The Analytical Calculation Of The Four-Loop Cusp Anomalous Dimensions Of QCD

Robert M. Schabinger

with Andreas von Manteuffel and Erik Panzer, based on: Phys. Lett. **B744** (2015) 101; **JHEP** 1502 (2015) 120; Phys. Rev. **D93** (2016) no.12, 125014; Phys. Rev. **D95** (2017) no.3, 034030 and work in progress

Trinity College Dublin

Outline

What We're Doing

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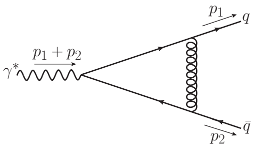

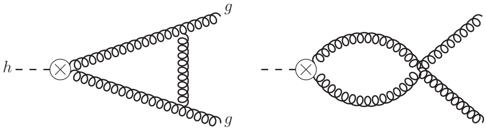

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$$\left(\mu^2 \frac{\partial}{\partial \mu^2} + \beta(\alpha_s)\frac{\partial}{\partial \alpha_s}\right)\mathcal{G}\left(q^2/\mu^2, \alpha_s, \epsilon\right) = \Gamma(\alpha_s)$$

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At L loops, Γ_L characterizes the leading IR divergences which cannot be understood as exponentiated lower-loop contributions.

$\Longrightarrow$ Γ_4 is the last unknown ingredient needed for N^3LL resummation!

A Dipole Formula For Gauge Theory IR Divergences?

S. Catani, Phys. Lett. **B427** (1998) 161; S. Mert Aybat *et. al.*, Phys. Rev. **D74** (2006) 074004

T. Becher and M. Neubert, **JHEP** 0906 (2009) 081; E. Gardi and L. Magnea, **JHEP** 0903 (2009) 079

The IR divergences of the simplest non-Abelian gauge theory, planar $SU(N_c)$ $\mathcal{N} = 4$ super Yang-Mills, are believed to be of the form:

$$\mathcal{A}_1^{\mathcal{N}=4}(p_1, \ldots, p_n) = \exp\left\{-\frac{1}{2}\sum_{L=1}^{\infty}\left(\frac{\alpha_s}{4\pi}\right)^L \mu_\epsilon^{2L\epsilon}\int_0^{\mu_\epsilon^2} \mathrm{d}\mu^2 \left(\mu^2\right)^{-1-L\epsilon}\right.$$

$$\left.\sum_{\substack{\mathrm{i,j}=1\\ \mathrm{i}<\mathrm{j}}}^{n}\left(\Gamma_{1;L}^{\mathcal{N}=4}\ln\left(\frac{\mu^2}{-s_{ij}}\right) + \mathcal{G}_{1;L}^{\mathcal{N}=4}\right)\frac{\mathbf{T}_\mathrm{i}\cdot\mathbf{T}_\mathrm{j}}{N_c}\right\}\sum_{L=0}^{\infty}\mathbf{H}_{1;L}^{\mathcal{N}=4}(\epsilon; p_1, \ldots, p_n)$$

At four points, this structure has been realized explicitly at strong coupling (L. F. Alday and J. Maldacena, **JHEP** 0706 (2007) 064). In principle, the above structure could hold for more general gauge theories like QCD.

When Something Sounds Too Good To Be True...

Although some three-loop evidence was collected by Dixon (Phys. Rev. **D79** (2009) 091501) for the n_f terms, it is now clear that the dipole conjecture fails in general.

S. Caron-Huot, **JHEP** 1505 (2015) 093; Ø. Almelid *et. al.*, Phys. Rev. Lett. **117** (2016) no.17, 172002;

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$$\Gamma^g_L \stackrel{?}{=} C_A/C_F \Gamma^q_L$$

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J. Davies *et. al.*, Nucl. Phys. **B915** (2017) 335; S. Moch *et. al.*, arXiv:1707.08315

How To Survive The Calculation

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- Use a decent-sized cluster to do numerator algebra. ($\sim$ 50,000 diagrams with `QGraf` + `FORM`/`Mathematica`)

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- Evaluate all finite master integrals analytically using `HyperInt`.

  F. C. S. Brown, Commun. Math. Phys. **287** (2009) 925; arXiv:0910.0114

  E. Panzer, arXiv:1506.07243; Comput. Phys. Commun. **188** (2015) 148

From Conventional To Finite Integral Bases

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- For $x = \Delta d/2$ (the dimension shift divided by two), $y = \nu - N$ (the number of "extra" powers of the propagators or "dots"), and all fixed non-negative integers $n = x + y$, this test is carried out in practice by considering the integrals which correspond to all possible non-negative integer solutions $\{x, y\}$, beginning with the $n = 0$ case corresponding to the basic scalar integral in $d = 4 - 2\epsilon$.

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- Rotate from the old basis to the new basis using auxiliary IBPs.
- The computationally expensive part at this stage is to perform a Tarasov shift (Phys. Rev. **D54** (1996) 6479) on the old basis and then IBP reduce the resulting linear combination of integrals in $d + 2$ with a number of additional dots equal to the loop order. This connects the "conventional" integral bases in d and $d + 2$; it can be used iteratively if multiple dimension shifts are required.

What About The Auxiliary Reductions Needed For The Basis Rotation?

Consider the three-loop gluon form factor, where $s_{\max} = 5$:

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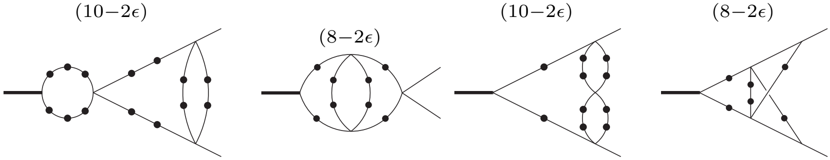

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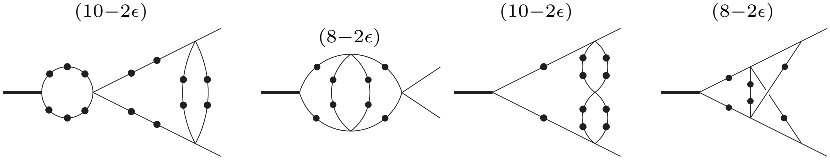
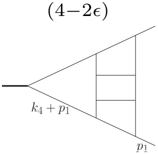


$\Longrightarrow$ Auxiliary reductions are a subleading problem!

An Illustrative Comparison

J. M. Henn *et. al.*, **JHEP** 1605 (2016) 066

ϵ^8 

$$\left[(k_4^2)^2\right] = \frac{1}{576} + \frac{1}{36}\zeta_2\epsilon^2 + \frac{151}{864}\zeta_3\epsilon^3 + \frac{173}{288}\zeta_2^2\epsilon^4 + \left(\frac{505}{216}\zeta_2\zeta_3 + \frac{5503}{1440}\zeta_5\right)\epsilon^5 + \left(\frac{6317}{720}\zeta_2^3 + \frac{9895}{2592}\zeta_3^2\right)\epsilon^6 + \mathcal{O}\left(\epsilon^7\right)$$

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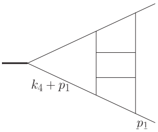
$(4-2\epsilon)$

ϵ^8 

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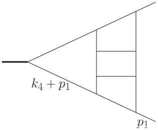
ϵ^8 

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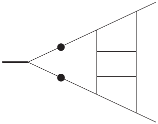
$(6-2\epsilon)$



$$= -\frac{3}{5}\zeta_2^2 + 5\zeta_2\zeta_3 + \frac{25}{2}\zeta_5 - \frac{7}{10}\zeta_2^3 - \frac{3}{10}\zeta_2^2\zeta_3 - \frac{5}{2}\zeta_2\zeta_5 - \frac{147}{16}\zeta_7 + \mathcal{O}(\epsilon)$$

The Correlation Of The Maximal Weight At Leading Order With The Number Of Edges Of The Graph

R. N. Lee and V. A. Smirnov, **JHEP** 1004 (2010) 020; T. Gehrmann *et. al.*, **JHEP** 1006 (2010) 094

Data for the 22 three-loop form factor master sectors:

# Edges	# Wt. 0	# Wt. 2	# Wt. 3	# Wt. 4	# Wt. 5
4	1	0	0	0	0
5	4	0	0	0	0
6	3	1	2	0	0
7	0	0	3	0	2
8	0	0	1	1	1
9	0	0	0	0	3

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P. A. Baikov and K. G. Chetyrkin, Nucl. Phys. **B837** (2010) 186;

R. N. Lee *et. al.*, Nucl. Phys. **B856** (2012) 95

Data for the 197 genuine four-loop form factor master sectors:

# Edges	# Wt. 2	# Wt. 3	# Wt. 4	# Wt. 5	# Wt. 6	# Wt. 7	# Wt. 8
7	6	4	0	0	0	0	0
8	2	17	2	19	0	0	0
9	0	2	5	18	8	7	2
10	0	0	1	23	6	19	10
11	0	0	0	0	4	14	8
12	0	0	0	0	0	11 + (0-5)	4 + (0-5)

Based on our experience, we expect that just 117 of the above are relevant to the calculation of the cusp anomalous dimensions!

The Compatibility Graph Algorithm

F. C. S. Brown, arXiv:0910.0114; E. Panzer, arXiv:1506.07243

For each integration order, the algorithm associates a cascade of polynomials and their compatibilities to the integral topology under consideration, starting with the "compatibility graph"

$$(\mathcal{U}) \longleftrightarrow (\mathcal{F})$$

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The algorithm assumes that, at each step, all factors are linear with respect to at least one of the remaining Feynman parameters:

$$\begin{aligned} f_j\left(x_{k_1}, \ldots, x_{k_m}\right) = {} & q_j^{(i)}\left(x_{k_1}, \ldots, x_{k_{i-1}}, x_{k_{i+1}}, \ldots, x_{k_m}\right) x_i \\ & + r_j^{(i)}\left(x_{k_1}, \ldots, x_{k_{i-1}}, x_{k_{i+1}}, \ldots, x_{k_m}\right) \end{aligned}$$

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One obtains a tight upper bound on the factors which are relevant to the integration! Non-trivial mathematics, but the intuition is clear.

The Compatibility Graph Algorithm

All $q_j^{(i)}$ and $r_j^{(i)}$ may appear as letters after x_i is integrated out, but that is not the end of the story. We have compatibility resultants

$$\{f_\ell, f_n\}_{x_i} = \det\begin{pmatrix} q_\ell^{(i)} & r_\ell^{(i)} \\ q_n^{(i)} & r_n^{(i)} \end{pmatrix} \qquad \{f_j, f_\infty\}_{x_i} = q_j^{(i)} \qquad \{f_j, f_0\}_{x_i} = r_j^{(i)}$$

Any set of compatibility resultants with indices in common, including 0 and ∞, generate polynomial factors which are then considered to be compatible at the next iteration of the algorithm.

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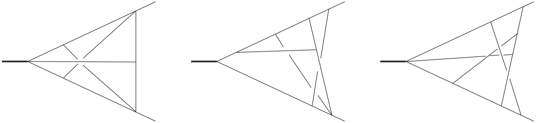
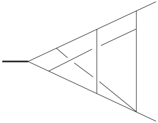
$$\frac{1}{f_1^{\nu_1}\cdots f_N^{\nu_N}} = \sum_{k=1}^{N}\sum_{j=1}^{\nu_k}\frac{(-1)^{\nu_k-j}(\{f_k,\infty\}_{x_i})^{\nu_{\bar{k}}}}{(\nu_k-j)!f_k^j}\sum_{\substack{\sum_{s=1, s\neq k}^{N}\ell_s = \nu_k - j}}$$

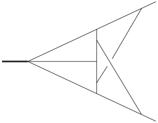
$$\begin{pmatrix} \nu_k - j \\ \ell_1\cdots\ell_{k-1}\ell_{k+1}\cdots\ell_N \end{pmatrix}\prod_{\substack{r=1\\r\neq k}}^{N}\begin{pmatrix} \nu_r+\ell_r-1 \\ \ell_r \end{pmatrix}\frac{(\{f_r,\infty\}_{x_i})^{\ell_r}}{(\{f_k,f_r\}_{x_i})^{\nu_r+\ell_r}}$$

“Universality Classes” Of Variable Changes

Remarkably, making certain simple variable changes in $\mathcal{U}$ and $\mathcal{F}$ can dramatically improve the linear reducibility of most tough sectors:

$x_i = x'_i x_j x_k \Longrightarrow$ 

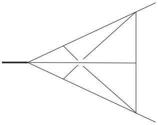
$x_i = x'_i x_j \quad x_k = x'_k x'_i \Longrightarrow$ 

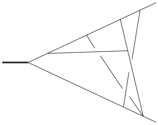
$x_i = x'_i x_j \Longrightarrow$ 

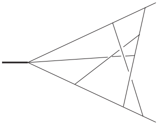
"Universality Classes" Of Variable Changes

Remarkably, making certain simple variable changes in $\mathcal{U}$ and $\mathcal{F}$ can dramatically improve the linear reducibility of most tough sectors:

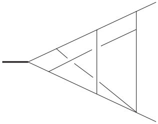
$x_i = x_i' x_j x_k \Longrightarrow$



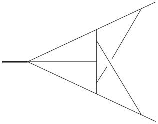




$x_i = x_i' x_j \quad x_k = x_k' x_i' \Longrightarrow$



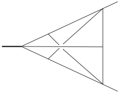
$x_i = x_i' x_j \Longrightarrow$



Only two top-level sectors left which we cannot access analytically!

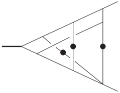
Selected Results

$(4-2\epsilon)$



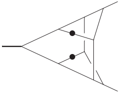
$$= \frac{612}{5}\zeta_2^2\zeta_3 - 300\zeta_2\zeta_5 - \frac{147}{2}\zeta_7 + \mathcal{O}(\epsilon)$$

$(6-2\epsilon)$



$$= -12\zeta_2\zeta_3 + 30\zeta_5 + \frac{418}{105}\zeta_2^3 + 12\zeta_3^2 - \frac{204}{5}\zeta_2^2\zeta_3 + 100\zeta_2\zeta_5 + \frac{49}{2}\zeta_7 - \frac{49151}{5250}\zeta_2^4 - 3\zeta_2\zeta_3^2 - 15\zeta_3\zeta_5 + \frac{72}{5}\zeta_{5,3} + \mathcal{O}(\epsilon)$$

$(6-2\epsilon)$



$$= -\frac{128}{15}\zeta_2^3 - 48\zeta_3^2 - 4\zeta_2^2\zeta_3 - 76\zeta_2\zeta_5 + \frac{343}{2}\zeta_7 + \frac{50503}{2625}\zeta_2^4 + 18\zeta_2\zeta_3^2 - 80\zeta_3\zeta_5 - \frac{222}{5}\zeta_{5,3} + \mathcal{O}(\epsilon)$$

Outlook

Our to-do list looks as follows:

- Buy new computers to more effectively run Laporta's algorithm.
- Keep thinking about algorithmic improvements.
- Obtain analytical results for the cusp anomalous dimensions.
- Obtain analytical results for the finite parts of the form factors.