

The method of global R^* and its applications

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Outline

1 Introduction

2 Global R^*

3 Renormalisation of gauge theories

4 Conclusion and outlook

Origin of the method

In MS scheme, the UV counterterm $Z(\gamma)$ of the Feynman diagram γ is a polynomial in the masses (Collins, 1977).

- If γ is logarithmically divergent $Z(\gamma)$ is mass-independent.
 γ can always be made log-divergent by taking derivatives.
- *Infrared rearrangements* (IRRs), acting on masses and external momenta, simplify the calculation of $Z(\gamma)$ (Vladimirov, 1980).



Double lines are massive propagators, dotted lines are squared propagators.

Factorized massive tadpoles

Dropping external momenta reduces the complexity **1 loop lower**

Example: three-loop propagator counterterm

$$Z \left[\begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \right] \rightarrow Z \left[\begin{array}{c} \text{---} \\ \bullet \\ \text{---} \end{array} \right]$$

The integral becomes a factorized tadpole

$$\bullet = \frac{1}{2+2\epsilon} * \text{---} = \int \frac{d^d q}{(2\pi)^d} \frac{1}{(q^2 + M^2)^2 (q^2)^{2+2\epsilon}} * \Pi_2$$

Not the end of the story...

Dealing with IR singularities

Spurious IR divergences can arise after rearrangements

$$\text{---} \bigcirc \text{---} \rightarrow \text{---} \bigcirc \text{---} \simeq \int \frac{d^d q}{(2\pi)^d} \cdots \int \frac{d^d k}{(2\pi)^d} \frac{1}{(q+k)^2 [k^2]^2}$$

Two strategies have been adopted

- Auxiliary mass regulating the IR (see Luthe, Maier, Marquard, Schröder 2016; Larin, van Ritbergen, Vermaseren '97) → A. Maier's talk
- IR counterterms: apply the R^* operation (Chetyrkin, Tkachov '82; Chetyrkin, Smirnov '84-'85)

$$R^*(\Gamma) = \tilde{R} \circ R(\Gamma) \quad (1.1)$$

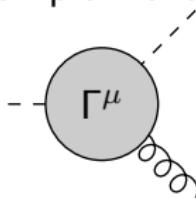
\tilde{R} recursively subtracts IR divergences (see Herzog, Ruijl, Ueda, Vermaseren, Vogt 2017) → B. Ruijl's talk

Why Global R^*

- R^* acts diagram-by-diagram:
 - many IR counterterms per diagram \longrightarrow bottleneck.
- Problems in gauge theories at high orders, due to the large number of diagrams.
- The **Global R^*** operation (Chetyrkin 1991) bypasses this complication introducing overall IR counterterms.

Global infrared counterterm

Specific example: renormalising the ghost-gluon vertex



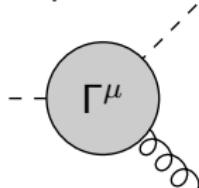
$$\Gamma^\mu = -g f^{abc} p^\mu \left(1 + \delta\Gamma(g, p) \right),$$

The sum of all the diagrams renormalises multiplicatively.

$$\left[Z_1 \left(1 + \delta\Gamma(g_0, p) \right) \right] = \text{finite.}$$

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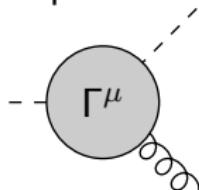
$$\left[Z_1 \left(1 + \delta\Gamma(g_0, p) \right) \right] = \text{finite.}$$

We drop the external momentum and apply \tilde{R} to the whole $\delta\Gamma$

$$\left[Z_1 \left(1 + \tilde{R} \left[\delta\Gamma(g_0, p=0) \right] \right) \right] = \text{finite.}$$

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$$\tilde{R} \left[\delta\Gamma(g_0, p=0) \right] = -\frac{Z_1 - 1}{Z_1} \equiv -\frac{\delta Z_1}{Z_1}.$$

A global infrared rearrangement

$$\delta\Gamma_M(g, p, M) = - \text{---} \text{---} + \cdots + \cdots + \cdots$$

Only subdivergences change: separate renormalisation of the **massive vertex**.

$$K_\epsilon \left[Z \left[\text{---} \text{---} + \cdots \right] + \cdots + Z \left[\text{---} \text{---} \right] * \cdots \right] = 0$$

$$\delta Z_1 = -K_\epsilon \left[\delta\Gamma_M(g_0, p, M) + \delta Z_1 * \delta\Gamma(g_0, p) \right]. \quad (2.2)$$

K_ϵ extracts the pole part.

Global R^* at work

$$\delta Z_1 = -K_\epsilon \left[\delta\Gamma_M(g_0, p, M) + \delta Z_1 * \delta\Gamma(g_0, p) \right]$$

Global R^* at work

1 Hard mass expansion $M \gg p$ (Smirnov 1996)

$$\delta\Gamma_M(g_0, p, M) \sim \delta\Gamma_M(g_0, 0, M) + \delta\Gamma_M(g_0, 0, M) * \delta\Gamma(g_0, p),$$

$$\delta Z_1 = -K_\epsilon \left[\delta\Gamma_M(g_0, p, M) + \delta Z_1 * \delta\Gamma(g_0, p) \right]$$

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- 2 We nullify p introducing the proper IR counterterm

$$\tilde{R}[\delta\Gamma(p=0)] = -\frac{\delta Z_1}{Z_1}.$$

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- 2 We nullify p introducing the proper IR counterterm

$$\tilde{R}[\delta\Gamma(p=0)] = -\frac{\delta Z_1}{Z_1}.$$

$$\boxed{\delta Z_1 = -K_\epsilon \left[\frac{\delta\Gamma_M(g_0, 0, M)}{Z_1} - \frac{(\delta Z_1)^2}{Z_1} \right]}$$

Towards 5-loop renormalisation

$$\delta Z_1 = -K_\epsilon \left[\frac{\delta \Gamma_M(g_0, 0, M)}{Z_1} - \frac{(\delta Z_1)^2}{Z_1} \right] \quad (3.3)$$



Counterterms from lower orders.

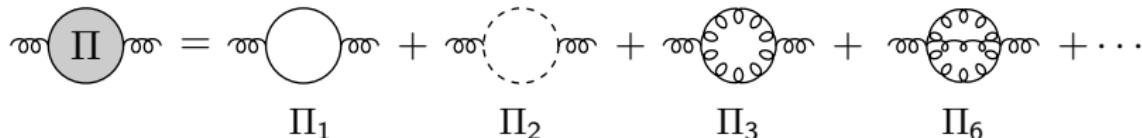
- We reduced the calculations to factorized L -loop tadpoles.
- We automated the computation of factorized tadpoles using **FORCER** (Ruijl, Ueda, Vermaseren 2017) and determined **Z_1 to 5 loops**.

Highly efficient approach

Renormalisation of QCD

- Same approach for the field renormalisations Z_2 , Z_3^c .
Unique IRR operation on a **single** interaction vertex.
- The gluon renormalisation Z_3 is complicate: first time undertaken in 2016 for $SU(3)$ ([Baikov, Chetyrkin, Kühn](#)).

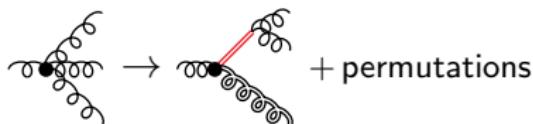
$$K_\epsilon \left[Z_3 \left(1 + \sum_{i=1,2,3,6} \Pi_i(g_0, q) \right) \right] = 0, \quad (3.4)$$



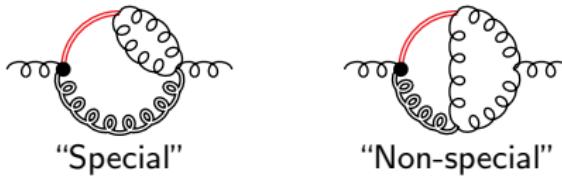
- (a) New rearrangements of 4-point vertices.
- (b) External gluons have many interactions.

The gluon renormalisation (a)

- Mass insertion in the 4-gluon vertex via an auxiliary field



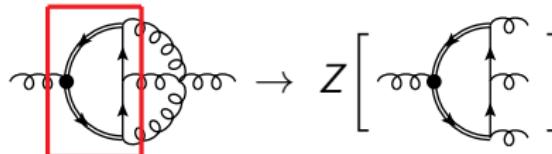
- New “special”, self-energy-like, subdivergences



The gluon renormalisation (b)

Highly non-trivial renormalisation of the modified vertices:

- vertices mix among each other. *E.g.*



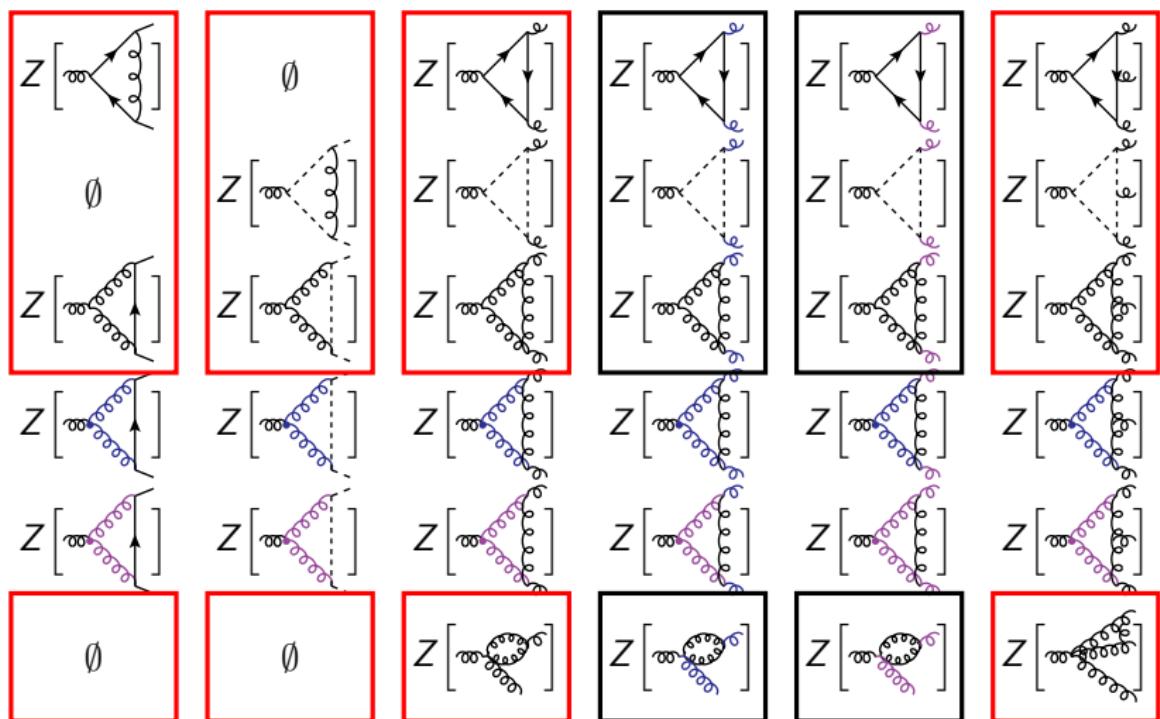
counterterm mixing quark
and 4-gluon operators.

- Mixing into new operators (not in the QCD Lagrangian).
We analysed the operators appearing for **general gauge group**
 - Two new 3-gluon vertices

$$O_4 = \text{---} \quad O_5 = \text{---}$$

- New 4-gluon vertices appear at each loop order: we selected a basis of six new vertices O_7, \dots, O_{12} needed to this order. Note that each of them is splitted with the auxiliary field.

A 12x12 mixing matrix



Solution

Only a small subset of the matrix elements z_{ij} contributes and

$$\sum_{i=1,2,3,6} z_{ij} = \begin{cases} 0, & \text{if } j \in \{4, 5, 7, \dots, 12\} \text{ is a gauge variant operator} \\ Z_1^j, & \text{if } j \in \{1, 2, 3, 6\} \text{ is a QCD operator} \end{cases}$$

where $Z_1^j \in \{Z_1^{ccg}, Z_1^{qqg}, Z_1^{ggg}, Z_1^{gggg}\}$ are QCD vertex renormalisation constant.

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Exploiting these features we write the gluon renormalisation constant in terms of factorized tadpoles $\Pi_i(M)$ and lower order counterterms

$$\begin{aligned} \delta Z_3 &= -K_\epsilon \left\{ \sqrt{Z_3} \sum_{i=1,2,3,6} \left\{ \sum_{j,k} \left[z_{ij}^{sp} \mathbf{Z}_j \left(\Pi_j(M) + \delta \Gamma_{jk}^{ns} \Pi_k(q) \right) \right] \right. \right. \\ &\quad \left. \left. + (\sqrt{Z_3} - \mathbf{Z}_i) \Pi_i(q) - \sum_j \delta z_{ij}^{sp} \mathbf{Z}_j \Pi_j(q) \right\} \right\}. \end{aligned}$$

Results

- Z_1^{ccg} , Z_2 , Z_3^c to 5 loops with all the powers of the gauge parameter ξ .
 - Verified $Z_1^{ccg} \propto (1 - \xi)$.
- Z_3 to 5 loops, retaining linear terms in ξ .
- We derived the coupling renormalisation

$$Z_\alpha = \frac{(Z_1^{ccg})^2}{Z_3(Z_3^c)^2}, \quad (4.5)$$

checking its independence on ξ to first order.

Complete renormalisation of QCD to 5 loops in covariant gauges
for general gauge group.

Landau gauge quark anomalous dimension

$$\begin{aligned}
 (\gamma_2)_4 &= C_F \frac{d_A^{abcd} d_A^{abcd}}{N_A} \left[-\frac{1985}{24} + \frac{781753}{192} \zeta_7 - \frac{1458845}{384} \zeta_5 + \frac{135731}{192} \zeta_3 + \frac{3577}{64} \zeta_3^2 \right] \\
 &\quad + T_R \eta_f^4 \frac{d_R^{abcd} d_A^{abcd}}{N_R} \left[\frac{6200}{9} - \frac{1425}{4} \zeta_6 + \frac{27377}{6} \zeta_7 + \frac{1113}{4} \zeta_4 - \frac{9915}{2} \zeta_5 \right. \\
 &\quad \left. - \frac{2468}{3} \zeta_3 + \frac{91}{2} \zeta_3^2 \right] + T_R \eta_f^2 \frac{d_R^{abcd} d_R^{abcd}}{N_R} \left[-\frac{7360}{9} + 640 \zeta_5 + \frac{704}{3} \zeta_3 \right] \\
 &\quad + C_F \eta_f^4 T_R^4 \left[\frac{1328}{243} - \frac{256}{27} \zeta_3 \right] + C_F \frac{d_R^{abcd} d_A^{abcd}}{N_R} \left[\frac{113}{6} - \frac{125447}{8} \zeta_7 \right. \\
 &\quad \left. + 1015 \zeta_5 + 17554 \zeta_3 - 4884 \zeta_3^2 \right] + C_F \eta_f^4 \frac{d_R^{abcd} d_R^{abcd}}{N_R} \left[-\frac{5984}{3} - 8680 \zeta_7 \right. \\
 &\quad \left. + 18080 \zeta_5 - 12096 \zeta_3 + 3648 \zeta_3^2 \right] + C_F^2 \eta_f^3 T_R^3 \left[-\frac{2636}{243} - 64 \zeta_4 + \frac{832}{9} \zeta_3 \right] \\
 &\quad + C_F^3 \eta_f^2 T_R^2 \left[-\frac{2497}{27} - 128 \zeta_4 + 320 \zeta_5 + \frac{400}{9} \zeta_3 \right] + C_F^4 \eta_f T_R \left[\frac{29209}{36} \right. \\
 &\quad \left. + \frac{6400}{3} \zeta_6 - 800 \zeta_4 - \frac{46880}{9} \zeta_5 + \frac{22496}{9} \zeta_3 + \frac{1024}{3} \zeta_3^2 \right] + C_F^5 \left[\frac{4977}{8} \right. \\
 &\quad \left. - 47628 \zeta_7 + 22600 \zeta_5 + 16000 \zeta_3 + 2496 \zeta_3^2 \right]
 \end{aligned}$$

$$\begin{aligned}
& + C_A \frac{d_R^{abcd} d_A^{abcd}}{N_R} \left[-\frac{173959}{144} + \frac{15675}{16} \zeta_6 + \frac{3016307}{256} \zeta_7 - \frac{12243}{16} \zeta_4 \right. \\
& \quad \left. + \frac{609425}{96} \zeta_5 - \frac{574393}{32} \zeta_3 + \frac{16935}{4} \zeta_3^2 \right] \\
& + C_A \eta_F \frac{d_R^{abcd} d_R^{abcd}}{N_R} \left[\frac{33464}{9} + \frac{23632}{3} \zeta_7 - \frac{48640}{3} \zeta_5 + 8992 \zeta_3 - 2320 \zeta_3^2 \right] \\
& + C_A C_F \eta_F^3 T_R^3 \left[-\frac{3566}{243} + 64 \zeta_4 - \frac{1984}{27} \zeta_3 \right] + C_A C_F^2 \eta_F^2 T_R^2 \left[\frac{101485}{162} \right. \\
& \quad \left. + \frac{1600}{3} \zeta_6 + 176 \zeta_4 - \frac{3712}{3} \zeta_5 - \frac{6160}{9} \zeta_3 + \frac{256}{3} \zeta_3^2 \right] + C_A C_F^3 \eta_F T_R \left[-\frac{167263}{108} \right. \\
& \quad \left. - 4800 \zeta_6 - 13944 \zeta_7 + 2120 \zeta_4 + \frac{58720}{3} \zeta_5 - \frac{25804}{9} \zeta_3 - 64 \zeta_3^2 \right] \\
& + C_A C_F^4 \left[-\frac{835739}{144} - \frac{17600}{3} \zeta_6 + 123977 \zeta_7 + 2200 \zeta_4 - \frac{248960}{9} \zeta_5 - \frac{530884}{9} \zeta_3 \right. \\
& \quad \left. - \frac{24632}{3} \zeta_3^2 \right] + C_A^2 C_F \eta_F^2 T_R^2 \left[\frac{120037}{162} - \frac{800}{3} \zeta_6 - \frac{441}{2} \zeta_7 - 179 \zeta_4 + \frac{3584}{9} \zeta_5 \right. \\
& \quad \left. + \frac{3140}{3} \zeta_3 - \frac{128}{3} \zeta_3^2 \right] + C_A^2 C_F^2 \eta_F T_R \left[\frac{717409}{432} + 1150 \zeta_6 + \frac{42203}{3} \zeta_7 - \frac{1411}{4} \zeta_4 \right. \\
& \quad \left. - \frac{95792}{9} \zeta_5 - \frac{14287}{24} \zeta_3 - 1214 \zeta_3^2 \right] + C_A^2 C_F^3 \left[\frac{827215}{72} + 13200 \zeta_6 - \frac{1789067}{16} \zeta_7 \right. \\
& \quad \left. - 4664 \zeta_4 - \frac{188795}{12} \zeta_5 + \frac{1365227}{18} \zeta_3 + \frac{18097}{2} \zeta_3^2 \right] + C_A^3 C_F \eta_F T_R \left[-\frac{31919039}{776} \right. \\
& \quad \left. + \frac{4825}{16} \zeta_6 - \frac{440419}{144} \zeta_7 - \frac{8705}{32} \zeta_4 + \frac{28721}{18} \zeta_5 - \frac{144377}{864} \zeta_3 + \frac{4067}{6} \zeta_3^2 \right]
\end{aligned}$$



$$\begin{aligned}
 & + C_A^3 C_F^2 \left[-\frac{42214139}{3888} - \frac{43175}{6} \zeta_6 + \frac{9074513}{192} \zeta_7 + \frac{3815}{4} \zeta_4 + \frac{5957573}{288} \zeta_5 \right. \\
 & - \frac{5503507}{144} \zeta_3 - \frac{78041}{24} \zeta_3^2 \Big] + C_A^4 C_F \left[\frac{368712343}{62208} + \frac{227975}{192} \zeta_6 \right. \\
 & \left. - \frac{312820991}{36864} \zeta_7 + \frac{87067}{128} \zeta_4 - \frac{16237513}{3072} \zeta_5 + \frac{46196783}{6912} \zeta_3 + \frac{23555}{128} \zeta_3^2 \right].
 \end{aligned}$$

- More complicated structure compared to the beta-function:
 - zeta-values up to weight 7.
- Applied to the definition of scale-invariant propagators

$$\mu^2 \frac{d}{d\mu^2} \hat{G}(\alpha_s) = 0$$

Outlook

- The global R^* method is highly efficient from the computational point of view.
- On the other hand, it is process-dependent → difficult to generalize and automate.
- We tackled the rearrangement of different operators → case study for more applications:
- Determination of the moments of the $N4LO$ splitting functions, whose calculation was shown to be highly demanding ([See B. Ruijl's talk](#)).

Thank you