Automated Matrix Element Generation

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Contents

- A computer program implementing the function

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- first robust and usable examples in the early 1990s: CompHEP, FeynArts, Grace, MadGraph, ...

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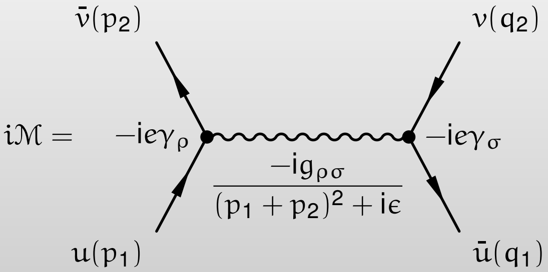

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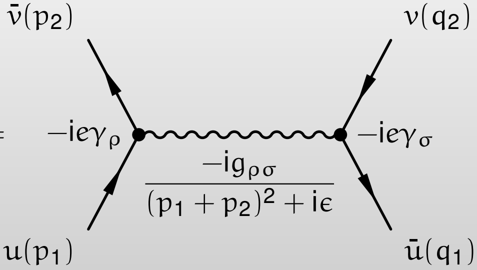

- analytical expression

$$\begin{aligned} i\mathcal{M} &= \bar{v}(p_2)(-ie\gamma^\rho)u(p_1)\frac{-ig_{\rho\sigma}}{(p_1+p_2)^2+i\epsilon}\bar{u}(q_1)(-ie\gamma^\sigma)v(q_2) \\ &= ie^2\frac{1}{s}\left[\bar{v}(p_2)\gamma_\rho u(p_1)\right]\left[\bar{u}(q_1)\gamma^\rho v(q_2)\right] \end{aligned}$$

- e. g. command line for O'Mega to compute $e^-e^+ \to \mu^-\mu^+$ in QED

```
$ f90_QED -scatter "e- e+ -> m- m+"
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  `$ f90_QED -scatter "e- e+ -> m- m+"`
- resulting Fortran95 code

```
pure function eleposmuoamu (k, s) result (amp)
  real(kind=omega_prec), dimension(0:,:), intent(in) :: k
  integer, dimension(:), intent(in) :: s
  complex(kind=omega_prec) :: amp
  type(momentum) :: p1, p2, p3, p4
  type(spinor) :: muo_4, ele_1
  type(conjspinor) :: amu_3, pos_2
  type(vector) :: gam_12
  type(momentum) :: p12
  p1 = - k(:,1) ! incoming e-
  p2 = - k(:,2) ! incoming e+
  p3 =   k(:,3) ! outgoing m-
  p4 =   k(:,4) ! outgoing m+
  ele_1 = u (mass(11), - p1, s(1))                    ! u_{s_1}(k_1)
  pos_2 = vbar (mass(11), - p2, s(2))                 ! \bar{v}_{s_2}(k_2)
  amu_3 = ubar (mass(13), p3, s(3))                   ! \bar{u}_{s_3}(k_3)
  muo_4 = v (mass(13), p4, s(4))                      ! v_{s_4}(k_4)
  p12 = p1 + p2
  gam_12 = pr_feynman(p12, + v_ff(qlep,pos_2,ele_1)) ! (1/s) e\bar{v}(k_2)\gamma_\mu u(k_1)
  amp = 0
  amp = amp + gam_12*( + v_ff(qlep,amu_3,muo_4))      ! (1/s) e\bar{v}(k_2)\gamma_\mu u(k_1) e\bar{u}(k_3)\gamma^\mu v(k_4)
  amp = - amp ! 2 vertices, 1 propagators
end function eleposmuoamu
```

(some additional interface routines suppressed)

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    ∴ helicity amplitudes win (eventually)!

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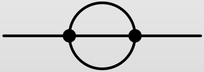

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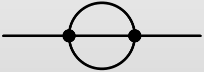

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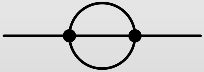
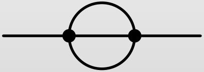
 $\rightarrow \frac{1}{3!} \cdot$ 

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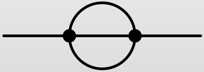
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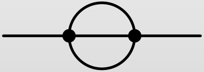

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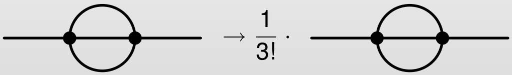

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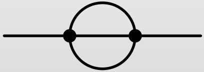

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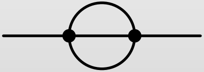

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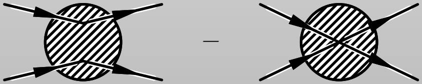

     - if Majorana fermions are present (MSSM!), use the clever algorithm of [Denner et al. Phys. Lett. **B291**, 278 (1992)]

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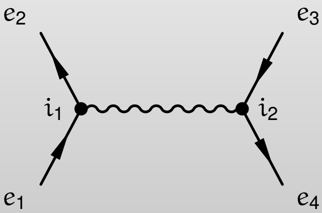

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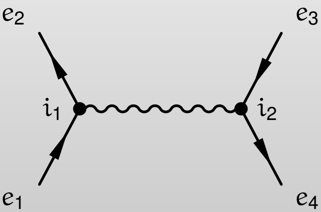

  1. sets of external and internal vertices and edges

$$\left(\begin{Bmatrix} (e_1, e^-, p_1) \\ (e_2, e^+, p_2) \\ (e_3, \mu^+, q_2) \\ (e_4, \mu^-, q_1) \end{Bmatrix}, \begin{Bmatrix} (i_1, -ie\gamma_\mu) \\ (i_2, -ie\gamma_\mu) \end{Bmatrix}, \begin{Bmatrix} (e_1, i_1, \{e^-\}) \\ (e_2, i_1, \{e^+\}) \\ (e_3, i_2, \{\mu^-\}) \\ (e_4, i_2, \{\mu^+\}) \\ (i_1, i_2, \{\gamma\}) \end{Bmatrix} \right)$$

- choices (cont'd)
  2. incidence matrix

$$\left\{\begin{array}{c}(e_1, i_1, \{e^-\}) \\ (e_2, i_1, \{e^+\}) \\ (e_3, i_2, \{\mu^-\}) \\ (e_4, i_2, \{\mu^+\}) \\ (i_1, i_2, \{\gamma\})\end{array}\right\} \sim \begin{array}{c|cccccc} & e_1 & e_2 & e_3 & e_4 & i_1 & i_2 \\ \hline e_1 & \emptyset & \emptyset & \emptyset & \emptyset & \{e^-\} & \emptyset \\ e_2 & \emptyset & \emptyset & \emptyset & \emptyset & \{e^+\} & \emptyset \\ e_3 & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \{\mu^-\} \\ e_4 & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \{\mu^+\} \\ i_1 & \{e^+\} & \{e^-\} & \emptyset & \emptyset & \emptyset & \{\gamma\} \\ i_2 & \emptyset & \emptyset & \{\mu^+\} & \{\mu^-\} & \{\gamma\} & \emptyset \end{array}$$

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☹ different egde set and incidence matrix for same graph

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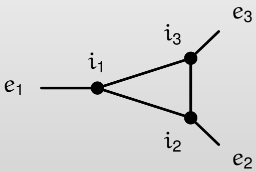

$$\sim \left\{ \begin{matrix} (e_1, i_1) \\ (e_2, i_2) \\ (e_3, i_3) \\ (i_1, i_2) \\ (i_2, i_3) \\ (i_3, i_1) \end{matrix} \right\} \sim \begin{array}{c|cccccc} & e_1 & e_2 & e_3 & i_1 & i_2 & i_3 \\ \hline e_1 & & & & 1 & & \\ e_2 & & & & & 1 & \\ e_3 & & & & & & 1 \\ i_1 & 1 & & & & 1 & \\ i_2 & & 1 & & & & 1 \\ i_3 & & & 1 & 1 & & \end{array}$$

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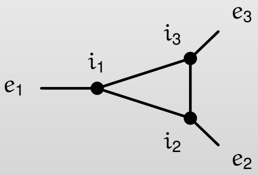

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	e_1	e_2	e_3	i_1	i_2	i_3
e_1				1		
e_2					1	
e_3						1
i_1	1				1	
i_2		1				1
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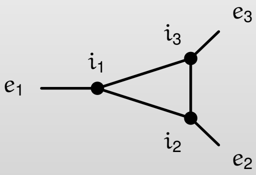

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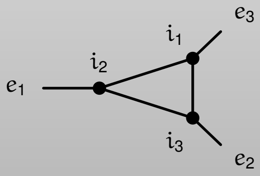

$$\sim \begin{Bmatrix} (e_1, i_2) \\ (e_2, i_3) \\ (e_3, i_1) \\ (i_1, i_2) \\ (i_2, i_3) \\ (i_3, i_1) \end{Bmatrix} \sim \begin{array}{c|cccccc} & e_1 & e_2 & e_3 & i_1 & i_2 & i_3 \\ \hline e_1 & & & & & 1 & \\ e_2 & & & & & & 1 \\ e_3 & & & & 1 & & \\ i_1 & & & 1 & & 1 & \\ i_2 & 1 & & & & & 1 \\ i_3 & & 1 & & 1 & & \end{array}$$

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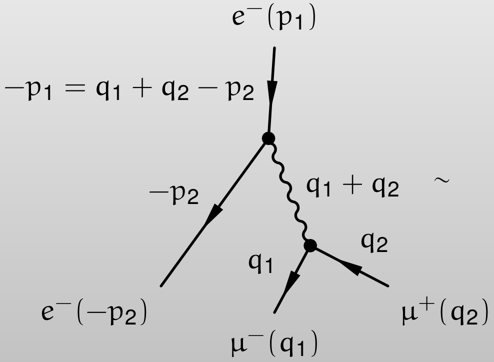

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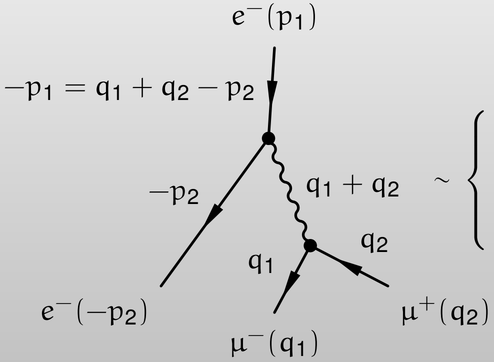

$$\sim \left\{ \begin{array}{l} \gamma(q_1+q_2) \mapsto \\ \qquad (\mu^-(q_1), \mu^+(q_2)) \end{array} \right\}$$

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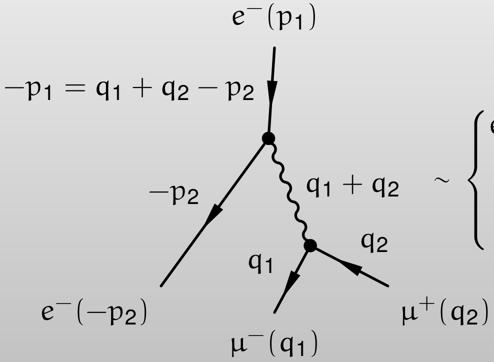

$$\sim \left\{ \begin{array}{l} e^-(q_1+q_2-p_2) \mapsto \\ \qquad (e^-(-p_2), \gamma(q_1+q_2)) \\ \gamma(q_1+q_2) \mapsto \\ \qquad (\mu^-(q_1), \mu^+(q_2)) \end{array} \right\}$$

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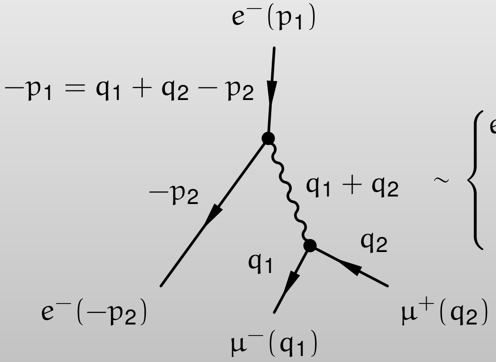

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- a harmless 2-fold ambiguity from overall momentum conservation can be avoided, if we don't use the momentum at the root, i. e. p_1

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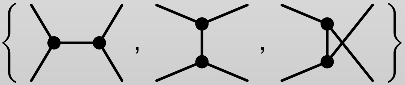

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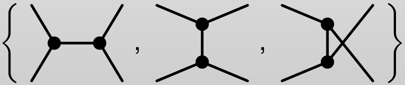

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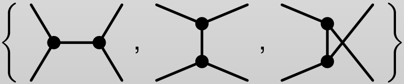

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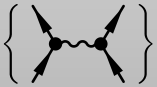

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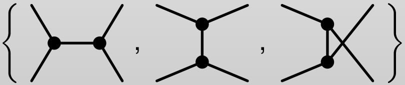

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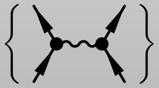

- avoids an enormous number of fruitless attempts

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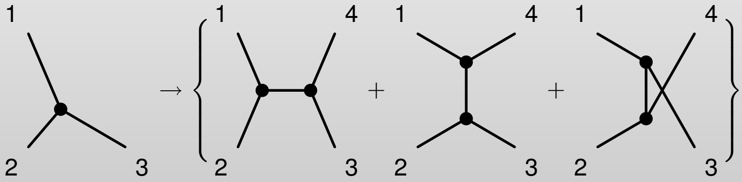

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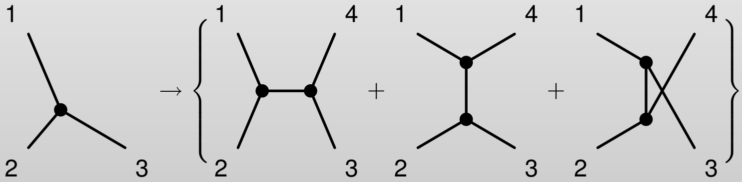
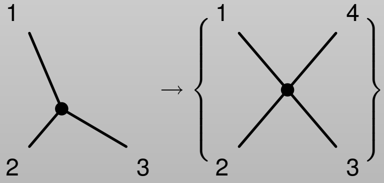


     - in very vertex of degree less than the maximum, e. g.



- corollary: there are

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- proof:

$$\begin{aligned} F(3) &= 1 \\ F(n) &= \Big(\underbrace{(n-1)}_{\text{external lines}} + \underbrace{(n-4)}_{\text{internal lines}}\Big)\cdot F(n-1) \\ &= (2n-5)\cdot F(n-1) = (2n-5)\cdot(2n-7)\cdot F(n-2) = \ldots \end{aligned}$$

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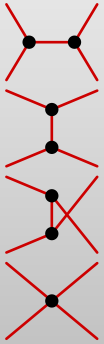

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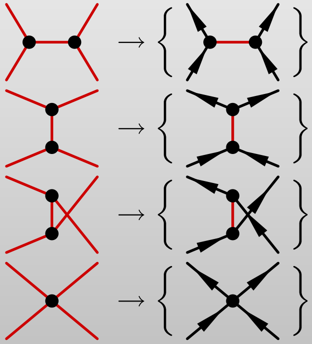

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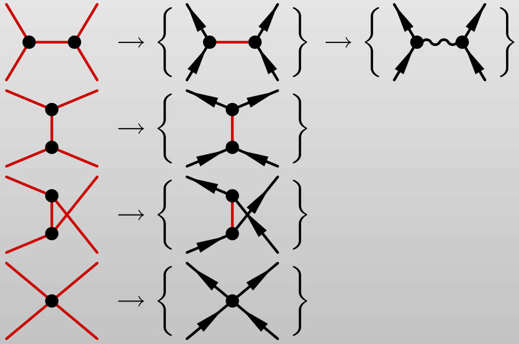

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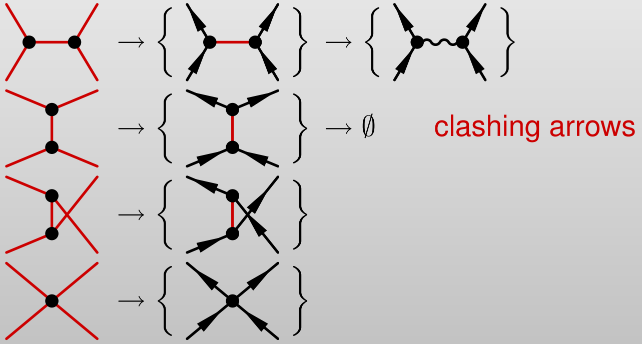

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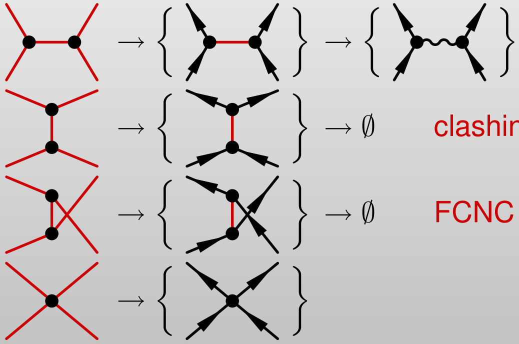

clashing arrows

FCNC

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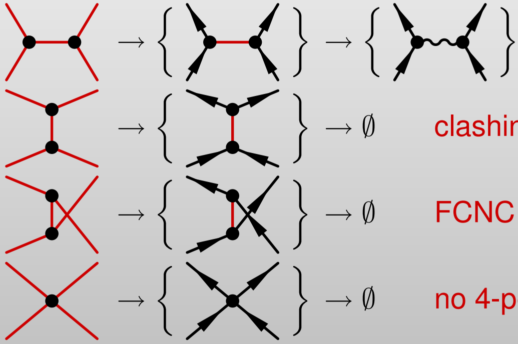

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no 4-point vertex

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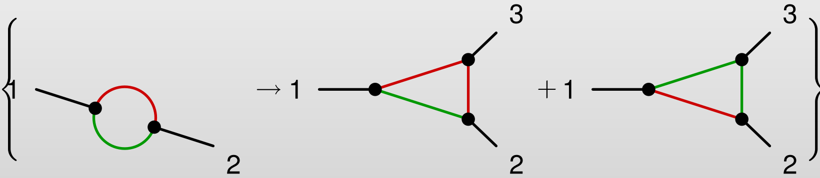

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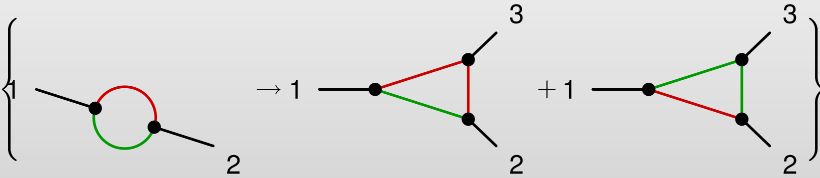

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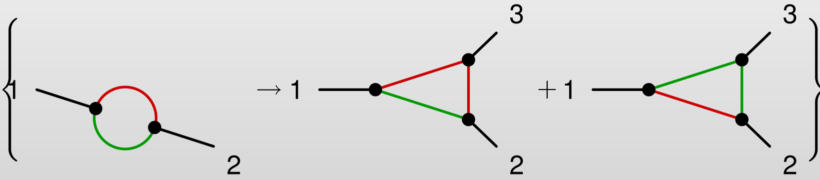

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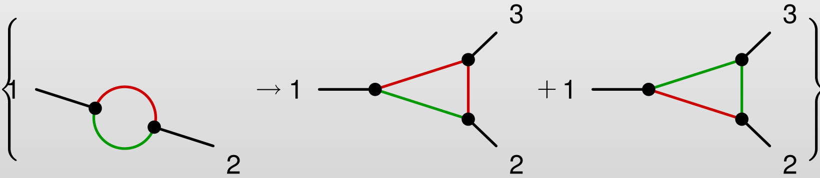

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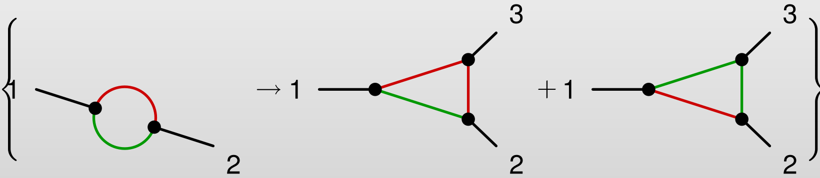

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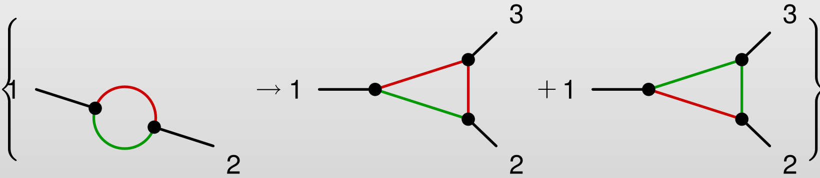

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- most efficient, but badly documented, public tool for multi loop diagrams QGRAF [Nogueira, 1991]

The number of tree Feynman diagrams w/ n legs grows like a factorial, e. g. in ϕ^3-theory: $F(n) = (2n-5)!! = (2n-5)\cdot(2n-7)\cdot\ldots\cdot3\cdot1$

n		
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5		
6		
7		
8		
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10		
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12		
13		
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8	10 395	
9	135 135	
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5	15	10
6	105	25
7	945	56
8	10 395	119
9	135 135	246
10	2 027 025	501
11	34 459 425	1 012
12	654 729 075	2 035
13	13 749 310 575	4 082
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n	$F(n)$	$P(n)$
4	3	3
5	15	10
6	105	25
7	945	56
8	10 395	119
9	135 135	246
10	2 027 025	501
11	34 459 425	1 012
12	654 729 075	2 035
13	13 749 310 575	4 082
14	316 234 143 225	8 177
15	7 905 853 580 625	16 368

- ☹ computational costs grow beyond all reasonable limits
- ☹ gauge cancellations cause loss of precision

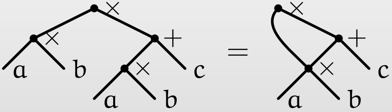
Number of possible momenta in tree diagrams grows only exponentially

$$P(n) = \frac{2^n - 2}{2} - n = 2^{n-1} - n - 1$$

∴ Feynman diagrams redundant for many external particles!

∴ Replace the forest of tree diagrams by the Directed Acyclical Graph (DAG) of the algebraic expression.

$$ab(ab+c) =$$



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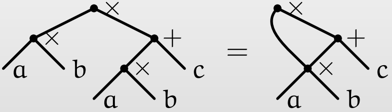

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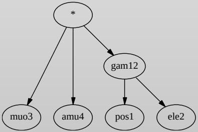

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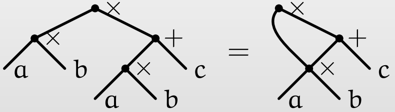

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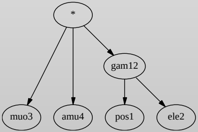

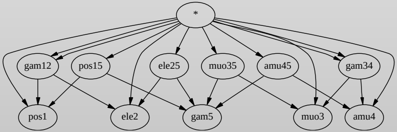

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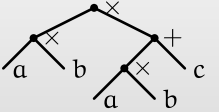
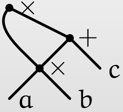
$ab(ab+c) =$  $=$ 

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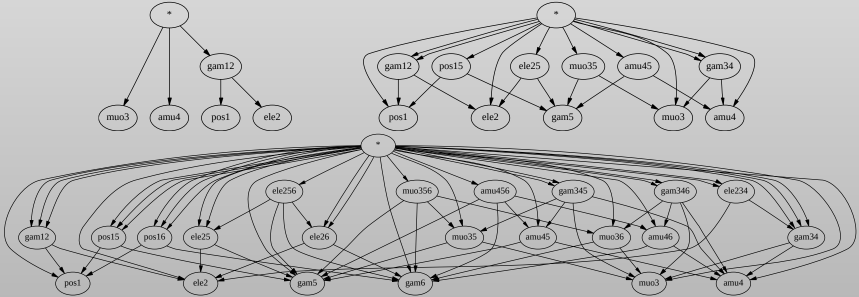

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- O'Mega [TO et al.]:
  - systematic elimination of all redundancies
  - symbolic, generation of compilable code

One particle off-shell wave functions (1POWs) are obtained from by applying the LSZ reduction formula to all but one line:

$$W(x; p_1, \ldots, p_n; q_1, \ldots, q_m) = \\ \langle \phi(q_1), \ldots, \phi(q_m); \text{out} | \Phi(x) | \phi(p_1), \ldots, \phi(p_n); \text{in} \rangle \; .$$

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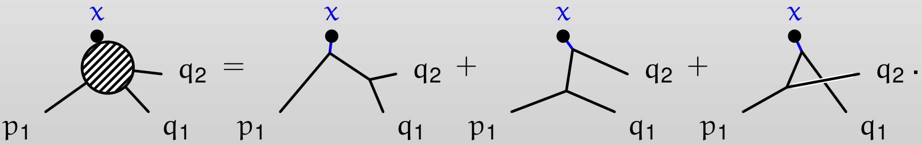
E. g. $\langle \phi(q_1), \phi(q_2); \text{out} | \Phi(x) | \phi(p_1); \text{in} \rangle$ in ϕ^3-theory at tree level



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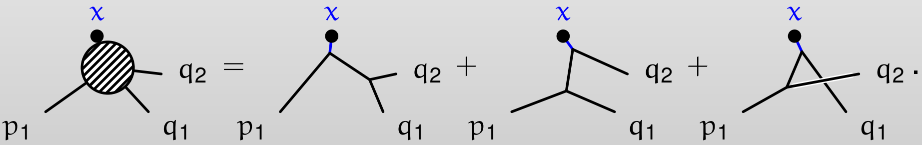

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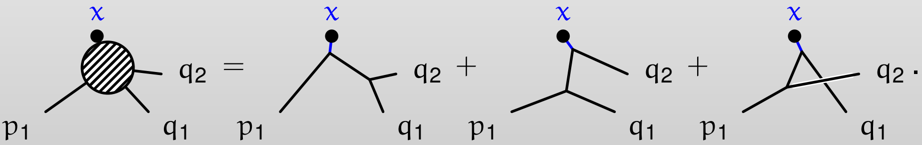

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There exists a well defined set of keystones K that allow to express the sum of Feynman diagrams through 1POWs:

$$T = \sum_{i=1}^{F(n)} D_i = \sum_{k,l,m=1}^{P(n)} K^3_{f_k f_l f_m}(p_k, p_l, p_m) W_{f_k}(p_k) W_{f_l}(p_l) W_{f_m}(p_m)$$

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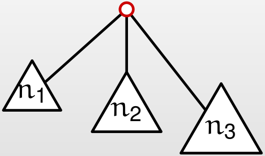

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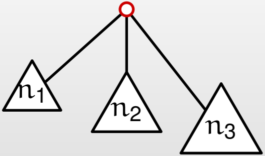

n	$\sum$	$\sum$
4	4	$1 \cdot (1,1,1,1) + 3 \cdot (1,1,2)$
5	26	$1 \cdot (1,1,1,1,1) + 10 \cdot (1,1,1,2) + 15 \cdot (1,2,2)$
6	236	$1 \cdot (1,1,1,1,1,1) + 15 \cdot (1,1,1,1,2) + 40 \cdot (1,1,1,3)$
		$+ 45 \cdot (1,1,2,2) + 120 \cdot (1,2,3) + 15 \cdot (2,2,2)$

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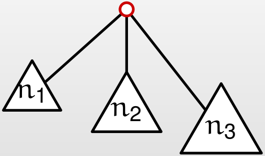

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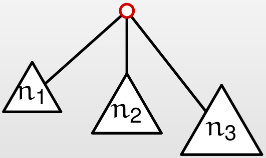

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Subtlety: some partitions for an even number of external lines are degenerate, e. g. $(1,1,1,3)$ and $(1,2,3)$ contain the same diagram

and representatives must be chosen consistently.

Non trivial cross check from self-consistency of counting
$F(d_{\text{max}}, n)$ = # of Feynman diagrams with n external legs in unflavored

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2 - \sum_{d=3}^{d_{\text{max}}} \frac{\lambda_d}{d!}\phi^d$$

theory. In a partition $N_{d,n} = \{n_1, n_2, \ldots, n_d\}$ with
$n = n_1 + n_2 + \cdots + n_d$, there are

$$\tilde{F}(d_{\text{max}}, N_{d,n}) = \frac{1}{(1+\delta_{n_d, n_1+n_2+\cdots+n_{d-1}})} \frac{n!}{|\mathcal{S}(N_{d,n})|} \prod_{i=1}^{d} \frac{F(d_{\text{max}}, n_i + 1)}{n_i!}$$

Feynman diagrams ($|\mathcal{S}(N)|$ the size of the symmetric group of N).

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☺

$$F(d_{\max}, n) = \sum_{d=3}^{d_{\max}} \sum_{\substack{N=\{n_1,n_2,\ldots,n_d\} \\ n_1+n_2+\cdots+n_d=n \\ 1\leqslant n_1\leqslant n_2\leqslant\cdots\leqslant n_d\leqslant\lfloor n/2\rfloor}} \tilde{F}(d_{\max}, N)$$

can be be checked numerically up to $n = \mathcal{O}(100)$.

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☺ the resulting expression contains no more redundancies!

Even for vector particles, the 1POWs are 'almost' physical objects and satisfy simple Ward Identities in unbroken gauge theories

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Amplitudes can be continued off-shell:

- Slavnov-Taylor Identities can be checked numerically by adding operator insertions implementing BRS transformations.

Slightly simplified `Model.T` signature that all models must implement:

```
module type Model.T =
  sig
    type flavor (* all quantum numbers *)
    val flavor_symbol : flavor -> string
    val conjugate : flavor -> flavor (* antiparticles *)
    val lorentz : flavor -> Coupling.lorentz (* spin *)
    val fermion : flavor -> int (* fermion, boson, antifermion *)
    val width : flavor -> Coupling.width (* scheme, not value! *)
    type gauge (* parametrized gauges *)
    val gauge_symbol : gauge -> string
    val propagator : flavor -> gauge Coupling.propagator
    type constant (* coupling constants *)
    val constant_symbol : constant -> string
    val fuse2 : flavor -> flavor ->
     (flavor * constant Coupling.t) list (* A_μ(p_12) ← gψ̄(p_1)γ_μψ(p_2) *)
    val fuse3 : flavor -> flavor -> flavor ->
     (flavor * constant Coupling.t) list (* φ(p_123) ← gφ(p_1)φ(p_2)φ(p_3) *)
    val fuse : flavor list -> (flavor * constant Coupling.t) list
  end
```

$e^+e^- \to \tilde\chi_1^+\tilde\chi_1^-$:

```
$ f90_MSSM -scatter "e+ e- -> ch1+ ch1-"
```

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```
pure function l1bl1cp1cm1 (k, s) result (amp)
  real(kind=omega_prec), dimension(0:,:), intent(in) :: k
  integer, dimension(:), intent(in) :: s
  complex(kind=omega_prec) :: amp
  type(momentum) :: p1, p2, p3, p4
  type(bispinor) :: cp1_4, l1_2
  type(bispinor) :: cm1_3, l1b_1
  complex(kind=omega_prec) :: snc1_13
  type(vector) :: a_12, z_12
  type(momentum) :: p12, p13
  p1 = - k(:,1) ! incoming e+
  p2 = - k(:,2) ! incoming e-
  p3 =   k(:,3) ! outgoing ch1+
  p4 =   k(:,4) ! outgoing ch1-
  l1b_1 = u (mass(11), - p1, s(1))
  l1_2 = u (mass(11), - p2, s(2))
  cm1_3 = v (mass(69), p3, s(3))
  cp1_4 = v (mass(69), p4, s(4))
```

```
    p12 = p1 + p2
    a_12 = pr_feynman(p12, + v_ff(qlep,l1b_1,l1_2))
    z_12 = pr_unitarity(p12,mass(23),wd_tl(p12,width(23)), &
       + va_ff(gnclep(1),gnclep(2),l1b_1,l1_2))
    p13 = p1 + p3
    snc1_13 = pr_phi(p13,mass(54),wd_tl(p13,width(54)), &
       + sr_ff(g_yuk_ch1_sn1_1_c,l1b_1,cm1_3))
    amp = 0
    amp = amp + snc1_13*( - sl_ff(g_yuk_ch1_sn1_1,l1_2,cp1_4))
    amp = amp + z_12*( + va_ff(-gczc_1_1(1),-gczc_1_1(2),cm1_3,cp1_4))
    amp = amp + a_12*( + v_ff(qchar,cm1_3,cp1_4))
    amp = - amp ! 2 vertices, 1 propagators
  end function l1bl1cp1cm1
```

9 fusions, 3 propagators, 3 diagrams

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    z_12 = pr_unitarity(p12,mass(23),wd_tl(p12,width(23)), &
       + va_ff(gnclep(1),gnclep(2),l1b_1,l1_2))
    p13 = p1 + p3
    snc1_13 = pr_phi(p13,mass(54),wd_tl(p13,width(54)), &
       + sr_ff(g_yuk_ch1_sn1_1_c,l1b_1,cm1_3))
    amp = 0
    amp = amp + snc1_13*( - sl_ff(g_yuk_ch1_sn1_1,l1_2,cp1_4))
    amp = amp + z_12*( + va_ff(-gczc_1_1(1),-gczc_1_1(2),cm1_3,cp1_4))
    amp = amp + a_12*( + v_ff(qchar,cm1_3,cp1_4))
    amp = - amp ! 2 vertices, 1 propagators
  end function l1bl1cp1cm1
```

9 fusions, 3 propagators, 3 diagrams

☺ readable code, can edited for exotic models or NLO vertex functions

Adding a photon $e^+e^- \to \tilde{\chi}_1^+\tilde{\chi}_1^-\gamma$:

```
$ f90_MSSM -scatter "e+ e- -> ch1+ ch1- A"
```

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```
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```

```
pure function l1bl1cp1cm1a (k, s) result (amp)
  real(kind=omega_prec), dimension(0:,:), intent(in) :: k
  integer, dimension(:), intent(in) :: s
  complex(kind=omega_prec) :: amp
  type(momentum) :: p1, p2, p3, p4, p5
  type(bispinor) :: cp1_4, l1_2
  type(bispinor) :: cm1_3, l1b_1
  type(vector) :: a_5
  complex(kind=omega_prec) :: sn1_24, snc1_13
  type(bispinor) :: cp1_45, l1_25
  type(bispinor) :: cm1_35, l1b_15
  type(vector) :: a_34, a_12, z_34, z_12
  type(momentum) :: p12, p13, p15, p24, p25, p34, p35, p45
  p1 = - k(:,1) ! incoming e+
  p2 = - k(:,2) ! incoming e-
  p3 =   k(:,3) ! outgoing ch1+
  p4 =   k(:,4) ! outgoing ch1-
  p5 =   k(:,5) ! outgoing A
  l1b_1 = u (mass(11), - p1, s(1))
  l1_2 = u (mass(11), - p2, s(2))
  cm1_3 = v (mass(69), p3, s(3))
  cp1_4 = v (mass(69), p4, s(4))
  a_5 = conjg (eps (mass(22), p5, s(5)))
  p12 = p1 + p2
  a_12 = pr_feynman(p12, + v_ff(qlep,l1b_1,l1_2))
  z_12 = pr_unitarity(p12,mass(23),wd_tl(p12,width(23)), &
     + va_ff(gnclep(1),gnclep(2),l1b_1,l1_2))
```

```
    p13 = p1 + p3
    snc1_13 = pr_phi(p13,mass(54),wd_tl(p13,width(54)), &
       + sr_ff(g_yuk_ch1_sn1_1_c,l1b_1,cm1_3))
    p24 = p2 + p4
    sn1_24 = pr_phi(p24,mass(54),wd_tl(p24,width(54)), &
       + sl_ff(g_yuk_ch1_sn1_1,l1_2,cp1_4))
    p34 = p3 + p4
    a_34 = pr_feynman(p34, + v_ff(qchar,cm1_3,cp1_4))
    z_34 = pr_unitarity(p34,mass(23),wd_tl(p34,width(23)), &
       + va_ff(-gczc_1_1(1),-gczc_1_1(2),cm1_3,cp1_4))
    p15 = p1 + p5
    l1b_15 = pr_psi(p15,mass(11),wd_tl(p15,width(11)), + f_vf(-qlep,a_5,l1b_1))
    p25 = p2 + p5
    l1_25 = pr_psi(p25,mass(11),wd_tl(p25,width(11)), + f_vf(qlep,a_5,l1_2))
    p35 = p3 + p5
    cm1_35 = pr_psi(p35,mass(69),wd_tl(p35,width(69)), &
       + f_vf(-qchar,a_5,cm1_3))
    p45 = p4 + p5
    cp1_45 = pr_psi(p45,mass(69),wd_tl(p45,width(69)), + f_vf(qchar,a_5,cp1_4))
    amp = 0
    amp = amp + sn1_24*( + sr_ff(g_yuk_ch1_sn1_1_c,cm1_35,l1b_1))
    amp = amp + snc1_13*( - sl_ff(g_yuk_ch1_sn1_1,l1_25,cp1_4) &
       + sl_ff(g_yuk_ch1_sn1_1,cp1_45,l1_2))
    amp = amp + l1_25*( - f_vf(-qlep,a_34,l1b_1) &
       - f_vaf(-(gnclep(1)),gnclep(2),z_34,l1b_1))
    amp = amp + l1b_15*( - f_srf(g_yuk_ch1_sn1_1_c,sn1_24,cm1_3) &
       + f_vf(qlep,a_34,l1_2) + f_vaf(gnclep(1),gnclep(2),z_34,l1_2))
    amp = amp + z_12*( - va_ff(-(-gczc_1_1(1)),-gczc_1_1(2),cp1_45,cm1_3) &
       + va_ff(-gczc_1_1(1),-gczc_1_1(2),cm1_35,cp1_4))
    amp = amp + a_12*( - v_ff(-qchar,cp1_45,cm1_3) + v_ff(qchar,cm1_35,cp1_4))
  end function l1bl1cp1cm1a
```

28 fusions, 10 propagators, 12 diagrams

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  - NLO matching (anything automated public???)