# Towards analytic local sector subtraction at NNLO

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# Status

- Considerable and successful activity in NNLO subtraction in recent years. Different schemes face the problem in different ways (local vs non-local, analytic vs numerical, ...)
  - Antenna subtraction [Gehrmann De Ridder, Gehrmann, Glover, Heinrich, et al.]
  - Sector-improved residue subtraction [Czakon, Mitov, et al.], [Boughezal, Caola, Melnikov, Petriello, et al.]
  - Colourful subtraction [Del Duca, Duhr, Kardos, Somogyi, Troscanyi, et al.]
  - See also several NNLO computations with qT-slicing [Catani, Grazzini, et al.], N-jettiness slicing [Boughezal, Petriello, et al.], [Gaunt, Tackmann, et al.], projection to Born [Cacciari, Salam, Zanderighi, et al.], sector decomposition [Anastasiou, et al.], [Binoth, et al.], $\mathcal{E}$-prescription [Frixione, Grazzini].
- Complexity of the subtraction structures seems to increase substantially with respect to NLO.
- Problem most of the times tackled introducing various elements/techniques new with respect to commonly used analytic NLO solutions.

# Motivations

- Investigation on possible alternative NNLO schemes motivated by questions like
  - What are the simplest possible objects that achieve local analytic subtraction at NNLO? Are we maximally exploiting the freedom we have in defining the subtraction procedure, in order to simplify it?
  - What are the NLO-subtraction ideas we can advantageously export at NNLO, and what are instead the bottlenecks? Can we cure the latter?
  - In view of analytic NNLO subtraction with massive particles, or of future higher-order computations, do we have to expect a formidable complexity gradient (also in the definition of subtraction schemes), or can we hope still to deal with objects manageable analytically?
- Won't provide final answers to all these questions, but this study is showing promising directions to be investigated.
- In the following, preliminary results on massless and final-state-only QCD partons.

# Subtraction at NLO

- Schematic structure of NLO correction ($X$ = IRC safe observable, $\delta_{X_i} \equiv \delta(X - X_i)$), $X_i$ = observable computed with $i$-body kinematics.

$$\frac{d\sigma_{\rm NLO} - d\sigma_{\rm LO}}{dX} = \int d\Phi_n V\,\delta_{X_n} + \int d\Phi_{n+1} R\,\delta_{X_{n+1}} = \text{finite.}$$

  $V$ diverges as $\epsilon^{-2}$ (with $d = 4 - 2\epsilon$), $R$ diverges (doubly) in the radiation phase space.

- Add and subtract the counterterm

$$\int d\Phi_{n+1}\,K\,\delta_{X_n}.$$

  $K$ with the same singularities as $R$, but sufficiently simple to be integrated in $d$ dimensions. Possible thanks to the universality of IRC singularities.

- Subtracted cross section ($I = \int \frac{d\Phi_{n+1}}{d\Phi_n} K$) becomes

$$\frac{d\sigma_{\rm NLO} - d\sigma_{\rm LO}}{dX} = \int d\Phi_n (V + I)\,\delta_{X_n} + \int d\Phi_{n+1}\Big(R\,\delta_{X_{n+1}} - K\,\delta_{X_n}\Big)$$

- $V + I$ is finite for $\epsilon = 0$, $R - K$ has no phase-space singularities. Everything integrable numerically in $d = 4$.

# Subtraction at NLO: sectors

- Different well-established recipes to achieve subtraction at NLO. Most used Catani-Seymour (CS) dipole subtraction [Catani, Seymour, 9605323], [Catani et al., 0201036], FKS residue subtraction [Frixione, Kunszt, Signer, 9512328], [Frixione, 9706545], Nagy-Soper [Nagy, Soper, 0308127].

- FKS introduces a partition of the phase spaces into sectors, successful in reducing the complexity of the subtraction problem.

  - Transparent singularity structure: only a limited number of identified partons can go soft/collinear in a given sector.

  - Help disentangling overlapping singularities.

  - Each sector can be treated separately, intrinsic parallelisation.

# 'Standard' sector subtraction at NLO: FKS

- Divide the phase space with (smooth) sector functions $\mathcal{P}_{ij}$ (such that $\sum_{ij} \mathcal{P}_{ij} = 1$) dampening all real singularities but single soft ($\xi_i \equiv 2E_i/\sqrt{s} \to 0$), and single collinear ($y_{ij} \equiv \cos\theta_{ij} \to 1$). Angles and energies defined in the partonic CM frame.

- Properties:

$$\lim_{\xi_i \to 0} \sum_j \mathcal{P}_{ij} = 1, \qquad \lim_{y_{ij} \to 1} (\mathcal{P}_{ij} + \mathcal{P}_{ji}) = 1.$$

- Namely, by summing over the sectors that do not vanish in the IRC limits, the $\mathcal{P}_{ij}$ functions disappear. Key for the analytical integration of the counterterm: integrating over sector functions would be cumbersome/impossible analytically.

- Each sector parametrised differently, in terms of $\xi_i$ and $y_{ij}$. Local counterterm defined after the parametrisation has been chosen:

$$K_{ij} \sim R\,\mathcal{P}_{ij}\Big|_{\xi_i=0} + R\,\mathcal{P}_{ij}\Big|_{y_{ij}=1} - R\,\mathcal{P}_{ij}\Big|_{\substack{\xi_i=0\\ y_{ij}=1}},$$

where '$\xi_i = 0$', and '$y_{ij} = 1$' mean to retain only the residues in the Laurent expansion around the IRC limits.

# Potential bottlenecks in counterterm analytic integration

- Very successful and natural subtraction method, but a couple of suboptimal features.
- By defining the local counterterm after parametrisation, some flexibility in its integration is lost: e.g. soft limit ($\xi_i = 0$) features an eikonal sum $\sum_{kl} \frac{s_{kl}}{s_{ik}s_{il}}$ that leaves

$$\int \frac{d\Phi_{n+1}}{d\Phi_n} \sum_j K_{ij}^{(\mathrm{soft})} \;\sim\; \sum_{kl} \int d\bar{\Omega}_i \frac{1-\cos\bar{\theta}_{kl}}{(1-\cos\bar{\theta}_{ki})(1-\cos\bar{\theta}_{il})}.$$

  Simplifications (reparametrisations) after the energy variable is pulled out are limited.
- Privileged reference frame and non-invariant parametrisation: potentially complicated expressions in view of NNLO.
- Radiation phase space

$$\frac{d\Phi_{n+1}}{d\Phi_n} \;\sim\; d\xi_i dy_{ij} \left( \frac{1}{2-\xi_i(1-y_{ij})} \right)^{-2\epsilon} f(\xi_i) g(y_{ij})$$

  easily integrated in $d$ dimensions only because denominator 'accidentally' trivial in all IRC limits.

# ‘Modified’ sector subtraction at NLO (as a laboratory for NNLO)

- Singularity structure in sector $ij$ known in advance: build the easiest possible function containing all real poles in sector $ij$ in terms of dot products $s_{ab} = 2p_a \cdot p_b$, before parametrising.

- Action of singular limits on dot products

  - Soft limit $\mathbf{S}_i$ ($p_i^\mu \to 0$): $s_{ia}/s_{ib} \to$ constant, $s_{ia}/s_{bc} \to 0$, $\forall\, a, b, c \neq i$.

  - Collinear limit $\mathbf{C}_{ij}$ ($k_\perp \to 0$): $s_{ij}/s_{ia} \to 0$, $s_{ij}/s_{jb} \to 0$, $s_{ij}/s_{ab} \to 0$, $\forall\, a, b \neq i, j$. $s_{ia}/s_{ja} \to$ independent of $a$.

- $\mathbf{S}_i R$ ($\mathbf{C}_{ij} R$) are the most singular terms in $R$ as $p_i^\mu$ ($k_\perp$) goes to 0. They are universal kernels and limits commute: $\mathbf{S}_i \mathbf{C}_{ij} R = \mathbf{C}_{ij} \mathbf{S}_i R$.

- Function $K_{ij} \;=\; (\mathbf{S}_i + \mathbf{C}_{ij} - \mathbf{S}_i \mathbf{C}_{ij}) R\, \mathcal{P}_{ij} \;=\; [\mathbf{1} - (\mathbf{1} - \mathbf{S}_i)(\mathbf{1} - \mathbf{C}_{ij})] R\, \mathcal{P}_{ij}$ (limits applied both to $R$ and to $\mathcal{P}_{ij}$) contains all singularities of $R\, \mathcal{P}_{ij}$ in sector $ij$, and $R_{ij}^{\mathrm{sub}} \;=\; R\, \mathcal{P}_{ij} - K_{ij}$ is finite everywhere in the phase space.

- Structure of the local counterterm $K_{ij}$ in each sector is as minimal as in FKS (one-to-one correspondence), but parametrisation independent.

# Mapping to Born kinematics

- Momentum mapping $p_1, ..., p_{n+1} \quad \rightarrow \quad \bar{p}_1, ..., \bar{p}_n$ necessary to factorise Born phase space from radiation phase space, and integrate the counterterm only in the latter.

- Born (and colour-correlated Born) amplitudes implicitly appearing in the counterterm $(\mathbf{S}_i + \mathbf{C}_{ij} - \mathbf{S}_i \mathbf{C}_{ij}) R \, \mathcal{P}_{ij}$ written in terms of mapped $\bar{p}_i$ momenta.

- Convenient Catani-Seymour mappings

$$
\begin{aligned}
p_i + p_j + p_k \quad &= \quad \bar{p}_{[ij]} + \bar{p}_k, \\
\bar{p}_k \quad = \quad \frac{1}{1-y} p_k, \qquad\qquad & \bar{p}_{[ij]} \quad = \quad p_i + p_j - \frac{y}{1-y} p_k,
\end{aligned}
$$

  with $i, j, k$ chosen to simplify as much as possible counterterm integration. At variance with FKS, where the sector defines the mapping.

- Modified sector subtraction at NLO is like a bridge between FKS and CS, retaining the strengths of both (sector approach, and minimal structure from FKS; Lorentz invariance, and phase-space mappings from CS).

# Advantages in counterterm integration

- In a given sector, there is freedom to choose phase-space mappings and parametrisations differently for different contributions to the counterterm.
  - For example: in the soft contribution, each term in the eikonal sum $\sum_{kl} \frac{s_{kl}}{s_{ik}s_{il}}$ is mapped and parametrised differently, yielding straightforward integration

$$\int \frac{d\Phi_{n+1}}{d\Phi_n} \frac{s_{kl}}{s_{ik}s_{il}} \propto (\bar{p}_{[ik]} \cdot \bar{p}_l)^{-\epsilon} \int_0^1 dy\, dz \left( y(1-y)^2 z(1-z) \right)^{-\epsilon} \frac{(1-y)(1-z)}{yz}$$
$$\propto (\bar{p}_{[ik]} \cdot \bar{p}_l)^{-\epsilon} B(-\epsilon, 2-\epsilon) B(-\epsilon, 2-2\epsilon).$$

  - $y$ and $z$ Catani-Seymour variables for dipole $ikl$, in terms of which the phase space is simple (factorised) also far from the singular limits.

- This 'modified' sector subtraction method has been successfully applied at NLO: integrated counterterm shown analytically to reproduce all virtual poles.

- Method works as well as FKS and CS, but seems to be more easily exportable to NNLO (still, study limited to massless and final-state-only QCD partons).

# Subtraction at NNLO

- Schematic structure of NNLO cross section ($X$ = IRC safe observable, $\delta_{X_i} \equiv \delta(X - X_i)$)

$$\frac{d\sigma_{\mathrm{NNLO}} - d\sigma_{\mathrm{NLO}}}{dX} = \int d\Phi_n VV\,\delta_{X_n} + \int d\Phi_{n+1} RV\,\delta_{X_{n+1}} + \int d\Phi_{n+2}\,RR\,\delta_{X_{n+2}}$$

$VV$ diverges as $\epsilon^{-4}$ (with $d = 4 - 2\epsilon$), $RR$ diverges (quadruply) in the radiation phase space, $RV$ diverges as $\epsilon^{-2}$ and (doubly) in the radiation phase space.

- Add and subtract the counterterms

$$\int d\Phi_{n+2}\left(K^{(\mathbf{1})}\,\delta_{X_{n+1}} + K^{(\mathbf{2})}\,\delta_{X_n}\right), \qquad \int d\Phi_{n+2}\,K^{(\mathbf{RV})}\,\delta_{X_n}.$$

- Subtracted cross section ($I^{(\mathbf{j})} = \int \frac{d\Phi_{n+2}}{d\Phi_{n+2-j}} K^{(\mathbf{j})}$, $\;I^{(\mathbf{RV})} = \int \frac{d\Phi_{n+1}}{d\Phi_n} K^{(\mathbf{RV})}$) becomes

$$\begin{aligned}\frac{d\sigma_{\mathrm{NNLO}} - d\sigma_{\mathrm{NLO}}}{dX} &= \int d\Phi_n\left(VV + I^{(\mathbf{2})} + I^{(\mathbf{RV})}\right)\delta_{X_n} \\ &+ \int d\Phi_{n+1}\left(\left(RV + I^{(\mathbf{1})}\right)\delta_{X_{n+1}} - K^{(\mathbf{RV})}\delta_{X_n}\right) \\ &+ \int d\Phi_{n+2}\left(RR\,\delta_{X_{n+2}} - K^{(\mathbf{1})}\delta_{X_{n+1}} - K^{(\mathbf{2})}\delta_{X_n}\right).\end{aligned}$$

Everything integrable numerically in $d = 4$.

# Sector functions at NNLO

- Introduce (smooth) sector functions $\mathcal{P}_{ijkl}$ partitioning the phase space ($\sum_{ijkl} \mathcal{P}_{ijkl} = 1$) to dampen all double-real singularities but single soft and single collinear ($\mathbf{S}_i$, $\mathbf{C}_{ij}$), double soft and double collinear ($\mathbf{S}_{i,k}$, $\mathbf{C}_{ikj}$, $\mathbf{C}_{ij,kl}$), and soft collinear $\mathbf{SC}_{i,kl}$.
- Required properties:
  - In the double limits, upon summing over the sectors that do not vanish, the sector functions must disappear $\Longrightarrow$ analytic integration of the $K^{(\mathbf{2})}$ counterterms.

$$\mathbf{S}_{i,k} \sum_j (\mathcal{P}_{ijkj} + \mathcal{P}_{ikkj} + \sum_l \mathcal{P}_{ijkl}) \; = \; 1, \qquad \mathbf{C}_{ikj} \sum_{\mathrm{perm}\, ijk} (\mathcal{P}_{ijjk} + \mathcal{P}_{ijkj}) \; = \; 1.$$

  - In the single limits, NNLO sector functions must factorise NLO ones $\Longrightarrow$ combination of the single-unresolved $I^{(\mathbf{1})}$ with the subtracted $RV$ NLO sector by NLO sector.

$$\mathbf{C}_{ij}\mathcal{P}_{ijkj} \;\sim\; \bar{\mathcal{P}}_{k[ij]}\mathbf{C}_{ij}\mathcal{P}_{ij}, \qquad \mathbf{S}_i\mathcal{P}_{ijkj} \;\sim\; \bar{\mathcal{P}}_{kj}\mathbf{S}_i\mathcal{P}_{ij},$$
$$\mathbf{C}_{ij}\mathcal{P}_{ijkl} \;\sim\; \bar{\mathcal{P}}_{kl}\mathbf{C}_{ij}\mathcal{P}_{ij}, \qquad \mathbf{S}_i\mathcal{P}_{ijkl} \;\sim\; \bar{\mathcal{P}}_{kl}\mathbf{S}_i\mathcal{P}_{ij}.$$

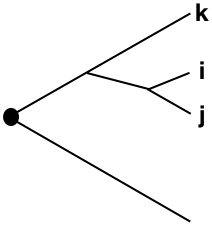
# Structure of the double-real counterterms

- Different combinations of indices in the sector functions select different singular limits:

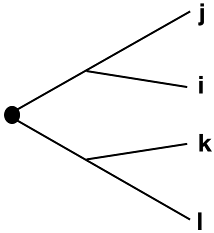
$$\begin{aligned} \text{sector} \;\; \mathcal{P}_{ijkj} : &\quad \mathbf{S}_i, \quad \mathbf{C}_{ij}, \quad \mathbf{SC}_{i,jk}, \quad \mathbf{C}_{ikj}, \quad \mathbf{S}_{i,k}, \\ \text{sector} \;\; \mathcal{P}_{ijkl} : &\quad \mathbf{S}_i, \quad \mathbf{C}_{ij}, \quad \mathbf{SC}_{i,kl}, \quad \mathbf{C}_{ij,kl}, \quad \mathbf{S}_{i,k}, \end{aligned}$$

(sector $\mathcal{P}_{ijjk}$ fully analogous to $\mathcal{P}_{ijkj}$ with $\mathbf{S}_{i,k} \leftrightarrow \mathbf{S}_{i,j}$).

- Roughly, they correspond to two topologies




$\mathcal{P}_{ijkj}$




$\mathcal{P}_{ijkl}$

## Structure of the double-real counterterms

- Local counterterm in sector $ijkj$ built from the singularities $\mathbf{S}_i, \mathbf{C}_{ij}, \mathbf{SC}_{i,jk}, \mathbf{C}_{ikj}, \mathbf{S}_{i,k}$ that are not killed by function $\mathcal{P}_{ijkj}$:

$$\begin{aligned} K^{(\mathbf{1})}_{ijkj} + K^{(\mathbf{2})}_{ijkj} &= RR\,\mathcal{P}_{ijkj} - RR^{\mathrm{sub}}_{ijkj} \\ &= [\mathbf{1} - (\mathbf{1} - \mathbf{S}_i)(\mathbf{1} - \mathbf{C}_{ij})(\mathbf{1} - \mathbf{S}_{i,k})(\mathbf{1} - \mathbf{C}_{ikj})(\mathbf{1} - \mathbf{SC}_{i,jk})]RR\,\mathcal{P}_{ijkj}. \end{aligned}$$

- Analogously for sector $ijkl$:

$$\begin{aligned} K^{(\mathbf{1})}_{ijkl} + K^{(\mathbf{2})}_{ijkl} &= RR\,\mathcal{P}_{ijkl} - RR^{\mathrm{sub}}_{ijkl} \\ &= [\mathbf{1} - (\mathbf{1} - \mathbf{S}_i)(\mathbf{1} - \mathbf{C}_{ij})(\mathbf{1} - \mathbf{S}_{i,k})(\mathbf{1} - \mathbf{C}_{ij,kl})(\mathbf{1} - \mathbf{SC}_{i,kl})]RR\,\mathcal{P}_{ijkl}. \end{aligned}$$

- $\mathbf{S}_{i,k}RR$, $\mathbf{C}_{ikj}RR$, and $\mathbf{SC}_{i,jk}RR$ are universal kernels [Catani, Grazzini, 9810389, 9908523], [Campbell, Glover, 9710255], [Berends, Giele, 1989].

- All limits commute ensuring that the above expressions collect all double-real poles in sectors $ijkj$ and $ijkl$ (and similarly for $ijjk$).

- See also [Frixione, Grazzini, 0411399] for a discussion on commutation and counterterm minimality.

## Simplification of the double-real counterterms

- Out of five singular limits, obviously only four are truly 'irreducible' (no more than four poles per sector in $RR$).

- Physical consistency (projection relations) eliminates redundancies in sector $ijkj$:

$$\mathbf{S}_i\mathbf{SC}_{i,jk}\{\mathbf{1},\mathbf{C}_{ij},\mathbf{S}_{i,k},...\}RR \quad = \quad \mathbf{SC}_{i,jk}\{\mathbf{1},\mathbf{C}_{ij},\mathbf{S}_{i,k},...\}RR$$

$$\Longrightarrow \qquad \begin{aligned} K^{(\mathbf{2})}_{ijkj} &= (\mathbf{S}_{i,k}+\mathbf{C}_{ikj}-\mathbf{S}_{i,k}\mathbf{C}_{ikj})RR\,\mathcal{P}_{ijkj}, \\ K^{(\mathbf{1})}_{ijkj} &= (\mathbf{1}-\mathbf{S}_{i,k})(\mathbf{1}-\mathbf{C}_{ikj})(\mathbf{S}_i+\mathbf{C}_{ij}-\mathbf{S}_i\mathbf{C}_{ij})RR\,\mathcal{P}_{ijkj}. \end{aligned}$$

- Analogously for sector $ijkl$:

$$\begin{aligned} \mathbf{S}_i\mathbf{SC}_{i,kl}\{\mathbf{1},\mathbf{C}_{ij},\mathbf{S}_{i,k},...\}RR &= \mathbf{SC}_{i,kl}\{\mathbf{1},\mathbf{C}_{ij},\mathbf{S}_{i,k},...\}RR, \\ \mathbf{C}_{ij}\mathbf{C}_{ij,kl}\{\mathbf{1},\mathbf{S}_{i,k},...\}RR &= \mathbf{C}_{ij,kl}\{\mathbf{1},\mathbf{S}_{i,k},...\}RR \end{aligned}$$

$$\Longrightarrow \qquad \begin{aligned} K^{(\mathbf{2})}_{ijkl} &= \mathbf{S}_{i,k}RR\,\mathcal{P}_{ijkl}, \\ K^{(\mathbf{1})}_{ijkl} &= (\mathbf{1}-\mathbf{S}_{i,k})(\mathbf{S}_i+\mathbf{C}_{ij}-\mathbf{S}_i\mathbf{C}_{ij})RR\,\mathcal{P}_{ijkl}. \end{aligned}$$

- Limits $\mathbf{C}_{ij,kl}$ and $\mathbf{SC}_{i,jk}$ completely disappear from the definition of the counterterm, accounted for by the overlap of other limits (see also [Caola, Melnikov, Roentsch, 1702.01352] about the redundancy of $\mathbf{SC}$).

# Integration of the double-unresolved counterterm $K^{(\mathbf{2})}$

$$\begin{aligned} I^{(\mathbf{2})} &= \int \frac{d\Phi_{n+2}}{d\Phi_n} \left\{ \sum_{ijk} (\mathbf{S}_{i,k} + \mathbf{C}_{ijk} - \mathbf{S}_{i,k}\mathbf{C}_{ijk}) RR\, (\mathcal{P}_{ijkj} + \mathcal{P}_{ikkj}) + \sum_{ijkl} \mathbf{S}_{i,k} RR\, \mathcal{P}_{ijkl} \right\} \\ &= \int \frac{d\Phi_{n+2}}{d\Phi_n} \left\{ \sum_{ik} \mathbf{S}_{i,k} + \sum_{ijk} (\mathbf{C}_{ijk} - \mathbf{S}_{i,k}\mathbf{C}_{ijk}) \right\} RR \end{aligned}$$

- All sector functions are gone owing to their sum rules in the IRC limits. Integration of kernels manageable analytically.

- Examples: double-soft $q\bar{q}$ (with two partons at Born) and collinear $qq'\bar{q}'$. Use Catani-Seymour-like variables $y, z, y', z'$. Phase-space measure factorises $d\mu = dx'dy'dz'dydz[x'(1-x')y'(1-y')^2 z'(1-z')y^2(1-y)^2 z(1-z)]^{-\epsilon}[x'(1-x')]^{-1/2}(1-y')y(1-y)$

$$\begin{aligned} \int \frac{d\Phi_{n+2}}{d\Phi_n} \mathbf{S}_{i,k} RR &= (4\pi\alpha_{\mathrm{S}}^u \mu_0^{2\epsilon})^2 T_R \sum_{l,m=1}^{2} \tilde{B}_{lm} \int \frac{d\Phi_{n+2}}{d\Phi_n} \frac{4(s_{il}s_{km} + s_{im}s_{kl} - s_{ik}s_{lm})}{s_{ik}^2 (s_{il} + s_{kl})(s_{im} + s_{km})} \\ &\propto \int_0^1 d\mu\, \frac{z'(1-z')}{y^2 y'^2} \frac{y'(1-z)}{y'(1-z) + z} \\ &= \tilde{B} \left(\frac{\alpha_{\mathrm{S}}}{2\pi}\right)^2 T_R C_F \left(\frac{\mu^2}{s}\right)^{2\epsilon} \left[ -\frac{1}{3\epsilon^3} - \frac{17}{9\epsilon^2} + \frac{1}{\epsilon}\left(\frac{7}{18}\pi^2 - \frac{232}{27}\right) + \left(\frac{38}{9}\zeta_3 + \frac{131}{54}\pi^2 - \frac{2948}{81}\right) \right] + \mathcal{O}(\epsilon), \end{aligned}$$

$$\begin{aligned} \int \frac{d\Phi_{n+2}}{d\Phi_n} \mathbf{C}_{ikj} RR &= (4\pi\alpha_{\mathrm{S}}^u \mu_0^{2\epsilon})^2 T_R C_F \tilde{B} \int \frac{d\Phi_{n+2}}{d\Phi_n} \frac{4}{s_{ijk}^2} \frac{1}{2} \frac{s_{ijk}}{s_{ik}} \left[ -\frac{t_{ik,j}^2}{s_{ik}s_{ikj}} + \frac{4z_j + (z_i - z_k)^2}{z_i + z_k} + (1-2\epsilon)\left(z_i + z_k - \frac{s_{ik}}{s_{ikj}}\right) \right] \\ &= \tilde{B} \left(\frac{\alpha_{\mathrm{S}}}{2\pi}\right)^2 T_R C_F \left(\frac{\mu^2}{s}\right)^{2\epsilon} \left[ -\frac{1}{3\epsilon^3} - \frac{31}{18\epsilon^2} + \frac{1}{\epsilon}\left(\frac{1}{2}\pi^2 - \frac{889}{108}\right) + \left(\frac{80}{9}\zeta_3 + \frac{31}{12}\pi^2 - \frac{23941}{648}\right) \right] + \mathcal{O}(\epsilon), \end{aligned}$$

# Real-virtual and integrated single-unresolved conuterterms

- Real-virtual contribution has NLO kinematics, and it is subtracted in sector $kl$ similarly to the real NLO by a counterterm

$$K_{kl}^{(\mathbf{RV})} = (\mathbf{S}_k + \mathbf{C}_{kl} - \mathbf{S}_k\mathbf{C}_{kl}) RV\, \bar{\mathcal{P}}_{kl}.$$

- $RV_{kl}^{\text{sub}} = RV\, \bar{\mathcal{P}}_{kl} - K_{kl}^{(\mathbf{RV})}$ finite in the radiation phase space, but featuring explicit $1/\epsilon$ poles (one loop).

- Combined sector by sector with the integrated single-unresolved counterterm:

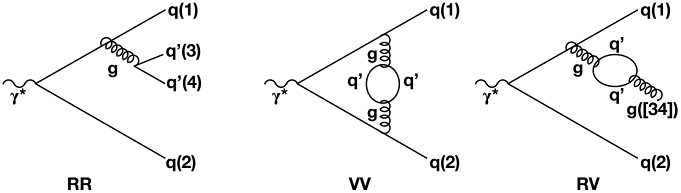
$$I_{kl}^{(\mathbf{1})} = \int \frac{d\Phi_{n+2}}{d\Phi_{n+1}} K_{kl}^{(\mathbf{1})}.$$

  $K_{kl}^{(\mathbf{1})}$ is the collection of all terms in $K^{(\mathbf{1})}$ featuring $\bar{\mathcal{P}}_{kl}$. Possible due to the factorisation properties of the $\mathcal{P}_{abcd}$ functions.

- $RV\, \bar{\mathcal{P}}_{kl} + I_{kl}^{(\mathbf{1})}$ finite in $d = 4$ (analogously to NLO subtraction, virtual plus integrated counterterm).

# A proof-of-concept example

- $T_R C_F$ NNLO contribution to the total cross section for $e^+e^- \to q\bar{q}$ (analytic matrix elements from [Hamberg, van Neerven, Matsuura, 1991], [Gehrmann De Ridder, Gehrmann, Glover, 0403057], [Ellis, Ross, Terrano, 1980])




$$
\begin{aligned}
VV &= \tilde{B}\left(\frac{\alpha_S}{2\pi}\right)^2 T_R C_F \left\{\left(\frac{\mu^2}{s}\right)^{2\epsilon}\left[\frac{1}{3\epsilon^3}+\frac{14}{9\epsilon^2}+\frac{1}{\epsilon}\left(-\frac{11}{18}\pi^2+\frac{353}{54}\right)+\left(-\frac{26}{9}\zeta_3-\frac{77}{27}\pi^2+\frac{7541}{324}\right)\right]\right. \\
&\qquad \left.+\left(\frac{\mu^2}{s}\right)^{\epsilon}\left[-\frac{4}{3\epsilon^3}-\frac{2}{\epsilon^2}+\frac{1}{\epsilon}\left(\frac{7}{9}\pi^2-\frac{16}{3}\right)+\left(\frac{28}{9}\zeta_3+\frac{7}{6}\pi^2-\frac{32}{3}\right)\right]\right\}, \\
\int\frac{d\Phi_{n+1}}{d\Phi_n} RV &= \frac{\alpha_S}{2\pi}\frac{2}{3}\frac{T_R}{\epsilon}\int\frac{d\Phi_{n+1}}{d\Phi_n}\,\bar{R} \\
&= \tilde{B}\left(\frac{\alpha_S}{2\pi}\right)^2 T_R C_F\left(\frac{\mu^2}{s}\right)^{\epsilon}\left[\frac{4}{3\epsilon^3}+\frac{2}{\epsilon^2}+\frac{1}{\epsilon}\left(-\frac{7}{9}\pi^2+\frac{19}{3}\right)+\left(-\frac{100}{9}\zeta_3-\frac{7}{6}\pi^2+\frac{109}{6}\right)\right], \\
\int\frac{d\Phi_{n+2}}{d\Phi_n} RR &= \tilde{B}\left(\frac{\alpha_S}{2\pi}\right)^2 T_R C_F\left(\frac{\mu^2}{s}\right)^{2\epsilon}\left[-\frac{1}{3\epsilon^3}-\frac{14}{9\epsilon^2}+\frac{1}{\epsilon}\left(\frac{11}{18}\pi^2-\frac{407}{54}\right)+\left(\frac{134}{9}\zeta_3+\frac{77}{27}\pi^2-\frac{11753}{324}\right)\right].
\end{aligned}
$$

# Double-real integrated counterterms $I^{(\mathbf{2})}$ and $I^{(\mathbf{1})}$

- $I^{(\mathbf{2})}$: in the case at hand only $\mathbf{S}_{3,4}RR$, $\mathbf{C}_{134}RR$, $\mathbf{C}_{234}RR$ are non-zero, so

$$
\begin{aligned}
I^{(\mathbf{2})} &= \int \frac{d\Phi_{n+2}}{d\Phi_n} \Big\{ \mathbf{S}_{3,4} + \mathbf{C}_{134} + \mathbf{C}_{234} - \mathbf{S}_{3,4}\mathbf{C}_{134} - \mathbf{S}_{3,4}\mathbf{C}_{234} \Big\} RR \\
&= \tilde{B} \left(\frac{\alpha_S}{2\pi}\right)^2 T_R C_F \left(\frac{\mu^2}{s}\right)^{2\epsilon} \left[ -\frac{1}{3\epsilon^3} - \frac{14}{9\epsilon^2} + \frac{1}{\epsilon}\left(\frac{11}{18}\pi^2 - \frac{425}{54}\right) + \left(\frac{122}{9}\zeta_3 + \frac{74}{27}\pi^2 - \frac{12149}{324}\right)\right].
\end{aligned}
$$

- $I^{(\mathbf{1})}$: in the case at hand only $\mathbf{C}_{34}$ is non-zero, so

$$
\begin{aligned}
I^{(\mathbf{1})} &= I^{(\mathbf{1})}_{12} + I^{(\mathbf{1})}_{1[34]} + I^{(\mathbf{1})}_{2[34]}, \\
I^{(\mathbf{1})}_{12} &= \int \frac{d\Phi_{n+2}}{d\Phi_{n+1}} \, \mathbf{C}_{34} RR \, (\bar{\mathcal{P}}_{12} + \bar{\mathcal{P}}_{21}), \\
I^{(\mathbf{1})}_{i[34]} &= \int \frac{d\Phi_{n+2}}{d\Phi_{n+1}} \, \mathbf{C}_{34} \Big[ RR \, (\bar{\mathcal{P}}_{i[34]} + \bar{\mathcal{P}}_{[34]i}) - \mathbf{C}_{i34} RR - \mathbf{S}_{3,4} RR \, \bar{\mathcal{P}}_{[34]i} + \mathbf{S}_{3,4}\mathbf{C}_{i34} RR \Big].
\end{aligned}
$$

# Real-virtual contribution

- Real virtual, subtracted in sector $ij$:

$$RV_{ij}^{\text{sub}} \;=\; RV\bar{\mathcal{P}}_{ij} - K_{ij}^{(\mathbf{RV})} \;=\; \frac{\alpha_S}{2\pi}\frac{2}{3}\frac{T_R}{\epsilon}\bar{R}_{ij}^{\text{sub}} \qquad \text{(gluon self-energy contribution to } T_R C_F).$$

- Combination with the single-unresolved integrated counterterm $I_{ij}^{(\mathbf{1})}$:

$$\begin{aligned}
RV_{12}^{\text{sub}} + I_{12}^{(\mathbf{1})} &= -\frac{\alpha_S}{2\pi}T_R\left(\frac{2}{3}\ln\frac{\mu^2}{\bar{s}_{1[34]}} + \frac{16}{9}\right)\bar{R}(\bar{\mathcal{P}}_{12} + \bar{\mathcal{P}}_{21}),\\
RV_{i[34]}^{\text{sub}} + I_{i[34]}^{(\mathbf{1})} &= -\frac{\alpha_S}{2\pi}T_R\left(\frac{2}{3}\ln\frac{\mu^2}{\bar{s}_{r[34]}} + \frac{16}{9}\right)\left[\bar{R}(\bar{\mathcal{P}}_{i[34]} + \bar{\mathcal{P}}_{[34]i}) - \mathbf{C}_{i[34]}\bar{R} - \mathbf{S}_{[34]}\bar{R}\,\bar{\mathcal{P}}_{[34]i} + \mathbf{S}_{[34]}\mathbf{C}_{i[34]}\bar{R}\right]
\end{aligned}$$

$I^{(\mathbf{1})}$ sector by sector factorises the same structure as the subtracted $RV$ (in this case proportional to the full subtracted real). Sum finite in $d = 4$ and integrated numerically.

- Real-virtual integrated counterterm (summed over NLO sectors)

$$\begin{aligned}
I^{(\mathbf{RV})} &= \sum_{ij}\int\frac{d\Phi_{n+1}}{d\Phi_n}K_{ij}^{(\mathbf{RV})} = \frac{\alpha_S}{2\pi}\frac{2}{3}\frac{T_R}{\epsilon}\int\frac{d\Phi_{n+1}}{d\Phi_n}\left(\mathbf{S}_{[34]} + \mathbf{C}_{1[34]} + \mathbf{C}_{2[34]} - \mathbf{S}_{[34]}\mathbf{C}_{1[34]} - \mathbf{S}_{[34]}\mathbf{C}_{2[34]}\right)\bar{R}\\
&= \tilde{B}\left(\frac{\alpha_S}{2\pi}\right)^2 T_R C_F\left(\frac{\mu^2}{s}\right)^{\epsilon}\left[\frac{4}{3\epsilon^3} + \frac{2}{\epsilon^2} + \frac{1}{\epsilon}\left(-\frac{7}{9}\pi^2 + \frac{20}{3}\right) + \left(-\frac{100}{9}\zeta_3 - \frac{7}{6}\pi^2 + 20\right)\right] + \mathcal{O}(\epsilon).
\end{aligned}$$

## Collection of results

- Example for $\mu/\sqrt{s} = 0.35$.
- Subtracted double-virtual (fully analytic):

$$\begin{aligned} VV + I^{(\mathbf{2})} + I^{(\mathbf{RV})} &= \tilde{B}\left(\frac{\alpha_S}{2\pi}\right)^2 T_R C_F \left(\frac{8}{3}\zeta_3 - \frac{1}{9}\pi^2 - \frac{44}{9} - \frac{4}{3}\ln\frac{\mu^2}{s}\right) \\ &= \tilde{B}\left(\frac{\alpha_S}{2\pi}\right)^2 T_R C_F \times 0.01949914. \end{aligned}$$

- Subtracted real-virtual and double-real (integrated numerically in $d = 4$):

$$\begin{aligned} \int \frac{d\Phi_{n+1}}{d\Phi_n}\left(RV + I^{(\mathbf{1})} - K^{(\mathbf{RV})}\right) &= \tilde{B}\left(\frac{\alpha_S}{2\pi}\right)^2 T_R C_F \times (-0.90635 \pm 0.00011), \\ \int \frac{d\Phi_{n+2}}{d\Phi_n}\left(RR - K^{(\mathbf{1})} - K^{(\mathbf{2})}\right) &= \tilde{B}\left(\frac{\alpha_S}{2\pi}\right)^2 T_R C_F \times (+2.29491 \pm 0.00038). \end{aligned}$$

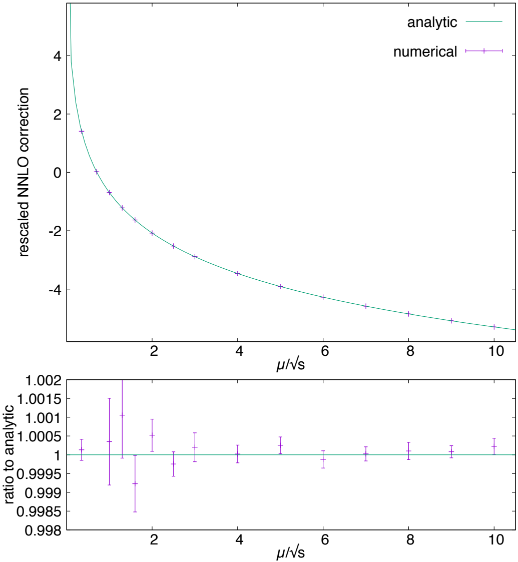
- NNLO correction, evaluated by means of the subtraction method, is

$$\frac{1}{\left(\frac{\alpha_S}{2\pi}\right)^2 T_R C_F}\frac{\sigma_{\rm NNLO} - \sigma_{\rm NLO}}{\sigma_{\rm LO}} = 1.40806 \pm 0.00040.$$

- Analytic result

$$-\frac{11}{2} + 4\zeta_3 - \ln\frac{\mu^2}{s} = 1.40787186.$$

# Renormalisation-scale dependence



## Conclusions

- Investigation on possible simplifications for local analytic subtraction at NNLO.
- Approach trying to conjugate minimality in counterterm definitions, and simplicity in their integrations.
- Use of sector functions to reduce the subtraction problem to its minimal constituent blocks, like FKS, but being more flexible and parametrisation independent.
- Method shown to work at NLO in general, and at NNLO in a proof-of-concept case. Compact local-counterterm structures, manageable analytically owing to sector-function sum rules. No sector decomposition involved in the subtraction scheme.
- Preliminary study performed in massless and final-state-only QCD. More general cases (initial-state radiation, masses) not yet investigated; hopefully complexity scaling under control.
- Structural simplicity may display links to fundamental concepts, like factorisation, or may be considered for automation.

# Thank you for your attention

# Backup: soft/collinear commutation at NLO

- Soft limit $\mathbf{S}_i$ ($p_i^\mu \to 0$): $s_{ia}/s_{ib} \to$ constant, $s_{ia}/s_{bc} \to 0$, $\forall\, a,b,c \neq i$.
- Collinear limit $\mathbf{C}_{ij}$ ($k_\perp \to 0$): $s_{ij}/s_{ia} \to 0$, $s_{ij}/s_{jb} \to 0$, $s_{ij}/s_{ab} \to 0$, $\forall\, a,b \neq i,j$. $s_{ia}/s_{ja} \to$ independent of $a$.
- Commutation in case $i =$ gluon and $j =$ quark.
- Altarelli-Parisi collinear kernel involved is $P(z_i) = [1+(1-z_i)^2]/z_i$, with $z_i = s_{ir}/(s_{ir}+s_{jr})$, with arbitrary $r \neq i,j$.

$$
\begin{aligned}
\mathbf{S}_i R &= -8\pi\alpha_{\mathrm{S}}\bar{\mu}^{2\epsilon} \sum_{l,k\neq l} \frac{s_{lk}}{s_{ik}s_{il}} B_{kl} \\
\implies \mathbf{C}_{ij}\mathbf{S}_i R &= -16\pi\alpha_{\mathrm{S}}\bar{\mu}^{2\epsilon} \sum_{k\neq j} \frac{s_{jk}}{s_{ik}s_{ij}} B_{kj} \;=\; -16\pi\alpha_{\mathrm{S}}\bar{\mu}^{2\epsilon} \frac{s_{jr}}{s_{ir}s_{ij}} (-C_j B), \\
\mathbf{C}_{ij} R &= 8\pi\alpha_{\mathrm{S}}\bar{\mu}^{2\epsilon} \frac{1}{s_{ij}} C_j B \frac{1+[1-s_{ir}/(s_{ir}+s_{jr})]^2}{s_{ir}/(s_{ir}+s_{jr})} \\
\implies \mathbf{S}_i\mathbf{C}_{ij} R &= -16\pi\alpha_{\mathrm{S}}\bar{\mu}^{2\epsilon} \frac{s_{jr}}{s_{ir}s_{ij}} (-C_j B).
\end{aligned}
$$