

QCD Correlators at High Orders



Adler function in
large- β_0

Scalar correlator in
large- β_0

Coupling evolution

C-scheme coupling

Adler function

Scalar correlator

Borel transform

IR renormalon poles

Borel models

Summary

Matthias Jamin
ICREA @ IFAE
Universitat Autònoma de Barcelona

RADCOR 2017
Sankt Gilgen
28 September 2017

Adler function in large- β_0

QCD Correlators
at High Orders

Matthias Jamin

Begin with the conventional Adler function $D(Q^2)$:

$$4\pi^2 D(Q^2) \equiv 1 + \hat{D}(Q^2) = 1 + a_Q + \mathcal{O}(a_Q^2)$$

where $a_Q \equiv \alpha_s(Q^2)/\pi$.

$\hat{D}(Q^2)$ can be expressed through a Borel-integral:

$$\hat{D}(Q^2) \equiv \frac{2}{\beta_1} \int_0^\infty du e^{-2u/(\beta_1 a_Q^C)} B[\hat{D}](u)$$

with the large- β_0 Borel-transform:

(Beneke 1993; Broadhurst 1993)

$$B[\hat{D}](u) = 8C_F \frac{e^{-Cu}}{(2-u)} \sum_{k=2}^{\infty} \frac{(-1)^k k}{[k^2 - (1-u)^2]^2}$$



Adler function in
large- β_0

Scalar correlator in
large- β_0

Coupling evolution

C-scheme coupling

Adler function

Scalar correlator

Borel transform

IR renormalon poles

Borel models

Summary

RADCOR 2017

Sankt Gilgen

28 September 2017

Introducing the scheme-invariant coupling A_Q in large- β_0 :

$$\frac{1}{A_Q} \equiv \frac{1}{a_Q^C} + \frac{\beta_1}{2} C = \frac{1}{a_Q^{\overline{\text{MS}}}} - \frac{5}{6} \beta_1$$

The Borel-integral can be expressed in scheme-invariant form:

$$\hat{D}(Q^2) \equiv \frac{2}{\beta_1} \int_0^\infty du e^{-2u/(\beta_1 A_Q)} \hat{B}[\hat{D}](u)$$

with

$$\hat{B}[\hat{D}](u) = \frac{8C_F}{(2-u)} \sum_{k=2}^{\infty} \frac{(-1)^k k}{[k^2 - (1-u)^2]^2}$$

Adler function in
large- β_0

Scalar correlator in
large- β_0

Coupling evolution

C-scheme coupling

Adler function

Scalar correlator

Borel transform

IR renormalon poles

Borel models

Summary

Scalar correlator in large- β_0

QCD Correlators
at High Orders

Matthias Jamin

The scalar correlator $\Psi(Q^2)$ is defined as:

$$\Psi(Q^2 = -q^2) \equiv i \int dx e^{iqx} \langle \Omega | T\{j(x) j^\dagger(0)\} | \Omega \rangle$$

where, for example,

$$j(x) = m : \bar{u}(x) u(x) :$$

and m is a generic quark mass.

In large- β_0 , $\Psi''(Q^2)$ can be expressed as: (MJ, Miravitllas 2016)

$$\begin{aligned} \Psi''(Q^2) &= \frac{N_c}{8\pi^2} \frac{\hat{m}^2}{Q^2} (\pi A_Q)^{2\gamma_m^{(1)}/\beta_1} \times \\ &\left\{ 1 + \frac{2}{\beta_1} \int_0^\infty du e^{-2u/(\beta_1 A_Q)} \hat{B}[\Psi''](u) \right\} \end{aligned}$$



Adler function in
large- β_0

Scalar correlator in
large- β_0

Coupling evolution

C-scheme coupling

Adler function

Scalar correlator

Borel transform

IR renormalon poles

Borel models

Summary

RADCOR 2017

Sankt Gilgen

28 September 2017

The RGI quark mass \hat{m} in full QCD is defined as:

$$m(\mu) \equiv \hat{m} [\alpha_s(\mu)]^{\gamma_m^{(1)} / \beta_1} \exp \left\{ \int_0^{a_\mu} da \left[\frac{\gamma_m(a)}{\beta(a)} - \frac{\gamma_m^{(1)}}{\beta_1 a} \right] \right\}$$

The scheme-invariant Borel-transform is found to be:

(Broadhurst, Kataev, Maxwell 2001)

$$\hat{B}[\Psi''](u) = \frac{3}{2} C_F \left[(1-u) G_D(u) - 1 \right]$$

with

$$G_D(u) = \frac{2}{1-u} - \frac{1}{2-u} + \frac{2}{3} \sum_{p=3}^{\infty} \frac{(-1)^p}{(p-u)^2} - \frac{2}{3} \sum_{p=1}^{\infty} \frac{(-1)^p}{(p+u)^2}$$

which explicitly displays the renormalon structure.

Adler function in large- β_0

Scalar correlator in large- β_0

Coupling evolution

C-scheme coupling

Adler function

Scalar correlator

Borel transform

IR renormalon poles

Borel models

Summary

RADCOR 2017

Sankt Gilgen

28 September 2017

Coupling evolution

Scale evolution of α_s is given by the β -function:

$$-Q \frac{da_Q}{dQ} \equiv \beta(a_Q) = \beta_1 a_Q^2 + \beta_2 a_Q^3 + \beta_3 a_Q^4 + \dots$$

with $a_Q = \alpha_s / \pi$.

The scale invariant parameter Λ can be defined by:

$$\frac{\Lambda}{Q} \equiv e^{-\frac{1}{\beta_1 a_Q}} [a_Q]^{-\frac{\beta_2}{\beta_1^2}} \exp \left\{ \int_0^{a_Q} \frac{da}{\tilde{\beta}(a)} \right\},$$

where

$$\frac{1}{\tilde{\beta}(a)} \equiv \frac{1}{\beta(a)} - \frac{1}{\beta_1 a^2} + \frac{\beta_2}{\beta_1^2 a}$$

is free of singularities as $a \rightarrow 0$.



Adler function in
large- β_0

Scalar correlator in
large- β_0

Coupling evolution

C-scheme coupling

Adler function

Scalar correlator

Borel transform

IR renormalon poles

Borel models

Summary

RADCOR 2017

Sankt Gilgen

28 September 2017

C-scheme coupling

However, Λ depends on the renormalisation scheme.

$$a' \equiv a + c_1 a^2 + c_2 a^3 + c_3 a^4 + \dots$$

Then, Λ transforms as:

(Celmaster, Gonsalves 1979)

$$\Lambda' = \Lambda e^{c_1/\beta_1}.$$

This suggests to define a “novel” “C-scheme” coupling \hat{a}_Q :

$$\begin{aligned} \frac{1}{\hat{a}_Q} + \frac{\beta_2}{\beta_1} \ln \hat{a}_Q - \frac{\beta_1}{2} C &\equiv \beta_1 \ln \frac{Q}{\Lambda} \\ &= \frac{1}{a_Q} + \frac{\beta_2}{\beta_1} \ln a_Q - \beta_1 \int_0^{a_Q} \frac{da}{\tilde{\beta}(a)} \end{aligned}$$

(Boito, MJ, Miravitllas 2016)



Adler function in
large- β_0

Scalar correlator in
large- β_0

Coupling evolution

C-scheme coupling

Adler function

Scalar correlator

Borel transform

IR renormalon poles

Borel models

Summary

RADCOR 2017

Sankt Gilgen

28 September 2017

The β -function of \hat{a}_Q takes the simple form:

$$-Q \frac{d\hat{a}_Q}{dQ} \equiv \hat{\beta}(\hat{a}_Q) = \frac{\beta_1 \hat{a}_Q^2}{\left(1 - \frac{\beta_2}{\beta_1} \hat{a}_Q\right)} = -2 \frac{d\hat{a}_Q}{dC}$$

Relating a general coupling a_Q and $\bar{a}_Q \equiv \hat{a}_Q^{C=0}$ reads:

$$\begin{aligned} a_Q = \bar{a}_Q &+ \left(\frac{\beta_3}{\beta_1} - \frac{\beta_2^2}{\beta_1^2} \right) \bar{a}_Q^3 + \left(\frac{\beta_4}{2\beta_1} - \frac{\beta_2^3}{2\beta_1^3} \right) \bar{a}_Q^4 \\ &+ \left(\frac{\beta_5}{3\beta_1} - \frac{\beta_2\beta_4}{6\beta_1^2} + \frac{5\beta_3^2}{3\beta_1^2} - \frac{3\beta_2^2\beta_3}{\beta_1^3} + \frac{7\beta_2^4}{6\beta_1^4} \right) \bar{a}_Q^5 + \mathcal{O}(\bar{a}_Q^6) \end{aligned}$$

The coupling \hat{a}_Q at arbitrary C is obtained from \bar{a}_Q via:

$$\begin{aligned} \bar{a}_Q = \hat{a}_Q &+ \frac{\beta_1}{2} C \hat{a}_Q^2 + \left(\frac{\beta_2}{2} C + \frac{\beta_1^2}{4} C^2 \right) \hat{a}_Q^3 \\ &+ \left(\frac{\beta_2^2}{2\beta_1} C + \frac{5\beta_1\beta_2}{8} C^2 + \frac{\beta_1^3}{8} C^3 \right) \hat{a}_Q^4 + \mathcal{O}(\hat{a}_Q^5) \end{aligned}$$

Adler function in
large- β_0

Scalar correlator in
large- β_0

Coupling evolution

C-scheme coupling

Adler function

Scalar correlator

Borel transform

IR renormalon poles

Borel models

Summary

Adler function in
large- β_0

Scalar correlator in
large- β_0

Coupling evolution

C-scheme coupling

Adler function

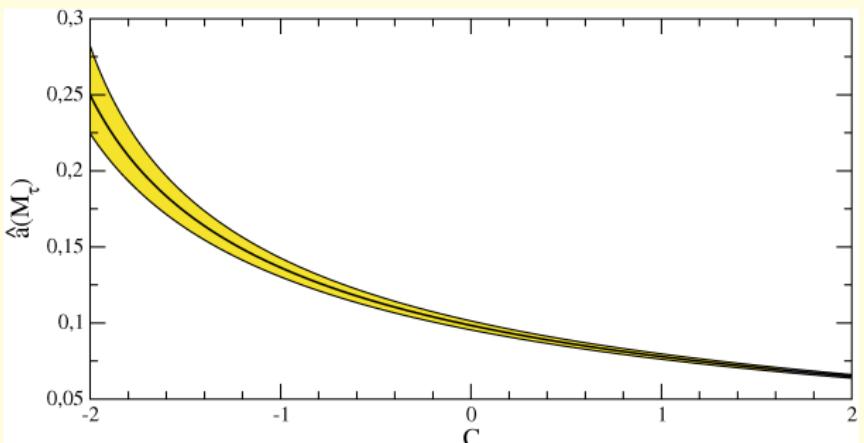
Scalar correlator

Borel transform

IR renormalon poles

Borel models

Summary



$\hat{a}(M_\tau)$ as a function of C for $\alpha_s(M_\tau) = 0.316(10)$.

Adler function

$$4\pi^2 D(a_Q) - 1 \equiv \hat{D}(a_Q) = \sum_{n=1}^{\infty} c_{n,1} a_Q^n$$
$$= a_Q + 1.640 a_Q^2 + 6.371 a_Q^3 + 49.08 a_Q^4 + \dots$$

Expressed in terms of the coupling \bar{a}_Q :

$$\hat{D}(a_Q) = \sum_{n=1}^{\infty} \bar{c}_{n,1} \bar{a}_Q^n$$
$$= \bar{a}_Q + 1.640 \bar{a}_Q^2 + 7.682 \bar{a}_Q^3 + 61.06 \bar{a}_Q^4 + \dots$$

Analytically, the coefficient $\bar{c}_{4,1}$ is given by:

(Baikov, Chetyrkin, Kühn 2008)

$$\bar{c}_{4,1} = \frac{357259199}{93312} - \frac{1713103}{432} \zeta_3 + \frac{4185}{8} \zeta_3^2 + \frac{34165}{96} \zeta_5 - \frac{1995}{16} \zeta_7$$

Scalar correlator

$$\psi''(Q^2) = \frac{N_c}{8\pi^2} \frac{m_Q^2}{Q^2} \left\{ 1 + \sum_{n=1}^{\infty} d_{n,1}'' a_Q^n \right\}$$

Expressed in terms of the RGI quark mass \hat{m} :

$$\psi''(Q^2) = \frac{N_c}{8\pi^2} \frac{\hat{m}^2}{Q^2} [\alpha_s(Q)]^{2\gamma_m^{(1)}/\beta_1} \left\{ 1 + \sum_{n=1}^{\infty} r_n a_Q^n \right\}$$

Expressed in terms of the coupling \bar{a}_Q :

$$\psi''(Q^2) = \frac{N_c}{8\pi^2} \frac{\hat{m}^2}{Q^2} [\bar{\alpha}_s(Q)]^{2\gamma_m^{(1)}/\beta_1} \left\{ 1 + \sum_{n=1}^{\infty} \bar{r}_n \bar{a}_Q^n \right\}$$

$$\bar{r}_1 = 5.457, \quad \bar{r}_2 = 25.45, \quad \bar{r}_3 = 142.4, \quad \bar{r}_4 = 932.7.$$

Analytically, the coefficient \bar{r}_4 is given by:

(Baikov, Chetyrkin, Kühn 2006)

$$\bar{r}_4 = \frac{49275071521973}{8264970432} - \frac{10679302931}{1889568} \zeta_3 + \frac{601705}{648} \zeta_3^2 + \frac{117947335}{209952} \zeta_5 - \frac{3285415}{20736} \zeta_7$$

The ζ_4 terms present in both r_4 and β_5 cancel each other.

Adler function in
large- β_0

Scalar correlator in
large- β_0

Coupling evolution

C-scheme coupling

Adler function

Scalar correlator

Borel transform

IR renormalon poles

Borel models

Summary

RADCOR 2017

Sankt Gilgen

28 September 2017

Borel transform

Conventional Borel transform for the Adler function:

$$\hat{D}(Q^2) = \int_0^\infty dt e^{-t/\hat{a}_Q^{C_a}} B[\hat{D}](t) \quad (t = 2u/\beta_1)$$

Modified Borel transform for the Adler function:

(Brown, Yaffe, Zhai 1992)

$$\hat{D}(Q^2) = \int_0^\infty dt e^{-t/\hat{a}_Q^{C_a}} \left(\frac{t}{\hat{a}_Q^{C_a}} \right)^{\beta_2/\beta_1 t} \hat{B}[\hat{D}](t)$$

Conventional Borel transform for the scalar correlator:

$$\Psi''(Q^2) = \frac{N_c}{8\pi^2} \frac{\hat{m}^2}{Q^2} (\hat{\alpha}_Q^{C_m})^{2\gamma_m^{(1)}/\beta_1} \left\{ 1 + \int_0^\infty dt e^{-t/(\hat{a}_Q^{C_a})} B[\Psi''](t) \right\}$$



Adler function in
large- β_0

Scalar correlator in
large- β_0

Coupling evolution

C-scheme coupling

Adler function

Scalar correlator

Borel transform

IR renormalon poles

Borel models

Summary

RADCOR 2017

Sankt Gilgen

28 September 2017

IR renormalon poles

(Beneke 1999)

QCD Correlators
at High Orders
Matthias Jamin

General term in the Operator Product Expansion:

$$\hat{C}_{O_d}(\hat{a}_Q) \frac{\langle \hat{O}_d \rangle}{Q^d} = \hat{C}_{O_d}^{(0)} [\hat{a}_Q]^\delta \left[1 + \tilde{C}_{O_d}^{(1)} \hat{a}_Q + \tilde{C}_{O_d}^{(2)} \hat{a}_Q^2 + \dots \right] \frac{\langle \hat{O}_d \rangle}{Q^d}$$

Express Q -dependence in terms of \bar{a}_Q :

$$\frac{\hat{C}_{O_d}(\bar{a}_Q)}{Q^d} = \frac{\hat{C}_{O_d}^{(0)}}{\Lambda^d} e^{-\frac{d}{\beta_1 \bar{a}_Q}} [\bar{a}_Q]^{\delta - d \frac{\beta_2}{\beta_1^2}} \left[1 + \tilde{C}_{O_d}^{(1)} \bar{a}_Q + \tilde{C}_{O_d}^{(2)} \bar{a}_Q^2 + \dots \right]$$

Take Ansatz for Borel transform of IR renormalon pole:

$$B[\hat{D}_p^{\text{IR}}](u) \equiv \frac{d_p^{\text{IR}}}{(p-u)^\gamma} \left[1 + b_1(p-u) + b_2(p-u)^2 + \dots \right]$$



Adler function in
large- β_0

Scalar correlator in
large- β_0

Coupling evolution

C-scheme coupling

Adler function

Scalar correlator

Borel transform

IR renormalon poles

Borel models

Summary

RADCOR 2017

Sankt Gilgen

28 September 2017

The **imaginary** ambiguity takes the **form**:

$$\text{Im} \left[\hat{D}_\rho^{\text{IR}}(\bar{a}_Q) \right] = \pm \frac{2\pi^2}{\beta_1} d_\rho^{\text{IR}} e^{-\frac{2p}{\beta_1 \bar{a}_Q}} (\bar{a}_Q)^{1-\gamma} \left[1 + b_1 \frac{\beta_1}{2} (\gamma - 1) \bar{a}_Q \right.$$

$$\left. + b_2 \frac{\beta_1^2}{4} (\gamma - 1)(\gamma - 2) \bar{a}_Q^2 + \dots \right]$$

One can **identify**:

$$p = \frac{d}{2}, \quad \gamma = 1 - \delta + 2p \frac{\beta_2}{\beta_1^2},$$

$$b_1 = \frac{2\tilde{C}_{O_d}^{(1)}}{\beta_1(\gamma - 1)}, \quad b_2 = \frac{4\tilde{C}_{O_d}^{(2)}}{\beta_1^2(\gamma - 1)(\gamma - 2)}.$$

Adler function in large- β_0

Scalar correlator in large- β_0

Coupling evolution

C-scheme coupling

Adler function

Scalar correlator

Borel transform

IR renormalon poles

Borel models

Summary

Assume ambiguity $\pm i\Delta_p^{\text{IR}} \wedge^d$ for matrix element $\langle \hat{O}_d \rangle$.

Cancellation of ambiguities with PT entails:

$$\hat{C}_{O_d}^{(0)} \Delta_p^{\text{IR}} = \frac{2\pi^2}{\beta_1} C_0^{(0)} d_p^{\text{IR}}$$

Universality of ambiguity for correlators A and B leads to:

$$\frac{C_0^{(0)}(A)}{\hat{C}_{O_d}^{(0)}(A)} d_p^{\text{IR}}(A) = \frac{C_0^{(0)}(B)}{\hat{C}_{O_d}^{(0)}(B)} d_p^{\text{IR}}(B)$$

Example: Gluon condensate renormalon in large- β_0 :

$$C_0^{(0)} = \frac{N_c}{12\pi^2}, \quad \hat{C}_{GG}^{(0)} = \frac{1}{6}, \quad d_2^{\text{IR}} = \frac{3C_F}{2} e^{-2C}$$

The invariant combination reads:

$$\frac{C_0^{(0)}}{\hat{C}_{GG}^{(0)}} d_2^{\text{IR}} = \frac{3}{8\pi^2} (N_c^2 - 1) e^{-2C}$$

Adler function in large- β_0

Scalar correlator in large- β_0

Coupling evolution

C-scheme coupling

Adler function

Scalar correlator

Borel transform

IR renormalon poles

Borel models

Summary

Borel models

(Beneke, MJ 2008)

QCD Correlators
at High Orders
Matthias Jamin

To incorporate known renormalon structure, use an Ansatz for the Adler function:

$$B[\hat{D}](u) = B[\hat{D}_2^{\text{IR}}](u) + B[\hat{D}_3^{\text{IR}}](u) + B[\hat{D}_1^{\text{UV}}](u) + d_0^{\text{PO}}$$

Fitting $\bar{c}_{1,1}$ to $\bar{c}_{4,1}$, the parameters are found to be:

$$\begin{aligned} d_2^{\text{IR}} &= 2.74, & d_3^{\text{IR}} &= -7.72, \\ d_1^{\text{UV}} &= -2.12 \cdot 10^{-2}, & d_0^{\text{PO}} &= 0.289. \end{aligned}$$

The Borel model predicts: $\bar{c}_{5,1} \approx 329 \Rightarrow c_{5,1} \approx 264$.
(BJ08: ≈ 280)

Imposing d_2^{IR} in the scalar correlator model yields $C_m \approx -1.6$.
(Boito, MJ, Miravilas: in preparation)



Adler function in large- β_0

Scalar correlator in large- β_0

Coupling evolution

C-scheme coupling

Adler function

Scalar correlator

Borel transform

IR renormalon poles

Borel models

Summary

RADCOR 2017

Sankt Gilgen

28 September 2017

Summary

- The C -scheme coupling \hat{a} was introduced to study scheme dependence in Borel models for QCD correlators.
- Its corresponding β -function $\hat{\beta}(\hat{a})$ is found to be manifestly scheme invariant.
- In the C -scheme, the ζ_4 term in \bar{r}_4 of the scalar correlator cancels against the corresponding ζ_4 term in β_5 .
- An appropriate choice of C_m resums dominant corrections in the scalar correlator.
The remaining corrections are more “Adler function like”.



Adler function in
large- β_0

Scalar correlator in
large- β_0

Coupling evolution

C -scheme coupling

Adler function

Scalar correlator

Borel transform

IR renormalon poles

Borel models

Summary

Summary

- The C -scheme coupling \hat{a} was introduced to study scheme dependence in Borel models for QCD correlators.
- Its corresponding β -function $\hat{\beta}(\hat{a})$ is found to be manifestly scheme invariant.
- In the C -scheme, the ζ_4 term in \bar{r}_4 of the scalar correlator cancels against the corresponding ζ_4 term in β_5 .
- An appropriate choice of C_m resums dominant corrections in the scalar correlator.
The remaining corrections are more “Adler function like”.



Adler function in
large- β_0

Scalar correlator in
large- β_0

Coupling evolution

C -scheme coupling

Adler function

Scalar correlator

Borel transform

IR renormalon poles

Borel models

Summary

Thank You!