N_R$	-123.90 ± 0.2	-143.5 ± 1.2
$n_f^2 C_F^2$	-21.31439	-5.775288
$n_f^2 C_F C_A$	58.36737	51.03056
$n_f^3 C_F$	2.454258	2.261237

Exact large- n_c limit: errors highly correlated. Quartic Casimirs: definitely $A_4 \neq 0$

see Gardi, Magnea ('09); Becher, Neubert ('09), ..., Grozin, Henn, Korchemsky, Marquard ('15); Boels, Huber, Yang ('17); Grozin, Henn, Stahlhofen ('17)

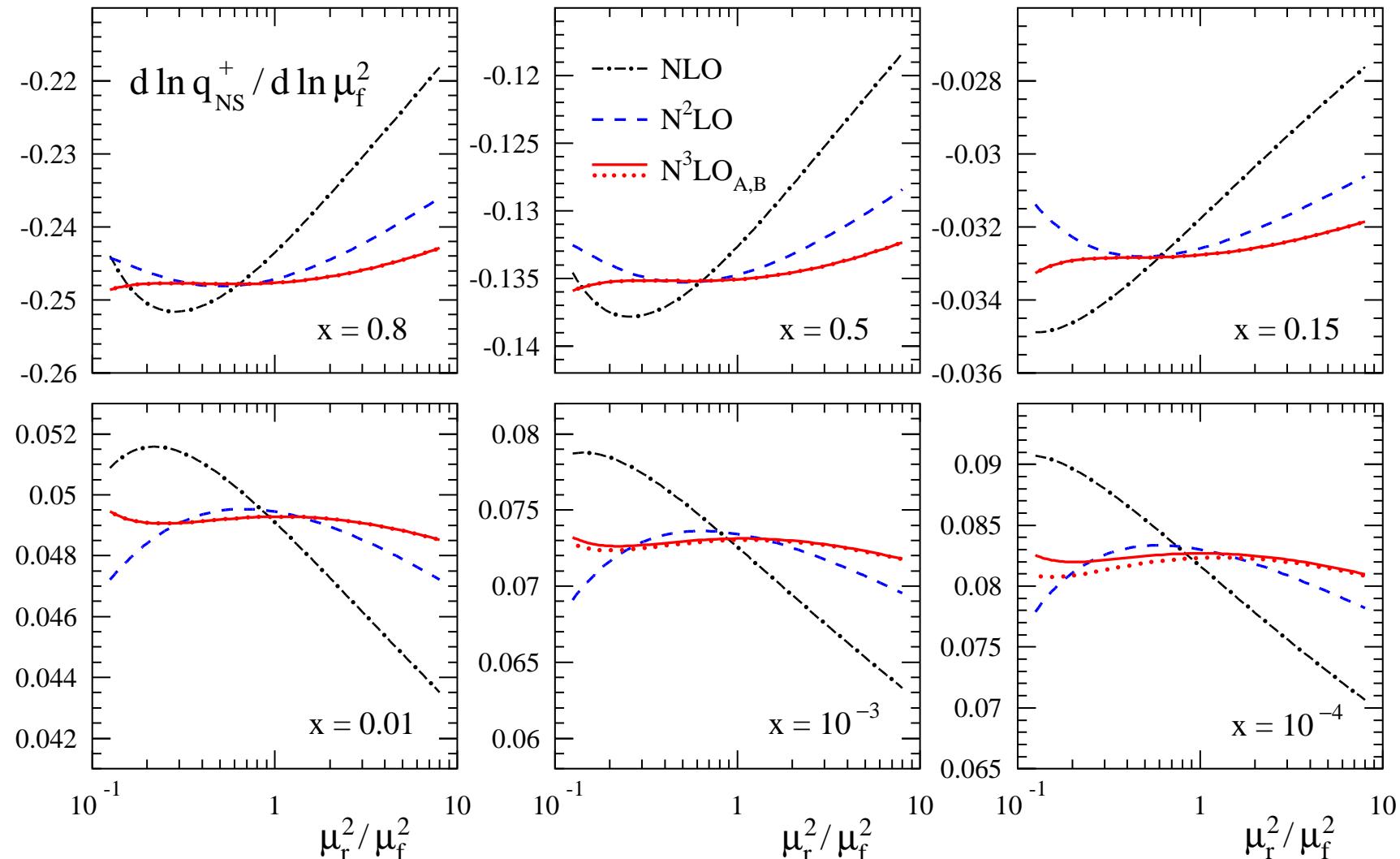
Higher-order corr's to the non-singlet evolutions

Logarithmic derivatives w.r.t. the factorization scale, $\dot{q}_{\text{ns}}^i \equiv d \ln q_{\text{ns}}^i / d \ln \mu_f^2$



Reference point: $xq_{\text{ns}}^{\pm, v}(x, \mu_0^2) = x^{0.5}(1 - x)^3$ with $\alpha_s(\mu_0^2) = 0.2$

NS⁺ evolution: renormalization scale dependence



$\frac{1}{2} \mu_f \leq \mu_r \leq 2 \mu_f$: scale uncertainty below 1% except close to sign change

Summary and Outlook

4-loop splitting functions/anomalous dimensions via DIS/OPE

$N=6$ and $N=16$ reached for complete non-singlet results

⇒ approximate results for $P_{\text{ns}}^{(3)a}(x)$ incl. cusp anomalous dimension

Large- n_c : $P_{\text{ns}}^+ = P_{\text{ns}}^-$, no alternating sums in γ_{ns}^\pm , RR, endpoints

$1 \leq N \leq 20$, LLL ⇒ all- N /all- x results (construction and checks)

4-loop singlet: high- N sufficient for approximations only via OPE:
theoretically and computationally much more challenging: next time

5-loop non-singlet: (very - for now) low N possible via local R^*

Herzog, Ruijl ('17); Ben's talk (Monday)

Check of the setup, with ξ^1 : $\gamma_{\text{ns}}^{(4)-}(N=1)$. First result: $N=2$. At $n_f=3/4$:

$$\gamma_{\text{ns}}^{(4)+} \cong 0.2829\alpha_s(1 + 0.8695\alpha_s + 0.7980\alpha_s^2 + 0.9258\alpha_s^3 + 1.7810\alpha_s^4 + \dots)$$

$$\gamma_{\text{ns}}^{(4)+} \cong 0.2829\alpha_s(1 + 0.7987\alpha_s + 0.5451\alpha_s^2 + 0.5215\alpha_s^3 + 1.2229\alpha_s^4 + \dots)$$

5-loop non-singlet anomalous dimension at $N=2$

$$\gamma_{\text{ns}}^{(4)+}(N=2) =$$

$$\begin{aligned}
 & C_F^5 \left[\frac{9306376}{19683} - \frac{802784}{729} \zeta_3 - \frac{557440}{81} \zeta_5 + \frac{12544}{9} \zeta_3^2 + 8512 \zeta_7 \right] \\
 & - C_A C_F^4 \left[\frac{81862744}{19683} - \frac{1600592}{243} \zeta_3 + \frac{59840}{81} \zeta_4 - \frac{142240}{27} \zeta_5 + 3072 \zeta_3^2 - \frac{35200}{9} \zeta_6 + 19936 \zeta_7 \right] \\
 & + C_A^2 C_F^3 \left[\frac{63340406}{6561} - \frac{1003192}{243} \zeta_3 - \frac{229472}{81} \zeta_4 + \frac{61696}{27} \zeta_5 + \frac{30976}{9} \zeta_3^2 - \frac{35200}{9} \zeta_6 + 15680 \zeta_7 \right] \\
 & - C_A^3 C_F^2 \left[\frac{220224724}{19683} + \frac{4115536}{729} \zeta_3 - \frac{170968}{27} \zeta_4 - \frac{3640624}{243} \zeta_5 + \frac{70400}{27} \zeta_3^2 + \frac{123200}{27} \zeta_6 + \frac{331856}{27} \zeta_7 \right] \\
 & + C_A^4 C_F \left[\frac{266532611}{39366} + \frac{2588144}{729} \zeta_3 - \frac{221920}{81} \zeta_4 - \frac{3102208}{243} \zeta_5 + \frac{74912}{81} \zeta_3^2 + \frac{334400}{81} \zeta_6 + \frac{178976}{27} \zeta_7 \right] \\
 & - \frac{d_A^{abcd} d_A^{abcd}}{N_A} C_F \left[\frac{15344}{81} - \frac{12064}{27} \zeta_3 - \frac{704}{3} \zeta_4 + \frac{58400}{27} \zeta_5 - \frac{6016}{3} \zeta_3^2 - \frac{19040}{9} \zeta_7 \right] \\
 & + \frac{d_R^{abcd} d_A^{abcd}}{N_R} C_F \left[\frac{23968}{81} - \frac{733504}{81} \zeta_3 + \frac{176320}{81} \zeta_5 + \frac{6400}{3} \zeta_3^2 + \frac{77056}{9} \zeta_7 \right] \\
 & - \frac{d_R^{abcd} d_A^{abcd}}{N_R} C_A \left[\frac{82768}{81} - \frac{555520}{81} \zeta_3 + \frac{10912}{9} \zeta_4 - \frac{1292960}{81} \zeta_5 + \frac{84352}{27} \zeta_3^2 + \frac{140800}{27} \zeta_6 + 12768 \zeta_7 \right] \\
 & + \text{terms with } n_f
 \end{aligned}$$