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q-Series and Modular Functions: a Basic Introduction

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Holonomic Functions and Sequences

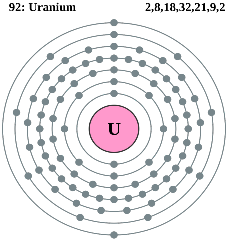
Example (relativistic Coulomb integrals):

Email to P. from Sergei Suslov, 27 Feb 2010:

"I am attaching two of my recent papers inspired by recent success in checking QED in strong fields [. . .].

It is a very complicated problem theoretically, and fantastically, enormously complicated (at the level of science fiction!) experimentally, which has been solved - after 20 years of hard work by theorists from Russia (Shabaev + 20 coauthors/students) and experimentalists from Germany.

Experimentally they took a uranium 92 atom, got rid of all but one electrons, and measured the energy shifts due to the quantization of the electromagnetic radiation field!



https://upload.wikimedia.org/wikipedia/commons/

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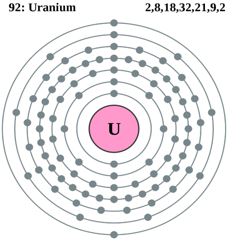

Mathematically, among other things, the precise structure of the energy levels of the U 91+ ion requires the evaluation of certain relativistic Coulomb integrals [...]:

https://upload.wikimedia.org/wikipedia/commons/

$$\begin{aligned} A_p &= \int_0^\infty r^{p+2} \left(F^2(r) + G^2(r)\right) \, dr, \\ B_p &= \int_0^\infty r^{p+2} \left(F^2(r) - G^2(r)\right) \, dr, \\ C_p &= \int_0^\infty r^{p+2} F(r)\, G(r) \, dr, \end{aligned}$$

with $p = 0, 1, \ldots$ and radial functions $F(r)$ and $G(r)$,

$$\begin{pmatrix} F(r) \\ G(r) \end{pmatrix} := a^2 \beta^{3/2} \sqrt{\frac{n!}{\gamma\ \Gamma(n+2\nu)}} (2a\beta r)^{\nu-1} e^{-a\beta r} \begin{pmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{pmatrix} \begin{pmatrix} L_{n-1}^{2\nu}(2a\beta r) \\ L_n^{2\nu}(2a\beta r) \end{pmatrix}$$

where $L_n^a(x)$ stands for the Laguerre polynomial of order n.

NOTE. Suslov evaluated these integrals in terms of linear combinations of 3 special generalized hypergeometric ${}_3F_2$ series related to the Chebyshev polynomials. — Suslov was inspired by work of L. Davis (1939) who concluded his article by saying:

"In conclusion I wish to thank Professors H. Bateman, P.S. Epstein, W.V. Houston and J.R. Openheimer."

$${}_3F_2\left(\begin{matrix} a_1,\ a_2,\ a_3 \\ b_1,\ b_2 \end{matrix};\ z\right) = \sum_{k=0}^{\infty} \frac{(a_1)_k(a_2)_k(a_3)_k}{(b_1)_k(b_2)_k}\frac{z^k}{k!},$$

where $(a)_k := a(a+1)\ldots(a+k-1)$, $k \geq 1$; $(a)_0 := 1$.

Email to P. from Sergei Suslov cont'd:

"These integrals have numerous recurrence relations found by physicists on the basis of virial theorems. They are also sums of 3 (linearly dependent) ${}_3F_2$ series[2]

Now you can imagine what a mess it is if one tries to derive those relations at the level of hypergeometric series (3 times 3 = 9 functions usually!).

It looks as a perfect job for the G-Z algorithm in a realistic (important) classical problem of relativistic quantum mechanics [...]."

[2] This independency can be determined algorithmically.

For example, Suslov computed:

$$2\mu\,(2a\beta)^p \;\frac{\Gamma\,(2\nu+1)}{\Gamma\,(2\nu+p+1)}\; A_p = 2p\varepsilon a n\ {}_3F_2\left(\begin{array}{c} 1-n,\ -p,\ p+1 \\ 2\nu+1,\quad 2 \end{array}\right)$$
$$+\,(\mu+a\kappa)\ {}_3F_2\left(\begin{array}{c} 1-n,\ -p,\ p+1 \\ 2\nu+1,\quad 1 \end{array}\right) + (\mu-a\kappa)\ {}_3F_2\left(\begin{array}{c} -n,\ -p,\ p+1 \\ 2\nu+1,\quad 1 \end{array}\right),$$

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Now packages like C. Schneider's `Sigma` deliver:

$$\begin{aligned} A_{p+1} &= \frac{\mu\,P\,(p)}{a^2\beta\,\left(4\mu^2\,(p+1)+p\,(2\varepsilon\kappa+p)\,(2\varepsilon\kappa+p+1)\right)\,(p+2)}\ A_p \\ &\quad - \frac{\left(4\nu^2-p^2\right)\left(4\mu^2\,(p+2)+(p+1)\,(2\varepsilon\kappa+p+1)\,(2\varepsilon\kappa+p+2)\right)p}{(2a\beta)^2\,\left(4\mu^2\,(p+1)+p\,(2\varepsilon\kappa+p)\,(2\varepsilon\kappa+p+1)\right)\,(p+2)}\ A_{p-1}, \end{aligned}$$

with

$$\begin{aligned} P\,(p) &= 2\varepsilon p\,(p+2)\,(2\varepsilon\kappa+p)\,(2\varepsilon\kappa+p+1) \\ &\quad +\varepsilon\left[4\left(\varepsilon^2\kappa^2-\nu^2\right)-p\left(4\varepsilon^2\kappa^2+p\,(p+1)\right)\right] \\ &\quad +(2p+1)\left[4\varepsilon^2\kappa+2\,(p+2)\left(2\varepsilon\mu^2-\kappa\right)\right], \end{aligned}$$

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He ingeniously combined

$$\int_0^\infty e^{-x}x^{a+s}L_m^a(x)L_n^b(x)dx = \frac{\Gamma(a+s+1)\Gamma(a+b+1)\Gamma(s+1)}{m!(n-m)!\Gamma(b+1)\Gamma(s-n+m+1)}$$
$$\times(-1)^{n-m}\ {}_3F_2\left(\begin{array}{c} -m,\ s+1,\ b-a-s \\ b+1,\quad n-m+1 \end{array};1\right), n \geq m \geq 0,$$

with properties of Hahn polynomials.

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HOLONOMIC FUNCTIONS and SEQUENCES.

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EXAMPLE. With C. Koutschan's package `HolonomicFunctions` one can derive linear recurrences (in p) for A_p, B_p, and C_p using as input only the integrands of these Coulomb integrals. $\Box$

Holonomic Sequences

$(a_n)_{n\geq 0}$ holonomic $:\Leftrightarrow$ there are polynomials $p, p_0, \ldots, p_r$, not all 0, such that

$$p_r(n)\, a_{n+r} + p_{r-1}(n)\, a_{n+1} + \cdots + p_0(n)\, a_n = p(n), \quad n \geq 0.$$

Examples.

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- $a_n = \sum_{k=0}^{n} \binom{n-k}{k}$: $a_{n+2} - a_{n+1} - a_n = 0$;
- $a_n = L_n^a(x)$ (Laguerre)
  $$(n+2)a_{n+2} - (2n+3+a-x)a_{n+1} + (n+a+1)a_n = 0.$$

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Example (Laguerre).

$$L_n^a(x) = \sum_{k=0}^{n} \binom{n+a}{n-k} \frac{(-x)^k}{k!} = \Gamma(n+a+1) \cdot \sum_{k=0}^{n} \frac{(-x)^k/k!}{\Gamma(k+a+1)} \cdot \frac{1}{(n-k)!}$$

with

$$a_n = \frac{(-x)^n/n!}{\Gamma(n+a+1)}, b_n = \frac{1}{n!}, c_n = \Gamma(n+a+1).$$

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NOTE (special case: hypergeometric sequences).

$$\frac{a_{n+1}}{a_n} = \frac{-x}{(n+1)(n+a+1)}.$$

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Examples.

- $f(x) = e^x, \log(x), \sin(x)$ and $\cos(x)$:
  $f'(x) - f(x) = 0, (1+x)f'(x) = 1, f''(x) - f(x) = 0;$

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  $xf''(x) + (a+1-x)f'(x) + nf(x) = 0;$
- $f(x) = {}_2F_1\left(\begin{matrix} a\ b \\ c \end{matrix}; x\right) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{x^k}{k!}$:
  $x(1-x)f''(x) + (c-(a+b+1)x)f'(x) - abf(x) = 0.$

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Example (Laguerre). $f(x) = L_n^a(x)$:

$$\int_0^x e^{-z} z^{a+s} L_m^a(z) L_n^b(z) dz \quad \text{is holonomic.}$$

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Recall

$$\int_0^\infty e^{-x} x^{a+s} L_m^a(x) L_n^b(x) dx = \frac{\Gamma(a+s+1)\Gamma(a+b+1)\Gamma(s+1)}{(m!(n-m)!\Gamma(b+1)\Gamma(s-n+m+1)}$$

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In[5]:= << RISC‘GeneratingFunctions‘

Package GeneratingFunctions version 0.7
written by Christian Mallinger © RISC-JKU

In[6]:= **RE2DE[{(n+2)a[n+2]−(2n+3+a−z)a[n+1]+(n+a+1)a[n] == 0, a[0] == 1, a[1] == a + 1 − z}, a[n], f[x]]   //Simplify**

In[7]:= $\mathbf{RE2DE[\{(n+2)a[n+2]-(2n+3+a-z)a[n+1]+(n+a+1)a[n] == 0,}$
$\mathbf{a[0] == 1, a[1] == a+1-z\}, a[n], f[x]] \quad //Simplify}$

Out[7]= $\{(-1+\mathtt{a}(-1+\mathtt{x})+\mathtt{x}+\mathtt{z})\mathtt{f}[\mathtt{x}]+(-1+\mathtt{x})^2\mathtt{f}'[\mathtt{x}] == 0, \mathtt{f}[0] == 1\}$

In[10]:= $\mathbf{RE2DE[\{(n+2)a[n+2]-(2n+3+a-z)a[n+1]+(n+a+1)a[n] == 0,}$
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In[11]:= $\mathbf{DSolve[\{(-1+a(x-1)+x+z)f[x]+(x-1)^2 f'[x] == 0,}$
$\mathbf{f[0] == 1\}, f[x], x]} \quad \mathbf{//Simplify}$

In[13]:= $\mathbf{RE2DE[\{(n{+}2)a[n{+}2]-(2n{+}3{+}a{-}z)a[n{+}1]+(n{+}a{+}1)a[n] == 0, a[0] == 1, a[1] == a+1-z\}, a[n], f[x]]}$ $\mathbf{//Simplify}$

Out[13]= $\{(-1+a(-1+x)+x+z)f[x]+(-1+x)^2 f'[x] == 0, f[0] == 1\}$

In[14]:= $\mathbf{DSolve[\{(-1+a(x-1)+x+z)f[x]+(x-1)^2 f'[x] == 0, f[0] == 1\}, f[x], x]}$ $\mathbf{//Simplify}$

Out[14]= $\left\{\left\{f(x) \to (x-1)^{-a-1} e^{\frac{xz}{x-1}+i\pi(a+1)}\right\}\right\}$

In[15]:= $\mathbf{DE2RE[\{(-1+\alpha(-1+x)+x+z)f[x]+(-1+x)^2 f'[x] == 0, f[0] == 1\}, f[x], a[n]]}$

In[16]:= $\mathbf{RE2DE[\{(n+2)a[n+2]-(2n+3+a-z)a[n+1]+(n+a+1)a[n] == 0,}$
$\mathbf{a[0] == 1, a[1] == a+1-z\}, a[n], f[x]] \quad //Simplify}$

Out[16]= $\{(-1+\mathtt{a}(-1+\mathtt{x})+\mathtt{x}+\mathtt{z})\mathtt{f}[\mathtt{x}]+(-1+\mathtt{x})^2\mathtt{f}'[\mathtt{x}] == 0, \mathtt{f}[0] == 1\}$

In[17]:= $\mathbf{DSolve[\{(-1+a(x-1)+x+z)f[x]+(x-1)^2f'[x] == 0,}$
$\mathbf{f[0] == 1\}, f[x], x] \quad //Simplify}$

Out[17]= $\left\{\left\{\mathtt{f}(\mathtt{x}) \to (\mathtt{x}-1)^{-\mathtt{a}-1}\mathtt{e}^{\frac{\mathtt{xz}}{\mathtt{x}-1}+\mathtt{i}\pi(\mathtt{a}+1)}\right\}\right\}$

In[18]:= $\mathbf{DE2RE[\{(-1+\alpha(-1+x)+x+z)f[x]+(-1+x)^2f'[x] == 0,}$
$\mathbf{f[0] == 1\}, f[x], a[n]]}$

Out[18]= $\{(1+\mathtt{n}+\alpha)\mathtt{a}[\mathtt{n}]+(-3-2\mathtt{n}+\mathtt{z}-\alpha)\mathtt{a}[1+\mathtt{n}]+(2+\mathtt{n})\mathtt{a}[2+\mathtt{n}] == 0,$
$\mathtt{a}[0] == 1, \mathtt{a}[1] == 1-\mathtt{z}+\alpha\}$

References for Part 1

All articles are available at

`www.risc.jku.at/research/combinat/publications`

- Peter Paule, Sergei K. Suslov: Relativistic Coulomb Integrals and Zeilbergers Holonomic Systems Approach I.
- Christoph Koutschan, Peter Paule, Sergei K. Suslov: Relativistic Coulomb Integrals and Zeilberger's Holonomic Systems Approach II.
- Manuel Kauers, Peter Paule: The Concrete Tetrahedron. Springer Texts and Monographs in Symbolic Computation. [Book introducing to the algorithmic theory of univariate holonomic sequences.]
- C. Mallinger: Algorithmic Manipulations and Transformations of Univariate Holonomic Functions and Sequences. [Describes the Mathematica package `GeneratingFunctions`.]

q-Holonomic Functions and Sequences

q-Holonomic Sequences

$(a_n)_{n\geq 0}$ q-holonomic :$\Leftrightarrow$ there are polynomials $p, p_0, \ldots, p_r$, not all 0, such that

$$p_r(q^n)\, a_{n+r} + p_{r-1}(q^n)\, a_{n+1} + \cdots + p_0(q^n)\, a_n = p(q^n), \quad n \geq 0.$$

Examples.

- $a_{n+1} - (1 - zq^n)a_n = 0, \;\; n \geq 0$, hence

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- $a_{n+1} - (1 - zq^n)a_n = 0, \;\; n \geq 0$, hence

$$a_n = (1 - zq^{n-1})a_{n-1} = (1 - zq^{n-1})(1 - zq^{n-2}) \ldots (1 - z)a_0.$$

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q-shifted factorials:

$$(z;q)_n := \begin{cases} (1-z)(1-zq)\ldots(1-zq^{n-1}), & \text{if } n \geq 1, \\ 1, & \text{if } n = 0, \\ \frac{1}{(1-zq^{-1})(1-zq^{-2})\cdots(1-zq^{n})}, & \text{if } n \leq 1. \end{cases}$$

q-Holonomic Sequences

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Examples cont'd.

q-Holonomic Sequences

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Examples cont'd.

- q-binomial coefficients $a_n = \binom{n}{k}_q := \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}$:

$$(1 - q^{n-k+1})a_{n+1} - (1 - q^{n+1})a_n = 0.$$

---

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$$p_r(q^n)\, a_{n+r} + p_{r-1}(q^n)\, a_{n+1} + \cdots + p_0(q^n)\, a_n = p(q^n), \quad n \geq 0.$$

Examples cont'd.

- q-binomial coefficients $a_n = \binom{n}{k}_q := \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}$:

$$(1 - q^{n-k+1}) a_{n+1} - (1 - q^{n+1}) a_n = 0.$$

NOTE: For $q \to 1$,

$$\binom{n}{k}_q = \frac{(1-q^n)(1-q^{n-1})\ldots(1-q^{n-k+1})}{(1-q^k)(1-q^{k-1})\ldots(1-q)} \to \binom{n}{k},$$

owing to the BASIC property (q is the "base"):

$$\lim_{q \to 1} \frac{1-q^a}{1-q} = a.$$

q-Holonomic Sequences

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Examples cont'd.

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Examples cont'd.

- $a_k = \binom{n}{k}_q := \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}$:

$$(1-q^{k+1})a_{k+1} - (1-q^{n-k})a_k = 0;$$

- $a_n = \sum_{k=0}^{n} q^{k^2} \binom{n-k}{k}_q$:

$$a_{n+2} - a_{n+1} - q^{n+1} a_n = 0.$$

q-Holonomic Sequences: Closure Properties

$(a_n)_{n\geq 0}, (b_n)_{n\geq 0}$ q-holonomic $\Rightarrow$

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- $(c_n)_{n\geq 0}$ with $c_n := \sum_{k=0}^{n} a_k b_{n-k}$ q-holonomic;

Example (q-binomial theorem).

$$S(n) := \sum_{k=0}^{n} \binom{n}{k}_q q^{\frac{1}{2}k^2-\frac{1}{2}k} x^k = (q;q)_n \cdot \sum_{k=0}^{n} \frac{q^{\frac{1}{2}k^2-\frac{1}{2}k} x^k}{(q;q)_k} \cdot \frac{1}{(q;q)_{n-k}}$$

with

$$a_n = \frac{q^{\frac{1}{2}n^2-\frac{1}{2}n} x^n}{(q;q)_n}, b_n = \frac{1}{(q;q)_n}, c_n = (q;q)_n.$$

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with

$$a_n = \frac{q^{\frac{1}{2}n^2-\frac{1}{2}n} x^n}{(q;q)_n}, b_n = \frac{1}{(q;q)_n}, c_n = (q;q)_n.$$

NOTE (special case: q-hypergeometric sequences).

$$\frac{a_{n+1}}{a_n} = \frac{q^n x}{1-q^{n+1}}.$$

Example (q-binomial theorem) cont'd.

Recall

$$S(n) := \sum_{k=0}^{n} \binom{n}{k}_q q^{\frac{1}{2}k^2-\frac{1}{2}k} x^k.$$

TASK. Derive the q-recurrence for $S(n)$.

Example (q-binomial theorem) cont'd.

Recall

$$S(n) := \sum_{k=0}^{n} \binom{n}{k}_q q^{\frac{1}{2}k^2 - \frac{1}{2}k} x^k.$$

TASK. Derive the q-recurrence for $S(n)$.

Apply C. Schneider's `Sigma` or A. Riese's `qZeil`:

In[21]:= $<<$ **RISC`qZeil`**

Package q-Zeilberger version 4.50
written by Axel Riese © RISC-JKU

In[22]:= $\mathbf{qZeil[qBinomial[n, k, q]\, q^{Binomial[k,2]} x^k, \{k, 0, n\}, n, 1]}$

Out[22]= $\mathsf{SUM}[n] == (1 + q^{n-1} x)\mathsf{SUM}[-1 + n]$

Example (q-binomial theorem) cont'd.

Recall

$$S(n) := \sum_{k=0}^{n} \binom{n}{k}_q q^{\frac{1}{2}k^2 - \frac{1}{2}k} x^k.$$

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Apply C. Schneider's `Sigma` or A. Riese's `qZeil`:

In[23]:= $<<$ **RISC`qZeil`**

> Package q-Zeilberger version 4.50
> written by Axel Riese © RISC-JKU

In[24]:= $\mathbf{qZeil[qBinomial[n, k, q]\, q^{Binomial[k,2]} x^k, \{k, 0, n\}, n, 1]}$

Out[24]= $\mathsf{SUM[n] == (1 + q^{n-1}x)SUM[-1 + n]}$

Hence,

$$S(n) = (1 + q^{n-1}x)(1 + q^{n-2}x) \ldots (1 + x)S(0) = (-x, q)_n.$$

q-Holonomic Functions

Let $f(x) := \sum_{n=0}^{\infty} a_n x^n$:

q-derivative: $$D_q f(x) := \frac{f(qx) - f(x)}{(q-1)x} = \sum_{n=1}^{\infty} a_n \frac{q^n - 1}{q - 1} x^{n-1}.$$

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$f(x)$ q-holonomic :$\Leftrightarrow$ there are polynomials $p, p_0, \ldots, p_r$, not all 0, such that

$$p_r(x)\, D_q^r f(x) + p_{r-1}(x)\, D_q^{r-1} f(x) + \cdots + p_0(x)\, f(x) = p(x).$$

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Example (q-binomial theorem):

$$f(x) := \sum_{k=0}^{\infty} \frac{(a;q)_k}{(q;q)_k} x^k = \lim_{n\to\infty} \frac{(ax;q)_n}{(x;q)_n} = \frac{(ax;q)_\infty}{(x;q)_\infty}$$

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$$\Rightarrow D_q f(x) = \frac{1-a}{(1-q)(1-ax)} f(x)$$

Recall

$$p_r(x)\,D_q^r f(x) + p_{r-1}(x)\,D_q^{r-1} f(x) + \cdots + p_0(x)\,f(x) = p(x).$$

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$$p_r(x)\, D_q^r f(x) + p_{r-1}(x)\, D_q^{r-1} f(x) + \cdots + p_0(x)\, f(x) = p(x).$$

Another example (basic hypergeometric ${}_2\phi_1$-series):

$${}_2\phi_1\left(\begin{matrix} a \;\; b \\ c \end{matrix}; q, x\right) = \sum_{k=0}^{\infty} \frac{(a;q)_k (b;q)_k}{(c;q)_k} \frac{x^k}{(q;q)_k}$$

NOTE.

$$f(x) := \sum_{k=0}^{\infty} \frac{(a;q)_k}{(q;q)_k} x^k = {}_2\phi_1\left(\begin{matrix} a \;\; 0 \\ 0 \end{matrix}; q, x\right) = {}_1\phi_0\left(\begin{matrix} a \\ - \end{matrix}; q, x\right).$$

q-Holonomic Sequences $\leftrightarrow$ q-Holonomic Functions

$(c_n)_{n\geq 0}$ q-holonomic $\Leftrightarrow$ $f(x) := \sum_{n=0}^{\infty} c_n x^n$ q-holonomic.

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In[25]:= $<<$ **RISC`qGeneratingFunctions`**

qGeneratingFunctions Package version 1.9.1
written by Christoph Koutschan © RISC-JKU

In[26]:= $\mathbf{QRE2DE[\{(1-q^{n+1})c[n+1]-(1-aq^n)c[n]==0,}$
$\mathbf{c[0]==1\},c[n],f[x]]}$

Out[26]= $\{(-1+a)f[x]+(-1+q)(-1+ax)f'[x]==0, f[0]==1\}$

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In[27]:= $\mathbf{QRE2SE[\{(1-q^{n+1})c[n+1]-(1-aq^n)c[n]==0, c[0]==1\}, c[n], f[x]]}$

Out[27]= $\{(1-x)f[x]+(-1+ax)f[qx]==0, \langle 1\rangle\,[f[x]]==1\}$

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In[28]:= $\mathbf{QRE2SE[\{(1 - q^{n+1})c[n+1] - (1 - aq^n)c[n] == 0, c[0] == 1\}, c[n], f[x]]}$

Out[28]= $\{(1 - x)f[x] + (-1 + ax)f[qx] == 0, \langle 1 \rangle [f[x]] == 1\}$

NOTE.

$$f(x) = \frac{1-ax}{1-x} f(qx) = \frac{1-ax}{1-x}\frac{1-axq}{1-xq} \cdots \frac{1-axq^{n-1}}{1-xq^{n-1}} f(q^n x).$$

q-Holonomic Functions: Closure Properties

$f(x), g(x)$ q-holonomic $\Rightarrow$

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- $f(cx^j)$ holonomic ($c \in \mathbb{K}, j \in \mathbb{Z}_{>0}$).

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Example. $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{(q;q)_n} \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n}{2}} x^n}{(q;q)_n}$ is q-holonomic:

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In[39]:= $\mathbf{SE1 = QRE2SE[\{(1-q^{n+1})c[n+1]-c[n] == 0, c[0] == 1\}, c[n], f[x]]}$

Out[39]= $\{(1-\mathtt{x})\mathtt{f}[\mathtt{x}] - \mathtt{f}[\mathtt{qx}] == 0, \langle 1 \rangle[\mathtt{f}[\mathtt{x}]] == 1\}$

In[40]:= $\mathbf{SE2 = QRE2SE[\{(1-q^{n+1})c[n+1] + q^n c[n] == 0,}$
$\mathbf{c[0] == 1\}, c[n], f[x]]}$

Out[40]= $\mathtt{f}[\mathtt{x}] + (-1+\mathtt{x})\mathtt{f}[\mathtt{qx}] == 0, \langle 1 \rangle[\mathtt{f}[\mathtt{x}]] == 1$

In[41]:= $\mathbf{SE1 = QRE2SE[\{(1-q^{n+1})c[n+1]-c[n] == 0, c[0] == 1\}, c[n], f[x]]}$

Out[41]= $\{(1 - \mathtt{x})\mathtt{f}[\mathtt{x}] - \mathtt{f}[\mathtt{qx}] == 0, \langle 1\rangle[\mathtt{f}[\mathtt{x}]] == 1\}$

In[42]:= $\mathbf{SE2 = QRE2SE[\{(1 - q^{n+1})c[n + 1] + q^{n}c[n] == 0,}$
$\mathbf{c[0] == 1\}, c[n], f[x]]}$

Out[42]= $\mathtt{f}[\mathtt{x}] + (-1 + \mathtt{x})\mathtt{f}[\mathtt{qx}] == 0, \langle 1\rangle[\mathtt{f}[\mathtt{x}]] == 1$

In[43]:= $\mathbf{QSECauchy[SE1, SE2, f[x]]}$

In[44]:= $\mathbf{SE1 = QRE2SE[\{(1-q^{n+1})c[n+1]-c[n] == 0, c[0] == 1\}, c[n], f[x]]}$

Out[44]= $\{(1-x)f[x] - f[qx] == 0, \langle 1\rangle[f[x]] == 1\}$

In[45]:= $\mathbf{SE2 = QRE2SE[\{(1-q^{n+1})c[n+1] + q^n c[n] == 0,}$
$\mathbf{c[0] == 1\}, c[n], f[x]]}$

Out[45]= $f[x] + (-1+x)f[qx] == 0, \langle 1\rangle[f[x]] == 1$

In[46]:= $\mathbf{QSECauchy[SE1, SE2, f[x]]}$

Out[46]= $\{-f[x] + f[qx] == 0, \langle 1\rangle[f[x]] == 1\}$

In[47]:= $\mathbf{SE1 = QRE2SE[\{(1-q^{n+1})c[n+1]-c[n] == 0, c[0] == 1\}, c[n], f[x]]}$

Out[47]= $\{(1-x)f[x] - f[qx] == 0, \langle 1\rangle[f[x]] == 1\}$

In[48]:= $\mathbf{SE2 = QRE2SE[\{(1-q^{n+1})c[n+1] + q^n c[n] == 0,}$
$\mathbf{c[0] == 1\}, c[n], f[x]]}$

Out[48]= $f[x] + (-1+x)f[qx] == 0, \langle 1\rangle[f[x]] == 1$

In[49]:= $\mathbf{QSECauchy[SE1, SE2, f[x]]}$

Out[49]= $\{-f[x] + f[qx] == 0, \langle 1\rangle[f[x]] == 1\}$

Hence

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n}{2}} x^n}{(q;q)_n} = \left(\sum_{n=0}^{\infty} \frac{x^n}{(q;q)_n}\right)^{-1} = \left(\sum_{n=0}^{\infty} \frac{(0;q)_n}{(q;q)_n} x^n\right)^{-1} = \frac{(x;q)_\infty}{(0\cdot x;q)_\infty}.$$

q-Hypergeometric Summation

q-Contiguous Relations; e.g., Ramanujan's ${}_1\psi_1$ summation:

$$F(a,b;x) := \sum_{n=-\infty}^{\infty} f_n(a,b;x) \text{ with } f_n(a,b;x) := \frac{(a;q)_n}{(b;q)_n} x^n.$$

NOTE. $$\frac{f_n(a,qb;x)}{f_n(a,b;x)} = \frac{1-b}{1-bq^n}.$$

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NOTE. $$\frac{f_n(a,qb;x)}{f_n(a,b;x)} = \frac{1-b}{1-bq^n}.$$

In[51]:= $\mathbf{qTelescope[f[a,b,n],\{n,-N,N\},qParameterized->\{1,\frac{1-b}{1-bq^n}\}]}$

Out[51]= $$\mathrm{Sum}\left(\frac{F_0(n)(b-ax)}{b} + \frac{x(b-a)F_1(n)}{(b-1)b}, \{n,-N,N\}\right) = \frac{a^{-N}b^N x^{-N}(q/b;q)_N}{(q/a;q)_N} - \frac{x^{N+1}(a;q)_{N+1}}{(b;q)_{N+1}}$$

The output of `qZeil` means this:

$$\left(1-\frac{ax}{b}\right)\sum_{n=-N}^{N} f_n(a,b;x)+\frac{ax}{b}\frac{1-b/a}{1-b}\sum_{n=-N}^{N} f_n(a,qb;x)$$
$$=\frac{a^{-N}b^{N}x^{-N}(q/b;q)_N}{(q/a;q)_N}-\frac{x^{N+1}(a;q)_{N+1}}{(b;q)_{N+1}}$$

Assume $|\frac{b}{a}|\leq|x|<1$, then in the limit $N\to\infty$:

$$\left(1-\frac{ax}{b}\right)F(a,b;x)=\frac{ax}{b}\frac{1-b/a}{1-b}F(a,qb;x).$$

Finally, we iterate this relation:

$$\begin{aligned}
F(a,b;x) &= \frac{1-\frac{b}{a}}{(1-b)(1-\frac{b}{ax})}F(a,qb;x)\\
&= \frac{(1-\frac{b}{a})(1-\frac{b}{a}q)}{(1-b)(1-bq)(1-\frac{b}{ax})(1-\frac{b}{ax}q)}F(a,q^2b;x)\\
&= \dots\\
&= \frac{\left(\frac{b}{a};q\right)_N}{(b;q)_N\left(\frac{b}{ax};q\right)_N}F(a,q^Nb;x)
\end{aligned}$$

In the limit $N \to \infty$:

$$
\begin{aligned}
F(a,b;x) &= \frac{1-\frac{b}{a}}{(1-b)(1-\frac{b}{ax})} F(a,qb;x) \\
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\end{aligned}
$$

In the limit $N \to \infty$:

$$
F(a,b;x) = \frac{\left(\frac{b}{a};q\right)_\infty}{(b;q)_\infty\left(\frac{b}{ax};q\right)_\infty} F(a,0;x)
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$$
\begin{aligned}
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&= \dots\\
&= \frac{\left(\frac{b}{a};q\right)_N}{(b;q)_N\left(\frac{b}{ax};q\right)_N}F(a,q^Nb;x)
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$$

In the limit $N \to \infty$:

$$F(a,b;x) = \frac{\left(\frac{b}{a};q\right)_\infty}{(b;q)_\infty\left(\frac{b}{ax};q\right)_\infty}F(a,0;x)$$

In this relation set $b = q$:

$$F(a,q;x) = \frac{\left(\frac{q}{a};q\right)_\infty}{(q;q)_\infty\left(\frac{q}{ax};q\right)_\infty}F(a,0;x).$$

Finally, we collect things:

$$\begin{aligned}
F(a,b;x) &= \frac{\left(\frac{b}{a};q\right)_\infty}{(b;q)_\infty\left(\frac{b}{ax};q\right)_\infty} F(a,0;x) \\
&= \frac{\left(\frac{b}{a};q\right)_\infty}{(b;q)_\infty\left(\frac{b}{ax};q\right)_\infty} \cdot \frac{(q;q)_\infty\left(\frac{q}{ax};q\right)_\infty}{\left(\frac{q}{a};q\right)_\infty} \cdot F(a,q;x) \\
&= \frac{\left(\frac{b}{a};q\right)_\infty}{(b;q)_\infty\left(\frac{b}{ax};q\right)_\infty} \cdot \frac{(q;q)_\infty\left(\frac{q}{ax};q\right)_\infty}{\left(\frac{q}{a};q\right)_\infty} \cdot \frac{(ax;q)_\infty}{(x;q)_\infty}; \quad \text{i.e.,}
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$$\sum_{n=-\infty}^{\infty} \frac{(a;q)_n}{(b;q)_n} x^n = \frac{(\frac{b}{a};q)_\infty (ax;q)_\infty (\frac{q}{ax};q)_\infty (q;q)_\infty}{(b;q)_\infty (\frac{b}{ax};q)_\infty (\frac{q}{a};q)_\infty (x;q)_\infty}.$$

Recall Ramanujan's ${}_1\psi_1$ summation:

$$\sum_{n=-\infty}^{\infty} \frac{(a;q)_n}{(b;q)_n} x^n = \frac{(\frac{b}{a};q)_\infty (ax;q)_\infty (\frac{q}{ax};q)_\infty (q;q)_\infty}{(b;q)_\infty (\frac{b}{ax};q)_\infty (\frac{q}{a};q)_\infty (x;q)_\infty}.$$

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NOTE. In the limit $a \to \infty$:

$$\frac{(a;q)_n}{a^n} = \left(\frac{1}{a} - 1\right)\left(\frac{1}{a} - q\right) \dots \left(\frac{1}{a} - q^{n-1}\right) \to (-1)^n q^{\binom{n}{2}}.$$

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Hence after $x \to \frac{x}{a}$, the ${}_1\psi_1$ summation for $a \to \infty$ turns into

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Finally $b=0$ gives Jacobi's triple product identity:

$$\sum_{n=-\infty}^{\infty} (-1)^n q^{\binom{n}{2}} x^n = (x;q)_\infty \left(\frac{q}{x};q\right)_\infty (q;q)_\infty.$$

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This, essentially, is a Jacobi theta function:

$$\sum_{n=-\infty}^{\infty} (-1)^n q^{\binom{n}{2}} e^{-(2n-1)iz} = iq^{-\frac{1}{8}} \theta_1(z,\tau). \tag{1}$$

References for Part 2

All references are available at

`www.risc.jku.at/research/combinat/publications`

- Manuel Kauers, Christoph Koutschan: A Mathematica Package for q-Holonomic Sequences and Power Series. [Describes the Mathematica package `GeneratingFunctions`.]
- Christoph Koutschan, Peter Paule: Holonomic Tools for Basic Hypergeometric Functions. [Describes q-applications of Koutschan's Mathematica package `HolonomicFunctions`.]
- Peter Paule, Axel Riese: A Mathematica q-Analogue of Zeilberger's Algorithm Based on an Algebraically Motivated Approach to q-Hypergeometric Telescoping. [Describes the Mathematica package `qZeil`.]

Modular Functions

Modular group $\mathrm{SL}_2(\mathbb{Z})$ and congruence subgroups for $N \in \mathbb{Z}_{>0}$:

$$\mathrm{SL}_2(\mathbb{Z}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{Z}^{2\times 2} : ad - bc = 1 \right\}.$$

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- f is meromorphic on upper half complex plane.

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$\Rightarrow \exists$ q-expansions at $\tau = i\infty$ with $\mathbf{q} := \mathbf{q}(\tau) = \mathbf{e}^{\mathbf{2\pi i}\tau}$:

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = \frac{a_{-N}}{q^N} + \frac{a_{-N+1}}{q^{N-1}} + \cdots + \frac{a_{-1}}{q} + a_0 + a_1 q + \cdots$$

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$$f\left(\frac{a}{c}\right) := \begin{cases} \infty, & \text{if } N > 0 \\ a_0, & \text{if } N = 0 \\ 0, & \text{otherwise} \end{cases}.$$

MODULAR FORMS of weight k for congruence subgroups Γ

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$$f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k f(\tau) \text{ for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma.$$

- Similar growth condition on $\lim_{\mathrm{Im}(\tau)\to\infty} |f(a/c)|$.

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Example for $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ (Dedekind eta function).

$$\eta(\tau) := q(\tau)^{\frac{1}{24}} \prod_{k=1}^{\infty} (1-q(\tau))^k :$$

$$\eta\left(\frac{a\,\tau+b}{c\,\tau+d}\right) = e^{2\pi i\,\rho(a,b,c,d)/24} \cdot \sqrt{\frac{c\tau+d}{i}} \cdot \eta(\tau).$$

Recall

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n}{2}} x^n}{(q;q)_n} = \left(\sum_{n=0}^{\infty} \frac{x^n}{(q;q)_n} \right)^{-1} = (x;q)_\infty.$$

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- Modular forms/functions as projections of q-holonomic series:

$$q^{-\frac{1}{24}} \eta(\tau) = (q;q)_\infty = \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n}{2}} x^n}{(q;q)_n} \Bigg|_{x=q} = \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n+1}{2}}}{(q;q)_n}.$$

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- q-Holonomic approximations of modular forms/functions:

$$q^{-\frac{1}{24}} \eta(\tau) = \lim_{n\to\infty} (q;q)_n = \lim_{n\to\infty} \sum_{k=0}^{n} \binom{n}{k}_q q^{\binom{k}{2}} x^k \Bigg|_{x=-q}$$

Projections of q-holonomic series

Recall

$$\frac{1}{(x;q)_\infty} = \frac{(0\cdot x;q)_\infty}{(x;q)_\infty} = \sum_{n=0}^{\infty} \frac{(0;q)_n}{(q;q)_n} x^n = \sum_{n=0}^{\infty} \frac{x^n}{(q;q)_n}.$$

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$$f(\tau) := \left(\frac{\eta(2\tau)}{\eta(\tau)}\right)^{24} = \frac{\Delta(2\tau)}{\Delta(\tau)}$$

is an analytic modular function for $\Gamma_0(2)$; for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(2)$:

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$$q^{-1/24}\frac{\eta(2\tau)}{\eta(\tau)} = \frac{(q^2;q^2)_\infty}{(q;q)_\infty} = \frac{1}{(q;q^2)_\infty} = \sum_{n=0}^{\infty} \frac{(0;q^2)_n}{(q^2;q^2)_n} x^n \Big|_{x=q}.$$

Example (Rogers-Ramanujan functions):

$$F(z) := \sum_{k=0}^{\infty} f_k(z) = \sum_{k=0}^{\infty} \frac{q^{k^2} z^k}{(q;q)_k};$$

the RR functions are the projections $z = 1$ and $z = q$,

$$F(1) = \sum_{k=0}^{\infty} \frac{q^{k^2}}{(q;q)_k} \text{ and } F(q) = \sum_{k=0}^{\infty} \frac{q^{k^2+k}}{(q;q)_k}$$

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Modular function versions:

$$G(\tau) := q^{-1/60} F(1) \text{ and } H(\tau) := q^{11/60} F(q).$$

Recall

$$G(\tau) := q^{-1/60} F(1) = q^{-1/60} \sum_{k=0}^{\infty} \frac{q^{k^2}}{(q;q)_k}.$$

For $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_1(5)$ with $\gcd(a,6) = 1$:

Recall

$$G(\tau) := q^{-1/60} F(1) = q^{-1/60} \sum_{k=0}^{\infty} \frac{q^{k^2}}{(q;q)_k}.$$

For $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_1(5)$ with $\gcd(a,6) = 1$:

$$G\left(\frac{a\,\tau + b}{c\,\tau + d}\right) = e^{2\pi i\,\alpha(a,b,c)/60}\,G(\tau),$$

where

$$\alpha(a,b,c) = a(9 - b + c) - 9.$$

Similar for $H(\tau)$.

The Rogers-Ramanujan quotient

$$r(\tau) := \frac{H(\tau)}{G(\tau)} = q^{\frac{1}{5}} \prod_{m=0}^{\infty} \frac{(1-q^{5m+1})(1-q^{5m+4})}{(1-q^{5m+2})(1-q^{5m+3})}$$

is an analytic modular function for $\Gamma(5)$: for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma(5)$,

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NOTE 1. To verify the modular transformation property the product presentation is essential!

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NOTE 1. To verify the modular transformation property the product presentation is essential!

NOTE 2. To obtain such product presentations usually the Jacobi triple product identity,or more general tools like the ${}_1\psi_1$-summation, are used.

q-Holonomic approximations

Example (Andrews/Watson version of Rogers-Ramanujan ids.):

$$a_n = b_n, \quad n \geq 0,$$

for

$$a_n := \sum_{k=0}^{n} \frac{q^{k^2}}{(q;q)_k (q;q)_{n-k}} \text{ and } b_n := \sum_{k=-n}^{n} \frac{(-1)^k q^{(5k^2-k)/2}}{(q;q)_{n+k}(q;q)_{n-k}}.$$

Proof. Use the `Sigma` or `qZeil` package. □

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Proof. Use the `Sigma` or `qZeil` package. □

$$\lim_{n\to\infty} (q;q)_n a_n = F(1) = q^{\frac{1}{60}} G(\tau);$$

$$\lim_{n\to\infty} (q;q)_n b_n = \frac{1}{(q;q)_\infty} \sum_{k=-\infty}^{\infty} (-1)^k q^{(5k^2-k)/2}$$

$$= \prod_{m=0}^{\infty} \frac{1}{(1-q^{5m+1})(1-q^{5m+4})}.$$

Generalized Lambert Series

Example (Ramanujan, Andrews):

$$r(\tau)^3 = q^{\frac{3}{5}} \cdot \frac{\sum_{n=-\infty}^{\infty} \frac{q^{2n}}{1-q^{5n+2}}}{\sum_{n=-\infty}^{\infty} \frac{q^{n}}{1-q^{5n+1}}}.$$

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Proof. (Andrews) Apply Ramanujan's ${}_1\psi_1$-summation using

$$\sum_{n=-\infty}^{\infty} \frac{(a;q)_n}{(b;q)_n} x^n \bigg|_{b=aq} = (1-a) \sum_{n=-\infty}^{\infty} \frac{x^n}{1-aq^n}$$

together with

$$\sum_{n=-\infty}^{\infty} \frac{x^n}{1-aq^n} \bigg|_{\substack{q \to q^\beta \\ x \to q^\alpha \\ a \to q^\gamma}} = \sum_{n=-\infty}^{\infty} \frac{q^{\alpha n}}{1-q^{\beta n+\gamma}}.$$

Conclusion

For symbolic computation treatment of modular forms/functions one might benefit from the two "basic" principles:

- Represent modular forms/functions as projections of q-holonomic/q-hypergeometric series.

- Represent modular forms/functions as q-holonomic/ q-hypergeometric approximations.

- The first principle is meant in view of algorithmic executable q-holonomic closure properties;
- the second principle is meant in view of powerful q-hypergeometric summation tools like `Sigma` or `qZeil`.

References for Part 3

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