Sub-leading power N-jet amplitudes

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Outline

- Introduction and SCET framework
- Operator basis
- Structure of the anomalous dimension matrix, soft and collinear mixing
- Example: fermion number two

MB, M. Garny, R. Szafron, J. Wang, in preparation

Motivations for NLP

- Next-to-leading power (NLP, $\tau \to 0$) at NNLO

$$\sigma^{\mathrm{NNLO}}(\tau) \overset{\tau \to 0}{\sim} \delta(\tau) + \left[\frac{\ln^{3,2,1,0}}{\tau}\right]_+ + \ln^{3,2,1,0}\tau + \mathcal{O}(\tau)$$

  - Drell-Yan process near threshold
    [Del Duca, Laenen, Magnea, Vernazza, White, 1706.04018; Bonocore, Laenen, Magnea, Vernazza, White, 1706.04018, 1610.06842 and earlier papers]
  - Improving N-jettiness subtraction
    [Moult, Rothen, Stewart, Tackmann, Zhu, 1612.00450; Boughezal, Liu, Petriello, 1612.02911; NLP leading log]

- All-order resummation of NLP logs

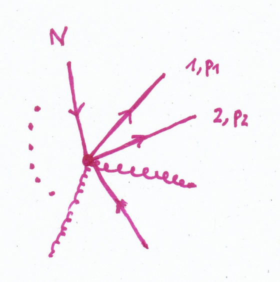
- Generality of Low's sub-leading soft theorem
  - Relation to an infinite-dimensional symmetry group of "large" (non-vanishing at infinity) gauge transformations
    [Strominger, 2013–]
  - But: LBKD theorem beyond tree-level not so simple.
    [SCET analysis: Larkowski, Neill, Stewart, 1412.3108]

N-jet amplitudes, leading power

Source of the hard process. N non-collinear directions defined by momenta

$$p_i^\mu = n_{+i} \cdot p_i \, \frac{n_{-i}^\mu}{2} + p_{\perp i} + n_{-i} \cdot p_i \, \frac{n_{+i}^\mu}{2}, \quad p_i^2 = 0, \quad \text{all } p_i \cdot p_j \sim Q^2$$

Log structure determined by IR singularities of the amplitude



N-jet operator in SCET

$$\mathcal{O}(x) = \int \prod_{i=1}^{N} dt_i \, C(\{t_i\}) \prod_{i=1}^{N} \psi_i(x + t_i n_{+i})$$

Log structure determined by the UV divergences of collinear and soft loops in SCET [Becher, Neubert, 2009]

$$Z_{\mathcal{O}} \prod_{i=1}^{N} \sqrt{Z}_i \langle 0 | \mathcal{O}(0) | \mathcal{M}(\{p_i\}) \rangle_{|\mathcal{L}^{(0)}_{\text{SCET}}} \stackrel{!}{=} \text{finite}$$

SCET, leading power

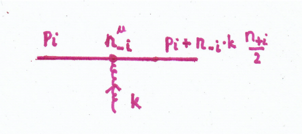
Employ the position-space SCET formalism [MB, Chapovsky, Diehl, Feldmann, 2002]

$$\mathcal{L}^{(0)}_{\text{SCET}} = \sum_{i=1}^{N} \mathcal{L}^{(0)}_{c_i} + \mathcal{L}_{\text{soft}}$$

$$\mathcal{L}^{(0)}_{c}(x) = \bar{\xi}\left(in_- D_c + g_s n_- A_s(x_-) + i\not{D}_{\perp c}\frac{1}{in_+ D_c}\, i\not{D}_{\perp c}\right)\frac{\not{n}_+}{2}\,\xi + \mathcal{L}^{(0)}_{c,\text{YM}}$$

$$iD_c = i\partial + g_s A_c\,, \qquad x^\mu_- = \frac{1}{2} n_+ \cdot x\, n^\mu_-$$

- Separate fields for every collinear direction, and one soft
- Different collinear sectors interact only via soft gluon, since hard virtualities are integrated out
- Soft interactions with collinear amounts to standard eikonal vertex

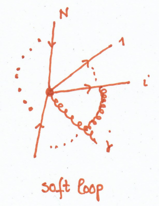


- Note multipole expansion of the soft field around x_-.
  Guarantees eikonal propagator and soft-gluon decoupling via Wilson line field redefinition
  $\xi \to Y(x_-)\xi^{(0)}$ [Bauer, Pirjol, Stewart, 2001]

N-jet amplitudes, leading power anomalous dimension

$$\langle 0|\mathcal{O}(0)|\mathcal{M}(\{p_i\})\rangle_{|\mathcal{L}^{(0)}_{\rm SCET}} = S(\{p_i\})\prod_{i=1}^{N} J_i(p_i^2)$$

$$= 1 - \frac{\alpha_s}{4\pi}\left(\sum_{i,j,i\neq j}\frac{\mathbf{T}_i\cdot\mathbf{T}_j}{2}\left[\frac{2}{\epsilon^2}+\frac{2}{\epsilon}\ln\frac{\mu^2}{-s_{ij}}\right] - \sum_i \mathbf{T}_i^2\,\frac{c_i}{\epsilon} + \mathcal{O}(\epsilon^0)\right)$$



SCET matrix element is scaleless without IR regulator, since all invariants are hard. Use small off-shellness p_i^2.
Colour conservation $\sum_i \mathbf{T}_i = 0$.

$$J_i(p_i^2) = 1 + \frac{\alpha_s}{4\pi}\mathbf{T}_i^2\left[\frac{2}{\epsilon^2}+\frac{2}{\epsilon}\ln\frac{\mu^2}{-p_i^2}+\frac{c_i}{\epsilon}\right]$$

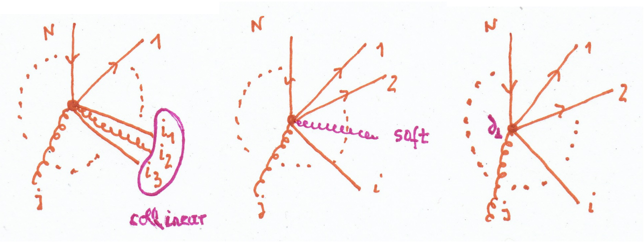
$$S(\{p_i\}) = 1 + \frac{\alpha_s}{4\pi}\sum_{i,j,i\neq j}\frac{\mathbf{T}_i\cdot\mathbf{T}_j}{2}\left[\frac{2}{\epsilon^2}+\frac{2}{\epsilon}\ln\frac{-\mu^2 s_{ij}}{p_i^2 p_j^2}\right]$$

Note cancellation of IR regulator in pole parts. Required by consistency. UV anomalous dimension must not depend on IR reg

N-jet amplitudes, sub-leading power

NLP N-jet operators are the basic objects to match onto for NLP calculations
If $p_\perp \sim \lambda Q$ and jet mass scale $p_J^2 \sim \lambda^2 Q^2$, need $\mathcal{O}(\lambda^2)$ in SCET expansion.

- Matrix elements of LP N-jet operators with sub-leading soft and collinear interactions from $\mathcal{L}^{(1)}$, $\mathcal{L}^{(2)}$
- N-jet operators with 1) more than one collinear field of the same type in one direction, or 2) with additional soft fields, or 3) with derivatives.



Building blocks and basis of N-jet operators

Put $x = 0$, i.e. $\mathcal{O}(0)$.
Building blocks

collinear quark	collinear gluon	soft fields
$(W_i^\dagger \xi_i)(t_{ik} n_{+i})$	$(W_i^\dagger i D^\mu_{c_i\perp} W_i)(t_{ik} n_{+i})$	$q_s(0), F^s_{\mu\nu}(0)$
$\mathcal{O}(\lambda)$	$\mathcal{O}(\lambda)$	$\mathcal{O}(\lambda^3, \lambda^4)$

- Collinear gluon operator always transverse. $n_{\pm i} \cdot D^\mu_{c_i}$ can be eliminated by Wilson line identities and equation of motion
- Sub-leading N-jet basis operators are constructed in the following way
  - operate with $i\partial^\mu_{\perp i}$ on collinear building block
  - take products of several collinear building blocks in the same collinear sector, e.g.

$$(W_i^\dagger \xi_i)(t_{i1} n_{+i})(W_i^\dagger \xi_i)(t_{i2} n_{+i})(W_i^\dagger i D^\mu_{c_i\perp} W_i)(t_{i3} n_{+i})$$

- At $\mathcal{O}(\lambda^2)$ up to two $\partial_\perp$ or up to three fields in one sector.
Notation: J^{Ai}, J^{Bi}, J^{Ci}, ...

  – $A, B, C, \ldots$ refers to 1,2,3, ... fields in a given collinear direction
  – i means $\mathcal{O}(\lambda^i)$ in a given collinear sector,

e.g. J^{B2} means two fields and one $\partial_\perp$ on a given sector.

$$(\bar{\xi}_j W_j)(t_j n_{+j}) iD_s^\mu(0) (W_i^\dagger \xi_i)(t_i n_{+i}) \quad \text{does not exist}$$

$\mathcal{L}_{\text{SCET}}$, including power-suppressed interactions

$$\begin{aligned}
\mathcal{L}^{(1)} &= \bar{\xi}\left(x_\perp^\mu n_-^\nu \, W_c \, gF_{\mu\nu}^{\text{us}} W_c^\dagger\right) \frac{\not{n}_+}{2}\,\xi + \mathcal{L}_{\text{YM}}^{(1)} \\
\mathcal{L}_\xi^{(2)} &= \frac{1}{2}\,\bar{\xi}\left((n_- x)\, n_+^\mu n_-^\nu \, W_c \, gF_{\mu\nu}^{\text{us}} W_c^\dagger + x_\perp^\mu x_{\perp\rho} n_-^\nu \, W_c \left[D_{\text{us}}^\rho, gF_{\mu\nu}^{\text{us}}\right] W_c^\dagger\right) \frac{\not{n}_+}{2}\,\xi \\
&+ \frac{1}{2}\,\bar{\xi}\left(i\not{D}_{\perp c}\,\frac{1}{in_+ D_c}\, x_\perp^\mu \gamma_\perp^\nu \, W_c \, gF_{\mu\nu}^{\text{us}} W_c^\dagger + x_\perp^\mu \gamma_\perp^\nu \, W_c \, gF_{\mu\nu}^{\text{us}} W_c^\dagger \,\frac{1}{in_+ D_c}\, i\not{D}_{\perp c}\right) \frac{\not{n}_+}{2}\,\xi + \mathcal{L}_{\text{YM}}^{(2)}
\end{aligned}$$

is invariant under the soft (and a separate collinear) gauge transformation

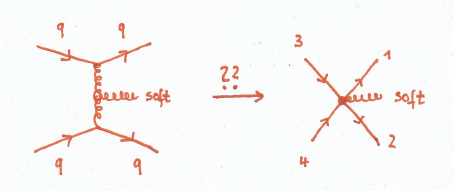
$$A_c \to U_s(x_-)\, A_c \, U_s^\dagger(x_-), \qquad \xi \to U_s(x_-)\,\xi \qquad A_s \to U_s A_s U_s^\dagger + \frac{i}{g}\, U_s\left[\partial, U_s^\dagger\right]$$

$$\begin{aligned}
[t_i n_{+i}]_- = 0 \qquad \Rightarrow \qquad & (W_i^\dagger \xi_i)(t_i n_{+i}) \to U_s(0) (W_i^\dagger \xi_i)(t_i n_{+i}) \\
\text{but} \qquad & iD_s^\mu(0) (W_i^\dagger \xi_i)(t_i n_{+i}) \not\to U_s(0) iD_s^\mu(0) (W_i^\dagger \xi_i)(t_i n_{+i})
\end{aligned}$$

since $i\partial^\mu U_s(0) = 0$.

Also would not have homogeneous power counting since $i\partial_\perp \sim \lambda$ while $A_s(0) \sim \lambda^2$.

$\mathcal{O}(\lambda^{1,2})$ NLP N-jet operator do not contain soft fields (2)



Does not generate a soft operator. Graph is reproduced in SCET by time-ordered products $T(J^{(A0)}, \mathcal{L}_\xi^{(2)}), T(J^{(A1)}, \mathcal{L}_\xi^{(1)})$

- NOTE: There are not subleading purely collinear interactions. Any term in $\mathcal{L}^{(n)}$ with $n \geq 1$ contains at least one soft field.
- Previous work on NLP operator bases
  - [MB, Campanario, Mannel, Pecjak, hep-ph/0411395]

    Operator with soft heavy quark fields as source of large energy. Different, there are $\mathcal{O}(\lambda^2)$ operators with soft gluon fields.
  - [Kolodrubetz, Moult, Stewart, 1601.02607; Feige, Kolodrubetz, Moult, Stewart, 1703.03411]

    Helicity basis in label SCET – rather different: subleading purely collinear interactions, two-point insertions, soft building blocks.
    Position space SCET looks simpler in this respect.

One-loop anomalous dimension of NLP N-jet operators

General form of the operator

$$\mathcal{O}(x) = \int \prod_{i=1}^{N} \prod_{k_i=1}^{n_i} dt_{ik_i}\, C(\{t_{ik_i}\}) \prod_{i=1}^{N} [i\partial_{i\perp}]^l \prod_{i_k=1}^{n_i} \psi_{ik_i}(x + t_{ik_i} n_{+i})$$

- Loop is either collinear *OR* soft. Mixed only from two loops
- Collinear renormalization only within one direction. Treat each sector separately, then sum over $i = 1, \ldots, N$
- Soft loops still connect exactly two directions. Treat each pair separately, then sum over pairs.
- LP: $J^{(A0)}$ for collinear, $J^{(A0,A0)}$ for soft. Omit labels for other sectors.

NLP – $\mathcal{O}(\lambda)$

Not by itself of interest, relevant as part of $\mathcal{O}(\lambda^2)$ renormalization and is observable is the square of an amplitude (for $\mathcal{O}(\lambda) \times \mathcal{O}(\lambda)$). Relevant operators:

collinear $\quad J^{(A1)}, J^{(B1)} \quad \rightsquigarrow \quad 2 \times 2$ matrix

soft $\quad J^{(A0,A1)}, J^{(A0,B1)}, T(J^{(A0,A0)}, \mathcal{L}^{(1)}) \quad \rightsquigarrow \quad 3 \times 3$ matrix

Soft part

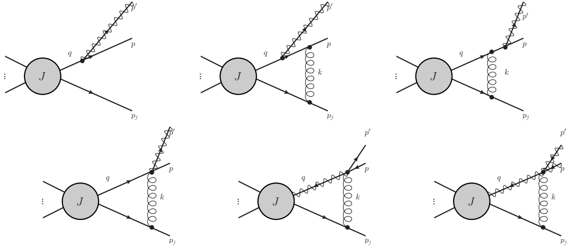
(1) No mixing. Anomalous dimension matrix diagonal.

(2) No time-ordered product contributions.

(3) Soft loops vanish, if gluon is attached to two collinear lines in the same direction.

(4) Non-zero renormalization constants $Z_s^{[(A0,A1),(A0,A1)]}$, $Z_s^{[(A0,B1),(A0,B1)]}$ can be expressed in terms of LP soft anomalous dimension.

(5) Requires consideration of $2 \to 3$ mixing. Soft IR singularities cancel with collinear contributions ($\mathcal{O}(\lambda^0)$) or vanish ($\mathcal{O}(\lambda^1)$).

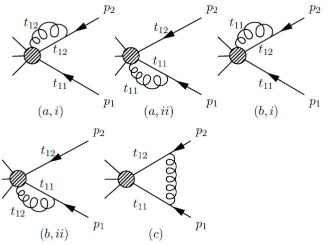


$\mathcal{O}(\lambda)$

Collinear part

(1) No mixing between A1 and B1. A1 can be expressed in terms of LP A0 anomalous dimension.

(2) Cases for B1: $\xi\xi$, $\xi\mathcal{A}$, and $\xi\bar{\xi}$ & $\mathcal{A}\mathcal{A}$
For two quarks in one direction



$$
\begin{aligned}
Z^c_{\xi_\alpha\xi_\beta,\xi_\gamma\xi_\delta}(x,y) &= \delta(x-y)\delta_{\alpha\gamma}\delta_{\beta\delta}\Bigg[J_q(p_1^2)^{-1}J_q(p_2^2)^{-1} \\
&\quad - \frac{\alpha\mathbf{T}_{11}\cdot\mathbf{T}_{12}}{2\pi}\left(\frac{2}{\epsilon^2}+\frac{1}{\epsilon}\ln\left(\frac{\mu^2}{-p_1^2}\right)+\frac{1}{\epsilon}\ln\left(\frac{\mu^2}{-p_2^2}\right)\right)\Bigg] \\
&\quad + \frac{\alpha\mathbf{T}_{11}\cdot\mathbf{T}_{12}}{2\pi\epsilon}\Bigg[\delta^{\alpha\gamma}\delta^{\beta\delta}\Bigg\{\theta(x-y)\left[\frac{1}{x-y}\right]_+ \\
&\quad +\theta(y-x)\left[\frac{1}{y-x}\right]_+ - \theta(x-y)\frac{1-\frac{\bar{x}}{2}}{\bar{y}} - \theta(y-x)\frac{1-\frac{x}{2}}{y}\Bigg\} \\
&\quad + \left(\theta(x-y)\frac{\bar{x}}{\bar{y}}+\theta(y-x)\frac{x}{y}\right)\frac{1}{4}\left(\sigma_\perp^{\nu\mu}\right)_{\alpha\gamma}\left(\sigma_\perp^{\nu\mu}\right)_{\beta\delta}\Bigg]
\end{aligned}
$$

(3) Previous results and checks

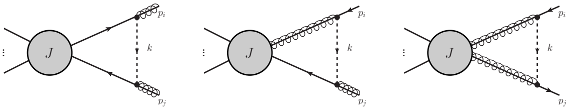
- $\xi\mathcal{A}$ agrees with calculation of $\bar{\xi}\mathcal{A}h_v$ anomalous dimension [Hill, Becher, Lee, Neubert, hep-ph/0404217; MB, Yang hep-ph/0508250] after subtracting soft terms related to h_v field.
- Some results on $\bar{\xi}_i\mathcal{A}_i\xi_j$ in the context of thrust in e^+e^- [Freedman, Goerke, 1408.6240] – different formalism, difficult to compare to.

One-loop anomalous dimension of NLP N-jet operators, $\mathcal{O}(\lambda^2)$

Operators

$$P \;=\; J^{(A0,A2)},\, J^{(A1,A1)},\, J^{(A1,B1)},\, J^{(A0,B2)},\, J^{(A0,C2)},\, J^{(B1,B1)},\\ T(J^{(A0,A0)}, \mathcal{L}^{(1)}, \mathcal{L}^{(1)}),\, T(J^{(A0,A0)}, \mathcal{L}^{(2)}),\, T(J^{(A0,A1)}, \mathcal{L}^{(1)}),\, T(J^{(A0,B1)}, \mathcal{L}^{(1)})$$

- Still generic. Can have several operators of one category.
- No T-product operators for collinear part.
- Non-trivial mixing matrix only for J into J and T-products into J.
- Operator mixing now also occurs in the soft sector through time-ordered products.
- Soft quark exchange could produce $q\bar{q} \to gg$ mixing, but vanishes.



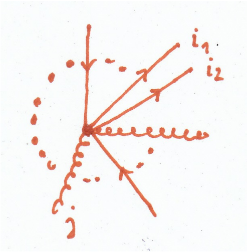
General structure of the result

$$
\begin{aligned}
Z_{PQ}(x,y) &= Z^s_{PQ}(x,y) + Z^c_{PQ}(x,y) \\
&= 1 - \frac{\alpha_s}{4\pi}\Bigg[\delta_{PQ}\delta(x-y)\sum_{i,j}\sum_{l,k}\mathbf{T}_{il}\cdot\mathbf{T}_{jk}\left\{\frac{1}{\epsilon}\ln\left(\frac{-s_{il,jk}}{\mu^2}\right)(1-\delta_{ij}) + \delta_{ij}\left[\frac{1}{\epsilon^2} + \delta_{lk}\frac{c_{jl}}{\epsilon}\right]\right\} \\
&\qquad - \sum_i \delta^{[i]}(x-y)\,\frac{\gamma^{i,\mathrm{coll}}_{PQ}(x,y)}{\epsilon} - \sum_{i,j,i\neq j}\sum_{l,k}\mathbf{T}_{il}\cdot\mathbf{T}_{jk}\,\delta(x-y)\,\frac{\gamma^{ij,\mathrm{soft}}_{PQ}}{\epsilon}\Bigg]_{sym}
\end{aligned}
$$

- Off-shell IR regulator p^2_{ik} cancels upon summing soft+collinear
- Note similarity of $1/\epsilon^2$ and $1/\epsilon\, \ln\dfrac{-s_{il,jk}}{\mu^2}$ to LP. Only total colour charge in every collinear sector matters as should be for terms related to soft.
- Complications hidden in collinear and soft single-pole terms

  Operator mixing, collinear anomalous dimension $\gamma^{i,\mathrm{coll}}_{PQ}(x,y)$ a matrix in the spin, colour and momentum labels within a collinear sector.

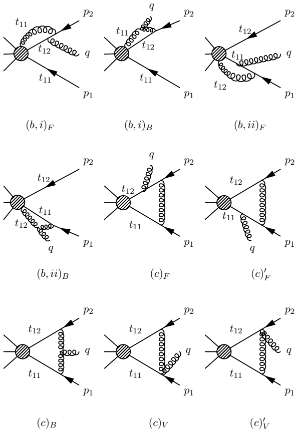
$$
\delta^{(i)}(x_i - y_i) = \prod_{k=1}^{n_i}\delta(x_{ik_i} - y_{ik_i}) \qquad \delta(x-y) = \prod_{i=1}^{N}\delta^{(i)}(x_i - y_i) \qquad \delta^{[i]}(x-y) = \prod_{j=1, j\neq i}^{N}\delta^{(j)}(x_j - y_j)
$$



$$
Z^c_{PQ} \;=\; \begin{array}{c||cc|c}
 & J^{B2}_{\xi\partial\xi} & J^{B2}_{\partial(\xi\xi)} & J^{C2}_{\mathcal{A}\xi\xi} \\
\hline\hline
J^{B2}_{\xi\partial\xi} & \star & \star & \star \\
J^{B2}_{\partial(\xi\xi)} & 0 & g_\perp^{\mu\nu} Z_{\xi\xi,\xi\xi} & 0 \\
\hline
J^{C2}_{\mathcal{A}\xi\xi} & 0 & 0 & \star
\end{array}
$$

- No A-operators in fermion number-two sector.
- Two-particle B2 mixing into three-particle C2.
- Three-particle collinear C2 anomalous dimension.
- In addition $\mathcal{O}(\lambda)$ B1 operator mixing in fermion-number-two sector times $\mathcal{O}(\lambda)$ B1 mixing in any other sector $j \neq i$.

$$Z^{c}_{\xi_s^{\alpha}\partial^{\mu}\xi_t^{\beta},\,\mathcal{A}^{\nu a}\xi_k^{\alpha'}\xi_l^{\beta'}}(x,y_1,y_2) = \frac{\alpha}{8\pi\epsilon}\, n_{+i}\cdot p_i \Big\{ -if^{abc}t^c_{sk}t^b_{tl}K^{\alpha\alpha'\beta\beta'}_{1,\mu\nu}(x,y_1,y_2) + (t^a t^b)_{sk}t^b_{tl}K^{\alpha\alpha'\beta\beta'}_{2,\mu\nu}(x,y_1,y_2) - (t^a t^b)_{tl}t^b_{sk}K^{\beta\beta'\alpha\alpha'}_{2,\mu\nu}(\bar{x},y_1,y_3)\Big\}$$



where the kernels $K^{\alpha\alpha'\beta\beta'}_{1/2,\mu\nu}(x,y_1,y_2)$ are a sum of terms of the form

$$I^{\alpha\alpha'\beta\beta'}_{\mu\nu}(x,y_1,y_2) \equiv \left(-\theta(x-y_1)\theta(\bar{y}_3-x)\,\frac{x^2\bar{y}_1+\bar{x}^2\bar{y}_3-\bar{y}_1\bar{y}_3}{\bar{y}_1 y_2\bar{y}_3} + \theta(y_1-x)\,\frac{x^2}{y_1\bar{y}_3} + \theta(x-\bar{y}_3)\,\frac{\bar{x}^2}{\bar{y}_1 y_3}\right) \times \left\{\frac{x+y_1}{x-y_1}(\gamma_\perp^\nu\gamma_\perp^\rho)^{\alpha\alpha'}(\gamma_\perp^\mu\gamma_\perp^\rho)^{\beta\beta'} + g_\perp^{\mu\nu}(\gamma_\perp^\sigma\gamma_\perp^\rho)^{\alpha\alpha'}(\gamma_\perp^\sigma\gamma_\perp^\rho)^{\beta\beta'} + (\gamma_\perp^\mu\gamma_\perp^\rho)^{\alpha\alpha'}(\gamma_\perp^\nu\gamma_\perp^\rho)^{\beta\beta'}\right\}$$

Three-particle collinear C2 anomalous dimension

$$
\begin{aligned}
Z^c_{\mathcal{A}^\mu \xi^\alpha \xi^\beta, \mathcal{A}^\nu \xi^{\alpha'} \xi^{\beta'}}(x_1, x_2, y_1, y_2) \;=\;& \\
& -(1 + J_g(p_1^2)^{-1} J_q(p_2^2)^{-1} J_q(p_3^2)^{-1})\delta(x_1 - y_1)\delta(x_2 - y_2)\,\delta^{\alpha\alpha'}\,\delta^{\beta\beta'}\,g_\perp^{\mu\nu} \\
& + \frac{1}{1-y_2}\delta(x_2 - y_2)\delta^{\beta\beta'} Z^c_{\mathcal{A}^\mu \xi^\alpha, \mathcal{A}^\nu \xi^{\alpha'}}\left(\frac{x_1}{1-x_2}, \frac{y_1}{1-y_2}\right) \\
& + \frac{1}{1-y_1}\delta(x_1 - y_1) g_\perp^{\mu\nu} Z^c_{\xi^\alpha \xi^\beta, \xi^{\alpha'} \xi^{\beta'}}\left(\frac{x_2}{1-x_1}, \frac{y_2}{1-y_1}\right) \\
& + \frac{1}{1-y_3}\delta(x_3 - y_3)\delta^{\alpha\alpha'} Z^c_{\mathcal{A}^\mu \xi^\beta, \mathcal{A}^\nu \xi^{\beta'}}\left(\frac{x_1}{1-x_3}, \frac{y_1}{1-y_3}\right)
\end{aligned}
$$

- At one-loop only two of the three lines can be connected.
- Anomalous dimension is a sum of $\mathcal{O}(\lambda)$ B1 anomalous dimensions for all pairs of lines with rescaled momentum fractions, since now $x_1 + x_2 + x_3 = 1$, $y_1 + y_2 + y_3 = 1$. First line removes graphs counted twice in the sum over pairs.

Summary & outlook

I Categorization of general basis of sub-leading power N-jet operators

II Computed (so far: most of) the one-loop anomalous dimension matrices

III Observables at NLP

$$d\sigma = \sum_{i_1,i_2,j,k} \underbrace{C^{(i_1)} C^{(i_2)}}_{\text{Log summation with ADM of NLP } N\text{-jet operators.}} \otimes \underbrace{J^{(j)} \otimes S^{(k)}}_{\text{Subleading jet and soft functions from time-ordered products.}} \qquad i_1 + i_2 + j + k \leq 2$$

IV Solving the RGE is complicated.
For a given process project on a spin and colour basis.
Even within a given collinear sector multi-variable integro-differential equations.
Numerical solution should be possible.