Solving differential equations for Feynman integrals by expansions near singular points

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Based on [R. Lee, A. Smirnov & V.S., arXiv:1709.07525]

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DE

$$\partial_x \boldsymbol{J} = M(x, \epsilon)\, \boldsymbol{J}\,,$$

where $\boldsymbol{J}$ is a column-vector of N primary master integrals, and M is an $N \times N$ matrix with elements which are rational functions of x and $\epsilon = (4 - D)/2$.

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However this form is possible:

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However, 'integrating out' the constant term appears to be an essentially more complicated problem.

Elliptic generalization of multiple polylogarithms motivated by two-loop examples, where the ε-form is impossible
[L. Adams, C. Bogner, A. Schweitzer & S. Weinzierl'16; E. Remiddi & L. Tancredi'17; talk by E. Remiddi]

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Elliptic functions appear only in two sectors and final results are expressed either in terms of multiple polylogarithms or, for the elliptic sectors, in terms of two and three-fold iterated integrals suitable for numerical evaluation.

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Some properties of the integrals are more accessible via DE.
Singularities of DE provide a way to examine the branching properties of integrals.
Numerical values of the integrals can be obtained from a numerical solution of DE.

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[B. A. Kniehl, A. F. Pikelner & O. L. Veretin'17]

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- A matching procedure which enables one to connect series expansions at two neighboring points.
- As a proof of concept: a computer code where this algorithm is implemented for a simple example of a family of four-loop Feynman integrals where the ϵ-form is impossible.

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General solution

$$\boldsymbol{J}(x) = U(x)\,\boldsymbol{C}\,,$$

where $\boldsymbol{C}$ is a column of constants, and U is an evolution operator

$$U(x) = P\exp\left[\int M(x)\,dx\right]\,.$$

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The expansion is

$$U(x) = \sum_{\lambda \in S} x^{\lambda} \sum_{n=0}^{\infty} \sum_{k=0}^{K_\lambda} \frac{1}{k!} C(n+\lambda, k)\, x^n \ln^k x\,,$$

where S is a finite set of powers of the form $\lambda = r\epsilon$ with integer r, $K_\lambda \geqslant 0$ is an integer number corresponding to the the maximal power of the logarithm.

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where S is a finite set of powers of the form $\lambda = r\epsilon$ with integer r, $K_\lambda \geqslant 0$ is an integer number corresponding to the the maximal power of the logarithm.
The goal is to determine S, K_λ, and the matrix coefficients $C(n+\lambda, k)$.

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In particular, the 'elliptic' cases, as a rule, can algorithmically be reduced to a global normalized Fuchsian form using, e.g., the algorithm of Lee [R.N. Lee'14].

Multiply both sides by the common denominator $xQ(x)$, where

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Define the polynomial matrix $B(x, \alpha)$ and its coefficients $B_m(\alpha)$ by

$$B(x, \alpha) = Q(x)(xM(x) - \alpha) = \sum_{m=0}^{N} B_m(\alpha) x^m .$$

with $B_0(\alpha) = q_0(A_0 - \alpha)$.

Then the recurrence relations read

$$
\begin{aligned}
&- \mathrm{BJF}(B_0(\lambda+n), -q_0, K_\lambda) C\left(\lambda+n, 0..K_\lambda\right) \\
&= \sum_{m=1}^{s} \mathrm{BJF}(B_m\left(\lambda+n-m\right), -q_m, K_\lambda) C\left(\lambda+n-m, 0..K_\lambda\right) \ .
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$C\left(\alpha,0..K\right)=\begin{bmatrix}C\left(\alpha,0\right)\\ \vdots\\ C\left(\alpha,K\right)\end{bmatrix}$ denotes a $(K+1)N\times N$ matrix

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$$\mathrm{BJF}(A,B,K)=\underbrace{\begin{bmatrix}A & B & 0 & 0\\ 0 & \ddots & \ddots & 0\\ 0 & 0 & \ddots & B\\ 0 & 0 & 0 & A\end{bmatrix}}_{K+1}$$

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We determine S, i.e. the set of distinct eigenvalues of A_0, and K_λ, i.e. the highest power of the logarithm, and the leading coefficients $C(\lambda, k)$, representing

$$x^{A_0} = \sum_{\lambda\in S} x^\lambda \sum_{k=0}^{K_\lambda} \frac{1}{k!} C(\lambda, k) \ln^k x\,.$$

The matrix $-\mathrm{BJF}(B_0(\lambda+n), -q_0, K_\lambda)$ on the lhs of the difference equation is invertible for $\lambda \in S$ and $n > 0$ because

$$\begin{aligned}\det \mathrm{BJF}(B_0(\lambda+n), -q_0, K_\lambda) &= (\det B_0(\lambda+n))^{K_\lambda+1} \\ &= q_0^{(K_\lambda+1)n} \left[\det(A_0 - \lambda - n)\right]^{K_\lambda+1}\end{aligned}$$

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The recurrence relation takes the form

$$C\left(\lambda+n, 0..K_\lambda\right) = \sum_{m=1}^{s} T(\lambda, n, m) C\left(\lambda+n-m, 0..K_\lambda\right) ,$$

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$$T(\lambda, n, m) = -\left[\mathrm{BJF}(B_0(\lambda+n), -q_0, K_\lambda)\right]^{-1} \times \mathrm{BJF}\left(B_m\left(\lambda+n-m\right), -q_m, K_\lambda\right).$$

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- The set $S = \{\lambda_1, \lambda_2, \ldots\}$ of the eigenvalues of the matrix residue A_0.
- For each $\lambda \in S$:
  - the maximal power of the logarithm K_λ and the leading coefficients $C(\lambda, 0..K_\lambda)$ determined by the boundary conditions.
  - the matrix coefficients $T(\lambda, n, 1), \ldots, T(\lambda, n, s)$ which are $(K_\lambda + 1)N \times (K_\lambda + 1)N$ matrices, where the dependence on n is explicit.

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In order to perform an analytical continuation to the whole complex plane, one may use the same approach for the expansion around other singular points.
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We can construct the evolution operator also in an expansion near this point. Let it be $\tilde{U}(x)$. Due to the freedom in definition of the evolution operator, we have

$$U(x) = \tilde{U}(x) L .$$

where L is a constant matrix.

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In the case where the singularities lie on the real axis and if we are interested in the evaluation for real x, we can avoid expansions near regular points. A sequence of the singular points

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then for each $0 \leqslant k \leqslant s$ we make the (Moebius) transformation

$$y_k(x) = \frac{ax + b}{cx + d}$$

which maps the points $x_{k-1},\ x_k,\ x_{k+1}$ to $\mp 1,\ 0,\ \pm 1$, respectively.

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Explicitly,

$$y_k(x) = \pm \frac{(x - x_k)(x_{k+1} - x_{k-1})}{(x - x_{k+1})(x_{k-1} - x_k) + (x - x_{k-1})(x_{k+1} - x_k)}$$

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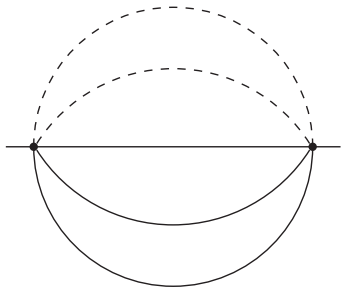
The boundary conditions are included at one of the points, e.g. $x = 0$ and then series expansions at other points can be obtained by matching, step by step, pairs of expansions at neighboring points.

Feynman integrals corresponding to the generalized sunset graph with two massless and three massive lines



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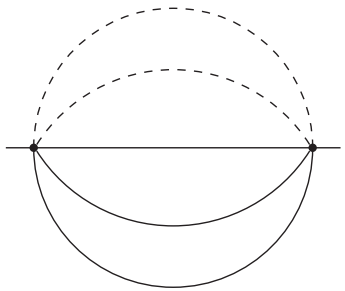
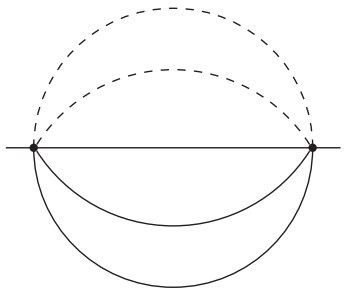


$$F_{a_1,\ldots,a_{14}} =$$

$$\int\ldots\int \frac{\mathrm{d}^D k_1 \ldots \mathrm{d}^D k_4 \; (k_1\cdot p)^{a_6}(k_2\cdot p)^{a_7}(k_3\cdot p)^{a_8}(k_4\cdot p)^{a_9}}{(-k_1^2)^{a_1}(-k_2^2)^{a_2}(m^2-k_3^2)^{a_3}(m^2-k_4^2)^{a_4}(m^2-(\sum k_i+p)^2)^{a_5}} \times (k_1\cdot k_2)^{a_{10}}(k_1\cdot k_3)^{a_{11}}(k_1\cdot k_4)^{a_{12}}(k_2\cdot k_3)^{a_{13}}(k_2\cdot k_4)^{a_{14}} \;,$$

with $x = p^2/m^2$.

Feynman integrals corresponding to the generalized sunset graph with two massless and three massive lines



$$F_{a_1,\ldots,a_{14}} =$$

$$\int\ldots\int \frac{\mathrm{d}^D k_1 \ldots \mathrm{d}^D k_4 \; (k_1\cdot p)^{a_6}(k_2\cdot p)^{a_7}(k_3\cdot p)^{a_8}(k_4\cdot p)^{a_9}}{(-k_1^2)^{a_1}(-k_2^2)^{a_2}(m^2-k_3^2)^{a_3}(m^2-k_4^2)^{a_4}(m^2-(\sum k_i+p)^2)^{a_5}} \times (k_1\cdot k_2)^{a_{10}}(k_1\cdot k_3)^{a_{11}}(k_1\cdot k_4)^{a_{12}}(k_2\cdot k_3)^{a_{13}}(k_2\cdot k_4)^{a_{14}} \;,$$

with $x = p^2/m^2$.

There are four master integrals in this family.

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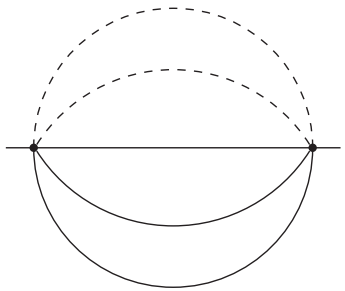

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There are four master integrals in this family. We choose

$$\boldsymbol{J_0} = \{F_{1,1,1,1,1,0,\ldots,0},\ F_{1,1,2,1,1,0,\ldots,0},\ F_{1,2,1,1,1,0,\ldots,0},\ F_{1,2,1,1,2,0,\ldots,0}\}\,.$$

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For adjacent regions i and $i+1$ we search the best possible matching point which is such x that it lies between x_i and x_{i+1} and that $|f_i(x)| = |f_{i+1}(x)|$.

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Matching points are $\{-3, 3(3-2\sqrt{2}), 3, 3(3+2\sqrt{2})\}$.

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The code `DESS.m` as well files with input data can be downloaded from `https://bitbucket.org/feynmanintegrals/dess`.

For example, at $x_0 = 25$, we obtain the following result (shown with a truncation to 10 digits) for the first primary integral:

$$
\begin{aligned}
&-\frac{0.25}{\epsilon^4}+\frac{2.125}{\epsilon^3}-\frac{0.2391337000}{\epsilon^2}-\frac{5.2663306926}{\epsilon}\\
&\quad -185.9464179437+6.5261388472\,\mathrm{i}\\
&\quad -(1825.1476432369-48.9550593728\,\mathrm{i})\epsilon\\
&\quad -(8406.8551978029-176.0638485153\,\mathrm{i})\epsilon^2\\
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We checked results at sample points (between singular point and matching points) with `FIESTA` [A.V. Smirnov'16].

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- This code is similar in spirit to the well-known existing codes to evaluate harmonic polylogarithms and multiple polylogarithms, where the problem of evaluation reduces to summing up appropriate series.
- Our public package includes tools for a decomposition of the real axis into domains, a subsequent mapping and an introduction of appropriate new variables.

- We are thinking of a more general package which would include an automation of as many steps of the presented algorithm as possible. The future package needs at least an implementation of the algorithm to solve difference equations for series expansions at the singular points.

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- One can hardy construct a general algorithm to fix boundary conditions because, usually, the choice of the corresponding point and the way to obtain data for the boundary conditions is done in every situation in a special way.
- Still we can suggest a format for including information about the boundary conditions for using it in our future package.

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- One can apply DE even in the case of one-scale integrals by introducing an extra scale, solving DE with the respect to the ratio of the two scales, x, and then picking a contribution to the expansion at the point where x tends to its primary value [J.M. Henn, A.V. Smirnov & V.S.'13].

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- We are thinking about adjusting our package to this case in general situation.