Differential equations for Feynman integrals beyond multiple polylogarithms

Stefan Weinzierl

Institut für Physik, Universität Mainz

in collaboration with L. Adams, Ch. Bogner, E. Chaubey and A. Schweitzer

I.: **Review of differential equations and multiple polylogarithms**

II.: **Beyond multiple polylogarithms: Single scale integrals**

III: **Towards multi-scale integrals beyond multiple polylogarithms**

Part I

Review of differential equations and multiple polylogarithms

Differential equations

Let t be an external invariant (e.g. $t = (p_i + p_j)^2$) or an internal mass. Let $I_i \in \{I_1, ..., I_N\}$ be a master integral. Carrying out the derivative

$$\frac{\partial}{\partial t} I_i$$

under the integral sign and using integration-by-parts identities allows us to express the derivative as a linear combination of the master integrals.

$$\frac{\partial}{\partial t} I_i = \sum_{j=1}^{N} a_{ij} I_j$$

(Kotikov ’90, Remiddi ’97, Gehrmann and Remiddi ’99)

Differential equations

More generally:

$\vec{I} = (I_1, ..., I_N)$, set of master integrals,

$\vec{x} = (x_1, ..., x_n)$, set of kinematic variables the master integrals depend on.

We obtain a system of differential equations of Fuchsian type

$$d\vec{I} = A\vec{I},$$

where A is a matrix-valued one-form

$$A = \sum_{i=1}^{n} A_i dx_i.$$

The matrix-valued one-form A satisfies the integrability condition

$$dA - A \wedge A = 0.$$

Multiple polylogarithms

Definition based on nested sums:

$$\mathrm{Li}_{m_1,m_2,\dots,m_k}(x_1,x_2,\dots,x_k) \quad = \quad \sum_{n_1>n_2>\dots>n_k>0}^{\infty} \frac{x_1^{n_1}}{n_1^{m_1}} \cdot \frac{x_2^{n_2}}{n_2^{m_2}} \cdot \ldots \cdot \frac{x_k^{n_k}}{n_k^{m_k}}$$

Definition based on iterated integrals:

$$G(z_1,\dots,z_k;y) \quad = \quad \int\limits_0^y \frac{dt_1}{t_1-z_1} \int\limits_0^{t_1} \frac{dt_2}{t_2-z_2} \dots \int\limits_0^{t_{k-1}} \frac{dt_k}{t_k-z_k}$$

Conversion:

$$\mathrm{Li}_{m_1,\dots,m_k}(x_1,\dots,x_k) \quad = \quad (-1)^k G_{m_1,\dots,m_k}\left(\frac{1}{x_1},\frac{1}{x_1x_2},\dots,\frac{1}{x_1\dots x_k};1\right)$$

Short hand notation:

$$G_{m_1,\dots,m_k}(z_1,\dots,z_k;y) \quad = \quad G(\underbrace{0,\dots,0}_{m_1-1},z_1,\dots,z_{k-1},\underbrace{0\dots,0}_{m_k-1},z_k;y)$$

The ε-form of the differential equation

If we change the basis of the master integrals $\vec{J} = U\vec{I}$, the differential equation becomes

$$d\vec{J} = A'\vec{J}, \qquad A' = UAU^{-1} - U\,dU^{-1}$$

Suppose one finds a transformation matrix U, such that

$$A' \quad = \quad \varepsilon \sum_j C_j \, d\ln p_j(\vec{x}),$$

where

- ε appears only as prefactor,
- C_j are matrices with constant entries,
- $p_j(\vec{x})$ are polynomials in the external variables,

then the system of differential equations is easily solved in terms of multiple polylogarithms.

Henn '13

Transformation to the ε-form

We may

- perform a rational / algebraic transformation on the kinematic variables

$$(x_1, ..., x_n) \quad \rightarrow \quad (x'_1, ..., x'_n),$$

often done to absorb square roots.

- change the basis of the master integrals

$$\vec{I} \quad \rightarrow \quad U\vec{I},$$

where U is rational in the kinematic variables

Henn ’13; Gehrmann, von Manteuffel, Tancredi, Weihs ’14; Argeri et al. ’14; Lee ’14; Meyer ’16; Prausa ’17; Gituliar, Magerya ’17; Lee, Pomeransky ’17;

Numerical evaluations of multiple polylogarithms

Multiple polylogarithms have branch cuts.

Numerical evaluation of multiple polylogarithms $\mathrm{Li}_{m_1,m_2,...,,m_k}(x_1,x_2,...,x_k)$ as a function of k complex variables x_1, x_2, ..., x_k:

- Use truncated sum representation within its region of convergence.
- Use integral representation to map arguments into this region.
- Acceleration techniques to speed up the computation.

Implementation in GiNaC, using arbitrary precision arithmetic in C++.

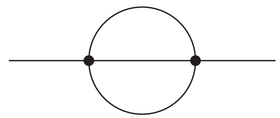
J. Vollinga, S.W. '04

Part II

Beyond multiple polylogarithms: Single scale integrals

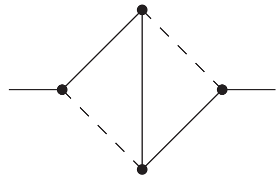
Single scale integrals beyond multiple polylogarithms

Starting from two-loops, there are integrals which cannot be expressed in terms of multiple polylogarithms.



Simplest example: Two-loop sunrise integral with equal masses.

Slightly more complicated: Two-loop kite integral.



Both integrals depend on a single scale t/m^2.

Change variable from t/m^2 to the nome q or the parameter τ with $q = e^{i\pi\tau}$.

Sabry, Broadhurst, Fleischer, Tarasov, Bauberger, Berends, Buza, Böhm, Scharf, Weiglein, Caffo, Czyz, Laporta, Remiddi, Groote, Körner, Pivovarov, Bailey, Borwein, Glasser, Adams, Bogner, Müller-Stach, Schweitzer, S.W, Zayadeh, Bloch, Vanhove, Tancredi, Pozzorini, Gunia, ...

The elliptic curve

How to get the elliptic curve?

- From the Feynman graph polynomial:

$$-x_1x_2x_3t + m^2\left(x_1+x_2+x_3\right)\left(x_1x_2+x_2x_3+x_3x_1\right) = 0$$

- From the maximal cut:

$$y^2 - \left(x - \frac{t}{m^2}\right)\left(x - \frac{t-4m^2}{m^2}\right)\left(x^2+2x+1-4\frac{t}{m^2}\right) = 0$$

Baikov '96; Lee '10; Frellesvig, Papadopoulos, '17; Bosma, Sogaard, Zhang, '17; Harley, Moriello, Schabinger, '17

The periods ψ_1, ψ_2 of the elliptic curve are solutions of the homogeneous differential equation.

Adams, Bogner, S.W., '13; Primo, Tancredi, '16

Set $\quad \tau = \frac{\psi_2}{\psi_1}, \qquad q = e^{i\pi\tau}.$

The elliptic dilogarithm

Recall the definition of the classical polylogarithms:

$$\mathrm{Li}_n(x) = \sum_{j=1}^{\infty} \frac{x^j}{j^n}.$$

Generalisation, the two sums are coupled through the variable q:

$$\mathrm{ELi}_{n;m}(x;y;q) = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \frac{x^j}{j^n} \frac{y^k}{k^m} q^{jk}.$$

Elliptic dilogarithm:

$$\mathrm{E}_{2;0}(x;y;q) = \frac{1}{i} \left[\frac{1}{2} \mathrm{Li}_2(x) - \frac{1}{2} \mathrm{Li}_2\left(x^{-1}\right) + \mathrm{ELi}_{2;0}(x;y;q) - \mathrm{ELi}_{2;0}\left(x^{-1};y^{-1};q\right) \right].$$

Various definitions of elliptic polylogarithms can be found in the literature

Beilinson '94, Levin '97, Wildeshaus '97, Brown, Levin '11, Bloch, Vanhove '13, Adams, Bogner, S.W. '14, Remiddi, Tancredi '17

Elliptic generalisations

In order to express the sunrise/kite integral to all orders in ε introduce

$$\mathrm{ELi}_{n_1,\ldots,n_l;m_1,\ldots,m_l;2o_1,\ldots,2o_{l-1}}(x_1,\ldots,x_l;y_1,\ldots,y_l;q) =$$

$$= \sum_{j_1=1}^{\infty}\cdots\sum_{j_l=1}^{\infty}\sum_{k_1=1}^{\infty}\cdots\sum_{k_l=1}^{\infty} \frac{x_1^{j_1}}{j_1^{n_1}}\cdots\frac{x_l^{j_l}}{j_l^{n_l}} \frac{y_1^{k_1}}{k_1^{m_1}}\cdots\frac{y_l^{k_l}}{k_l^{m_l}} \frac{q^{j_1k_1+\ldots+j_lk_l}}{\prod\limits_{i=1}^{l-1}\left(j_ik_i+\ldots+j_lk_l\right)^{o_i}}.$$

Numerical evaluation: G. Passarino '16

The all-order in ε result (ELi-representation)

Taylor expansion of the sunrise integral around $D = 2 - 2\varepsilon$:

$$S = \frac{\psi_1}{\pi} \sum_{j=0}^{\infty} \varepsilon^j E^{(j)}$$

Each term in the ε-series is of the form

$$E^{(j)} \sim \text{linear combination of } \mathrm{ELi}_{n_1,\dots,n_l;m_1,\dots,m_l;2o_1,\dots,2o_{l-1}} \text{ and } \mathrm{Li}_{n_1,\dots,n_l}$$

Using dimensional-shift relations this translates to the expansion around $4 - 2\varepsilon$.

$\Rightarrow$ The multiple polylogarithms extended by $\mathrm{ELi}_{n_1,\dots,n_l;m_1,\dots,m_l;2o_1,\dots,2o_{l-1}}$ are the class of functions to express the equal mass sunrise graph to all orders in ε.

Adams, Bogner, S.W., '15

Modular forms

Denote by $\mathbb{H}$ the complex upper half plane. A meromorphic function $f : \mathbb{H} \to \mathbb{C}$ is a modular form of modular weight k for $\mathrm{SL}_2(\mathbb{Z})$ if

(i) f transforms under Möbius transformations as

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k \cdot f(\tau) \qquad \text{for } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$$

(ii) f is holomorphic on $\mathbb{H}$,

(iii) f is holomorphic at ∞.

Iterated integrals of modular forms:

$$(2\pi i)^n \int\limits_{\tau_0}^{\tau} d\tau_1 f_1(\tau_1) \int\limits_{\tau_0}^{\tau_1} d\tau_2 f_2(\tau_2) \dots \int\limits_{\tau_0}^{\tau_{n-1}} d\tau_n f_n(\tau_n)$$

The all-order in ε result (iterated integrals)

$$
\begin{aligned}
S &= \frac{\psi_1}{\pi} e^{-\varepsilon I(f_2;q)+2\sum\limits_{n=2}^{\infty}\frac{(-1)^n}{n}\zeta_n\varepsilon^n} \\
&\quad \left\{ \left[\sum_{j=0}^{\infty} \left(\varepsilon^{2j} I\left(\{1,f_4\}^j;q\right) - \frac{1}{2}\varepsilon^{2j+1} I\left(\{1,f_4\}^j,1;q\right)\right)\right] \sum_{k=0}^{\infty} \varepsilon^k B^{(k)} \right. \\
&\quad \left. + \sum_{j=0}^{\infty} \varepsilon^j \sum_{k=0}^{\lfloor \frac{j}{2} \rfloor} I\left(\{1,f_4\}^k,1,f_3,\{f_2\}^{j-2k};q\right) \right\}
\end{aligned}
$$

Uniform weight: At order ε^j one has exactly $(j+2)$ integrations.

Alphabet given by modular forms 1, f_2, f_3, f_4.

Adams, S.W., ’17

The letters

Example: The modular form f_3 is given by

$$\begin{aligned} f_3 &= -\frac{1}{24}\left(\frac{\psi_1}{\pi}\right)^3 \frac{t\left(t-m^2\right)\left(t-9m^2\right)}{m^6} \\ &= \frac{3}{i}\left[\mathrm{ELi}_{0;-2}\left(r_3;-1;-q\right) - \mathrm{ELi}_{0;-2}\left(r_3^{-1};-1;-q\right)\right] \\ &= 3\sqrt{3}\frac{\eta\left(2\tau_2\right)^{11}\eta\left(6\tau_2\right)^7}{\eta\left(\tau_2\right)^5\eta\left(4\tau_2\right)^5\eta\left(3\tau_2\right)\eta\left(12\tau_2\right)} \\ &= 3\sqrt{3}\left[E_3\left(\tau_2;\chi_1,\chi_0\right) + 2E_3\left(2\tau_2;\chi_1,\chi_0\right) - 8E_3\left(4\tau_2;\chi_1,\chi_0\right)\right] \end{aligned}$$

with $\tau_2 = \tau/2$, $r_3 = \exp(2\pi i/3)$, Dedekind's eta function η, Dirichlet characters $\chi_0 = (\frac{1}{n})$, $\chi_1 = (\frac{-3}{n})$ and Eisenstein series E_3.

The ε-form of the differential equation for the sunrise/kite

It is not possible to obtain an ε-form by a rational/algebraic change of variables and/or a rational/algebraic transformation of the basis of master integrals.

However by the (non-algebraic) change of variables from t to τ and by factoring off the (non-algebraic) expression ψ_1/π from the master integrals in the sunrise sector one obtains an ε-form for the kite/sunrise family:

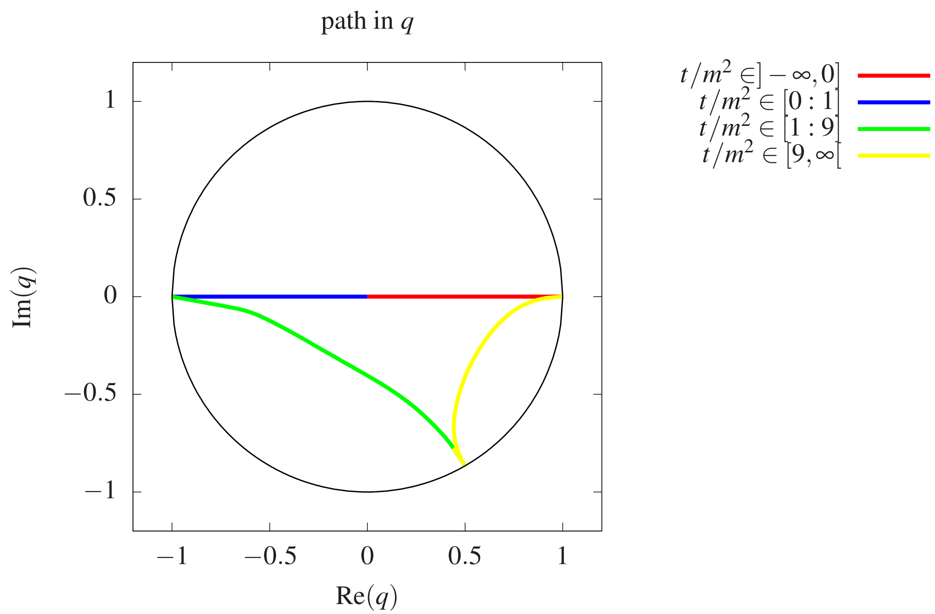
$$\frac{d}{d\tau}\vec{I} = \varepsilon \, A(\tau) \, \vec{I},$$

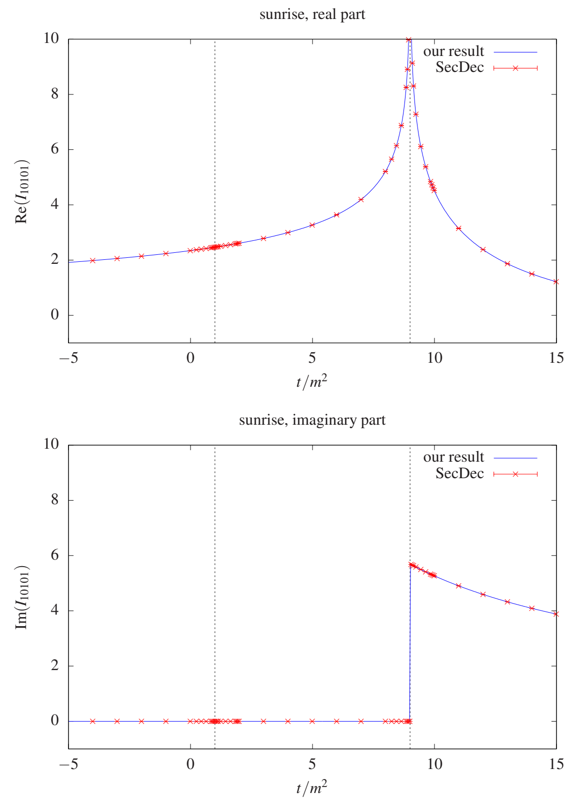
where $A(\tau)$ is an ε-independent 8×8-matrix whose entries are modular forms.

Analytic continuation and numerical evaluations of the kite and sunrise integral

Complete elliptic integrals efficiently computed from arithmetic-geometric mean.

q-series converges for all $t \in \mathbb{R}\backslash\{m^2, 9m^2, \infty\}$.





No need to distinguish the cases $t < 0$, $0 < t < m^2$, $m^2 < t < 9m^2$, $9m^2 < t$!

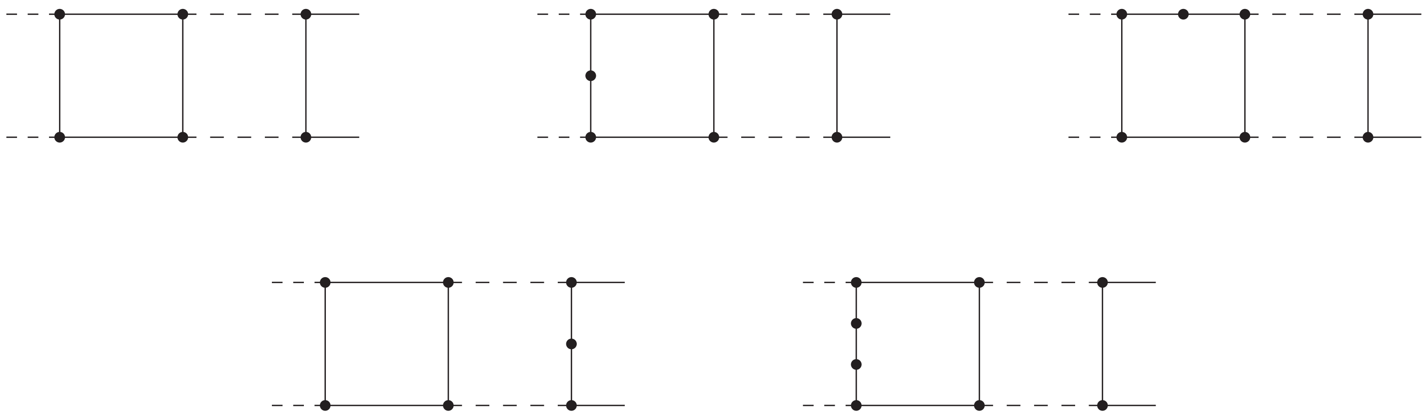
Bogner, Schweitzer, S.W., '17

Part III

Towards multi-scale integrals beyond multiple polylogarithms

A more complicated example

Let's look at a two-loop example from $t\bar{t}$-production. In the top topology we have 5 master integrals:



Multi-scale problem ($x_1 = s/m^2$, $x_2 = t/m^2$), contains sunrise as sub-topology.

Do we have to solve at order ε^0 a coupled system of 5 differential equations?

Reduction to a single-scale problem

Let $\alpha = [\alpha_1 : ... : \alpha_n] \in \mathbb{CP}^{n-1}$, without loss of generality take $\alpha_n = 1$.
Consider a path

$$x_i(\lambda) = \alpha_i \lambda, \qquad 1 \leq i \leq n.$$

View the master integrals as functions of λ. For the derivative with respect to λ we have

$$\frac{d}{d\lambda}\vec{I} = B\vec{I}, \qquad B = \sum_{i=1}^{n} \alpha_i A_i.$$

Let us write

$$B = B^{(0)} + \sum_{j>0} \varepsilon^j B^{(j)}.$$

The Picard-Fuchs operator

Consider the top sector and let us work modulo sub-topologies and ε-corrections.

Let I be one of the master integrals $\{I_1, ..., I_N\}$.

Determine the largest number r, such that the matrix which expresses I, $(d/d\lambda)I$, ..., $(d/d\lambda)^{r-1}I$ in terms of the original set $\{I_1, ..., I_N\}$ has full rank.

It follows that $(d/d\lambda)^r I$ can be written as a linear combination of $I, ..., (d/d\lambda)^{r-1}I$. This defines the Picard-Fuchs operator L_r for the master integral I with respect to λ:

$$L_r I = 0, \qquad L_r = \sum_{k=0}^{r} R_k \frac{d^k}{d\lambda^k}.$$

L_r is easily found by transforming to a basis which contains $I, ..., (d/d\lambda)^{r-1}I$.

Factorisation

Consider as an example the differential operator

$$L_2 = \frac{d^2}{d\lambda^2} - \left(\frac{1}{\lambda} + \frac{1}{\lambda - 1}\right)\frac{d}{d\lambda} + \left(\frac{1}{\lambda(\lambda-1)} + \frac{1}{(\lambda-1)^2}\right).$$

This operator factorises:

$$L_2 = \left(\frac{d}{d\lambda} - \frac{1}{\lambda}\right)\left(\frac{d}{d\lambda} - \frac{1}{\lambda - 1}\right)$$

Not every differential operator factorises into linear factors. We may decompose any differential operator into irreducible factors.

van Hoeij '97

Factorisation

Suppose L_r factorises as a differential operator

$$L_r = L_{1,r_1} L_{2,r_2} \dots L_{s,r_s},$$

where L_{i,r_i} denotes a differential operator of order r_i.

Then we may convert the system of differential equations at order ε^0 into block triangular form with blocks of size r_1, r_2, ..., r_s. A basis for block i is given by

$$J_{i,j} = \frac{d^{j-1}}{d\lambda^{j-1}} L_{i+1,r_{i+1}} \dots L_{s,r_s} I, \quad 1 \leq j \leq r_i.$$

This decouples the original system into sub-systems of size r_1, r_2, ..., r_s.

Adams, Chaubey, S.W. '17

Lifting

Let us write the transformation to the new basis as

$$\vec{J} = V(\alpha_1, ..., \alpha_{n-1}, \lambda) \vec{I}.$$

Setting

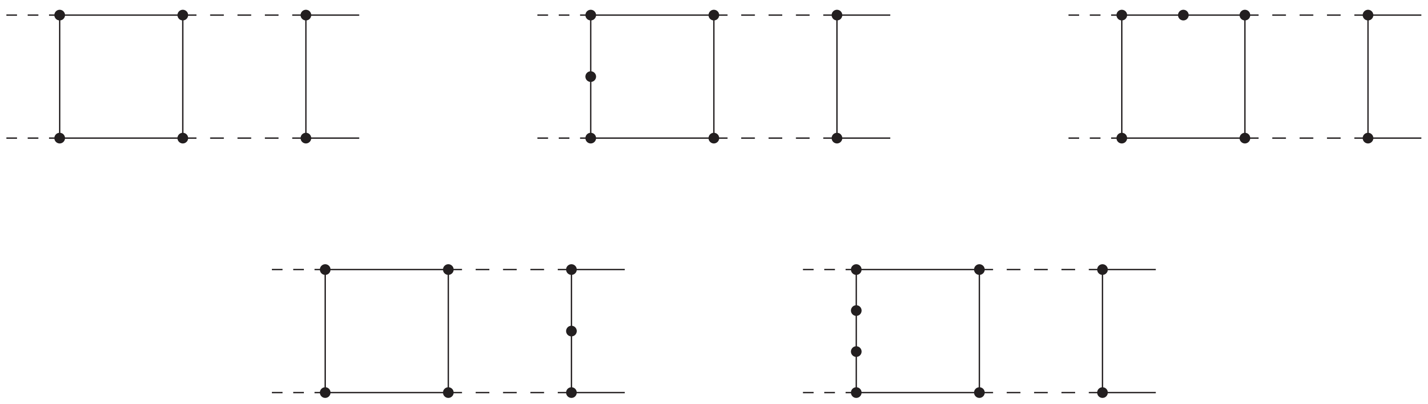
$$U = V\left(\frac{x_1}{x_n}, ..., \frac{x_{n-1}}{x_n}, x_n\right)$$

gives the transformation in terms of the original variables x_1, ..., x_n.

Remark: Terms in the original A of the form $d \ln Z(x_1, ..., x_n)$, where $Z(x_1, ..., x_n)$ is a rational function in $(x_1, ..., x_n)$ and homogeneous of degree zero in $(x_1, ..., x_n)$, map to zero in B. These terms are in many cases easily removed by a subsequent transformation.

Example

Let's return to the example of the double box integral for $t\bar{t}$-production:



Decoupling at ε^0 from the factorisation of the Picard-Fuchs operator:

$$5 \quad = \quad 1+2+1+1.$$

Need to solve only two coupled equations, not five!

Example

Let us look at the following sector with two master integrals:

The Picard-Fuchs operator factorises into two linear factors and we may transform to an ε-form:

$$
\begin{aligned}
A &= \varepsilon \left[\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} d\ln(x_1+1) - \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} d\ln(x_1-1) - \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} d\ln(x_2+1) \right. \\
&\quad \left. + \begin{pmatrix} 0 & 0 \\ -1 & 1 \end{pmatrix} d\ln(x_1+x_2) + \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} d\ln(x_1 x_2+1) \right],
\end{aligned}
$$

with $s = -\frac{m^2(1-x_1)^2}{x_1}$, $t = -m^2 x_2$.

Conclusions

- Differential equations are a powerful tool to compute Feynman integrals.

- If a system can be transformed to an ε-form, a solution in terms of multiple polylogarithm is easily obtained.

- There are system, where within rational transformations at order ε^0 two coupled equations remain.

  Kite/sunrise family:

  - Sum representation in terms of ELi-functions.
  - Iterated integral representation involving modular forms
  - Analytic continuation / numerical evaluation easy.

- Factorisation of the Picard-Fuchs operator allows us to find the irreducible blocks.