MC solution of DGLAP evolution equation

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Physics and Cookies

Equations we want to solve

DGLAP evolution equation for gluon with cut of on z already introduced:

$$t\frac{\partial\widetilde{g}(x,t)}{\partial t} = \frac{\alpha_s}{2\pi}\int_x^{z_{max}}\frac{\mathrm{d}z}{z}\left(z\widehat{P}_{gq}(z)\widetilde{q}\left(\frac{x}{z},t\right) + z\widehat{P}_{gg}(z)\widetilde{g}\left(\frac{x}{z},t\right)\right) - \frac{\alpha_s}{2\pi}\widetilde{g}(x,t)\int_0^{z_{max}} dzz\left(2n_f\widehat{P}_{qg}(z) + \widehat{P}_{gg}(z)\right). \tag{1}$$

and for quark:

$$t\frac{\partial\widetilde{q}(x,t)}{\partial t} = \frac{\alpha_s}{2\pi}\int_x^{z_{max}}\frac{\mathrm{d}z}{z}\left[z\widehat{P}_{qq}(z)\widetilde{q}\left(\frac{x}{z},t\right) + z\widehat{P}_{qg}(z)\widetilde{g}\left(\frac{x}{z},t\right)\right] - \frac{\alpha_s}{2\pi}\widetilde{q}(x,t)\int_0^{z_{max}} dzz(\widehat{P}_{qq} + \widehat{P}_{gq}(z)) \tag{2}$$

Here we will concentrate only on gluon case but quark is analogous.

Equations we want to solve

DGLAP evolution equation for gluon with cut of on z already introduced:

$$t\frac{\partial \widetilde{g}(x,t)}{\partial t} = \frac{\alpha_s}{2\pi}\int_x^{zmax}\frac{\mathrm{d}z}{z}\left(z\widehat{P}_{gq}(z)\widetilde{q}\left(\frac{x}{z},t\right)+z\widehat{P}_{gg}(z)\widetilde{g}\left(\frac{x}{z},t\right)\right) - \frac{\alpha_s}{2\pi}\widetilde{g}(x,t)\int_0^{zmax} dzz\left(2n_f\widehat{P}_{qg}(z)+\widehat{P}_{gg}(z)\right). \quad (1)$$

and for quark:

$$t\frac{\partial \widetilde{q}(x,t)}{\partial t} = \frac{\alpha_s}{2\pi}\int_x^{zmax}\frac{\mathrm{d}z}{z}\left[z\widehat{P}_{qq}(z)\widetilde{q}\left(\frac{x}{z},t\right)+z\widehat{P}_{qg}(z)\widetilde{g}\left(\frac{x}{z},t\right)\right] - \frac{\alpha_s}{2\pi}\widetilde{q}(x,t)\int_0^{zmax} dzz(\widehat{P}_{qq}+\widehat{P}_{gq}(z)) \quad (2)$$

Here we will concentrate only on gluon case but quark is analogous.
With Sudakov form factor defined

$$\Delta_g(t) = \exp\left(-\frac{\alpha_s}{2\pi}\int_{t_0}^{t}\frac{\mathrm{d}t'}{t'}\int_0^{zmax}\mathrm{d}z\, z\sum_j \widehat{P}(z)_{jg}\right) \quad (3)$$

we can rewrite this equation

$$t\frac{\partial \widetilde{g}(x,t)}{\partial t} = \frac{\alpha_s}{2\pi}\int_x^{zmax}\frac{\mathrm{d}z}{z}\left(z\widehat{P}_{gq}(z)\widetilde{q}\left(\frac{x}{z},t\right)+z\widehat{P}_{gg}(z)\widetilde{g}\left(\frac{x}{z},t\right)\right) + \widetilde{g}(x,t)\frac{t}{\Delta_g(t)}\frac{\partial \Delta_g(t)}{\partial t}. \quad (4)$$

and solved it:

$$\widetilde{g}(x,t) = \widetilde{g}(x,t_0)\Delta_g(t) + \frac{\alpha_s}{2\pi}\int_{t_0}^{t}\frac{\mathrm{d}t_1}{t_1}\frac{\Delta_g(t)}{\Delta_g(t_1)}\int_x^{zmax}\frac{\mathrm{d}z}{z}\left(z\widehat{P}_{gq}(z)\widetilde{q}\left(\frac{x}{z},t_1\right)+z\widehat{P}_{gg}(z)\widetilde{g}\left(\frac{x}{z},t_1\right)\right)+\dots \quad (5)$$

Backward evolution

Let's explain the equation: Zero branchings

$$\tilde{g}(x,t) = \underbrace{\tilde{g}(x,t_0)\Delta_g(t)}_{\text{see Fig.(1)}}$$

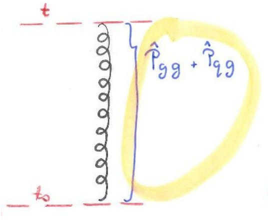

Figure 1 : No splitting between t_0 and t

Backward evolution

Let's explain the equation:
Zero and one branching

$$\tilde{g}(x,t) = \underbrace{\tilde{g}(x,t_0)\Delta_g(t)}_{\text{see Fig.(1)}} + \frac{\alpha_s}{2\pi}\int_{t_0}^{t}\frac{dt_1}{t_1} \underbrace{\frac{\Delta_g(t)}{\Delta_g(t_1)}}_{\text{no splitt. between } t_1 \text{ and } t} \int_x^{z_{max}}\frac{dz}{z} \underbrace{\left(z\hat{P}_{gq}(z)\tilde{q}\left(\frac{x}{z},t_1\right) + z\hat{P}_{gg}(z)\tilde{g}\left(\frac{x}{z},t_1\right)\right)}_{\text{splitting into a gluon at } t_1\text{, see Fig.(2)}} +$$

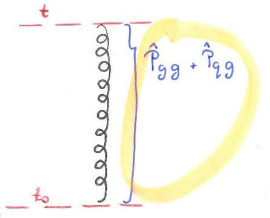

Figure 1: No splitting between t_0 and t

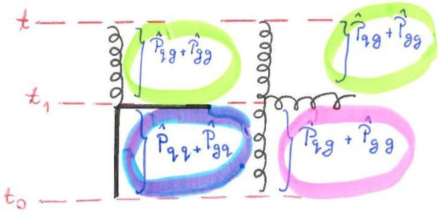

Figure 2 : Gluon appeared scale t_1. It could come from quark or gluon which appeared at t_0

Backward evolution

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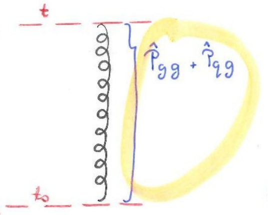

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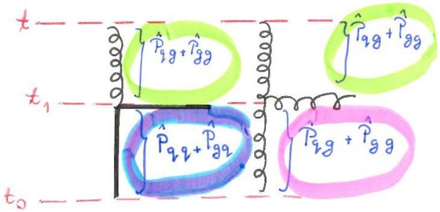

Figure 2: Gluon appeared scale t_1. It could come from quark or gluon which appeared at t_0

so we can rewrite this equation:

$$\tilde{g}(x,t) = \tilde{g}(x,t_0)\Delta_g(t) + \frac{\alpha_s}{2\pi}\int_{t_0}^{t}\frac{dt_1}{t_1}\frac{\Delta_g(t)}{\Delta_g(t_1)}\int_x^{z_{max}}\frac{dz}{z}\left(z\hat{P}_{gq}(z)\tilde{q}\left(\frac{x}{z},t_0\right)\Delta_q(t_1,t_0) + z\hat{P}_{gg}(z)\tilde{g}\left(\frac{x}{z},t_0\right)\Delta_g(t_1,t_0)\right)$$

Backward evolution

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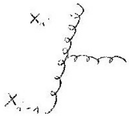
One technical issue:

1. where does the $\frac{\mathrm{d}z}{z}$ come from?



$$z = \frac{x_i}{x_{i-1}}$$

$$\tilde{g}(x,t) = \tilde{g}(x,t_0)\Delta_g(t) + \frac{\alpha_s}{2\pi}\int_{t_0}^{t}\frac{\mathrm{d}t_1}{t_1}\frac{\Delta_g(t)}{\Delta_g(t_1)}$$
$$\int \mathrm{d}x_{i-1}\int_x^{z_{max}}\mathrm{d}z\left(z\hat{P}_{gq}(z)\tilde{q}\left(x_{i-1},t_0\right)\Delta_q(t_1,t_0) + z\hat{P}_{gg}(z)\tilde{g}\left(x_{i-1},t_0\right)\Delta_g(t_1,t_0)\right)\delta(zx_{i-1}-x)$$

MC solution - forward evolution

Goal: to solve DGLAP with MC method

$$\widetilde{g}(x,t) = \widetilde{g}(x,t_0)\Delta_g(t) + \frac{\alpha_s}{2\pi}\int_{t_0}^{t}\frac{\mathrm{d}t_1}{t_1}\frac{\Delta_g(t)}{\Delta_g(t_1)}$$
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$$\widetilde{q}(x,t) = \widetilde{q}(x,t_0)\Delta_q(t) + \frac{\alpha_s}{2\pi}\int_{t_0}^{t}\frac{\mathrm{d}t_1}{t_1}\frac{\Delta_q(t)}{\Delta_q(t_1)}$$
$$\int \mathrm{d}x_{i-1}\int_x^{z_{max}}\mathrm{d}z\left(z\widehat{P}_{qg}(z)\widetilde{g}\left(x_{i-1},t_0\right)\Delta_g(t_1,t_0) + z\widehat{P}_{qq}(z)\widetilde{q}\left(x_{i-1},t_0\right)\Delta_q(t_1,t_0)\right)\delta(zx_{i-1}-x)$$

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Forward evolution: at the beginning of the evolution we don't know what we will have at the end.
Example:
We consider gluon $\tilde{g}\left(x_{i-1},t_0\right)$.
It started as a gluon at some t_0 and it evolves. We don't know into what it will split so maybe we are in the equation for gluon, maybe for quark.
But we know that if we are in gluon equation for sure we don't have blue parts

MC solution - forward evolution

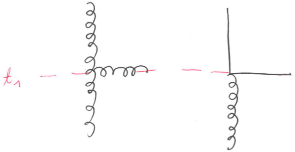
Goal: to solve with MC method

$$\tilde{g}(x,t) = \tilde{g}(x,t_0)\Delta_g(t) + \frac{\alpha_s}{2\pi}\int_{t_0}^{t}\frac{\mathrm{d}t_1}{t_1}\frac{\Delta_g(t)}{\Delta_g(t_1)}\int \mathrm{d}x_{i-1}\int_x^{z_{max}}\mathrm{d}z\left(z\hat{P}_{gg}(z)\tilde{g}\left(x_{i-1},t_0\right)\Delta_g(t_1,t_0)\right)\cancel{\delta(zx_{i-1}-x)}$$

or

$$\tilde{q}(x,t) = \tilde{q}(x,t_0)\Delta_q(t) + \frac{\alpha_s}{2\pi}\int_{t_0}^{t}\frac{\mathrm{d}t_1}{t_1}\frac{\Delta_q(t)}{\Delta_q(t_1)}\int \mathrm{d}x_{i-1}\int_x^{z_{max}}\mathrm{d}z\left(z\hat{P}_{qg}(z)\tilde{g}\left(x_{i-1},t_0\right)\Delta_g(t_1,t_0)\right)\cancel{\delta(zx_{i-1}-x)}$$

We generate t_1 and z according to $\hat{P}_{gg}+\hat{P}_{qg}$. We calculate $x_i = zx_{i-1}$ so we don't need $\cancel{\delta(zx_{i-1}-x)}$!



MC solution - forward evolution

Goal: to solve with MC method

$$\tilde{g}(x,t) = \tilde{g}(x,t_0)\Delta_g(t) + \frac{\alpha_s}{2\pi}\int_{t_0}^{t}\frac{dt_1}{t_1}\cancel{\frac{\Delta_g(t)}{\Delta_g(t_1)}}\int dx_{i-1}\int_x^{z_{max}} dz\left(z\hat{P}_{gg}(z)\tilde{g}\left(x_{i-1},t_0\right)\Delta_g(t_1,t_0)\right)$$

or

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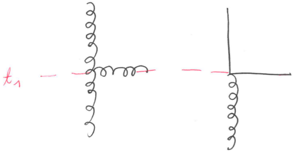
Moreover! if in the next step we will generate t_2 above the scale t we will know that we had only one branching and we will stop. If we will generate t_2 below the scale t than we will continue evolution with second branching. It means we don't need $\cancel{\frac{\Delta_g(t)}{\Delta_g(t_1)}}$ by constraction!



MC solution - forward evolution

Goal: to solve with MC method

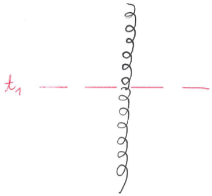
$$\tilde{g}(x,\, t) = \tilde{g}(x,\, t_0)\Delta_g(t) + \frac{\alpha_s}{2\pi}\int_{t_0}^{t}\frac{\mathrm{d}t_1}{t_1}\int \mathrm{d}x_{i-1}\int_x^{zmax}\mathrm{d}z\left(z\hat{P}_{gg}(z)\tilde{g}\left(x_{i-1},\, t_0\right)\Delta_g(t_1,\, t_0)\right)$$

or

$$\tilde{q}(x,\, t) = \tilde{q}(x,\, t_0)\Delta_q(t) + \frac{\alpha_s}{2\pi}\int_{t_0}^{t}\frac{\mathrm{d}t_1}{t_1}\int \mathrm{d}x_{i-1}\int_x^{zmax}\mathrm{d}z\left(z\hat{P}_{qg}(z)\tilde{g}\left(x_{i-1},\, t_0\right)\Delta_g(t_1,\, t_0)\right)$$

We make a decision about the splitting: The decision is made using a random numebr R. if

$$R \leq \frac{\int_x^{zmax}\mathrm{d}zz\left(\hat{P}_{gg}(z)\right)}{\int_x^{zmax}\mathrm{d}zz\sum_j\left(\hat{P}_{jg}(z)\right)}$$



continue as gluon $\longrightarrow$ now we know in which equation are we.
By making the decision we "perform this integral" and we don't need it anymore in the equation

MC solution - forward evolution

Goal: to solve with MC method

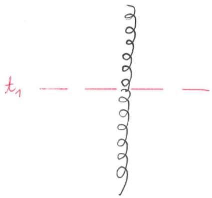
$$\tilde{g}(x,t) = \tilde{g}(x,t_0)\Delta_g(t) + \frac{\alpha_s}{2\pi}\int_{t_0}^{t}\frac{dt_1}{t_1}\int dx_{i-1}\left(\tilde{g}\left(x_{i-1},t_0\right)\Delta_g(t_1,t_0)\int_x^{z_{max}} dzz\sum_j\left(\hat{P}_{jg}(z)\right)\right)$$

We make a decision about the splitting: The decision is made using a random numebr R. if

$$R \leq \frac{\int_x^{z_{max}} dzz \left(\hat{P}_{gg}(z)\right)}{\int_x^{z_{max}} dzz \sum_j \left(\hat{P}_{jg}(z)\right)}$$

continue as gluon.

Additionaly we need to multiply our equation by $\int_x^{z_{max}} dzz\sum_j\left(\hat{P}_{jg}(z)\right)$



Goal: to solve with MC method

$$\widetilde{g}(x,\, t) = \widetilde{g}(x,\, t_0)\Delta_g(t) + \frac{\alpha_s}{2\pi}\int_{t_0}^{t}\frac{\mathrm{d}t_1}{t_1}\int \mathrm{d}x_{i-1}\left(\widetilde{g}\left(x_{i-1},\, t_0\right)\Delta_g(t_1,\, t_0)\int_x^{z_{max}}\mathrm{d}z z\sum_j\left(\widehat{P}_{jg}(z)\right)\right) weight_z$$

when we generate z: $weight_z$

Goal: to solve with MC method

$$\widetilde{g}(x,t) = \widetilde{g}(x,t_0)\Delta_g(t) + \frac{\alpha_s}{2\pi}\int_{t_0}^{t}\frac{\mathrm{d}t_1}{t_1}\int \mathrm{d}x_{i-1}\left(\widetilde{g}\left(x_{i-1},t_0\right)\Delta_g(t_1,t_0)\int_x^{z_{max}}\mathrm{d}zz\sum_j\left(\widehat{P}_{jg}(z)\right)\right) weight_z$$

when we generate z: $weight_z$
integrals $\int \mathrm{d}x_{i-1}$ and $\int_{t_0}^{t}\mathrm{d}t_1$ solved by MC integration method:

$$\int_a^b f(x)\mathrm{d}x = (b-a)\frac{1}{N}\sum_{i=1}^{N} f(x_i) \tag{6}$$

So at the end the pieces we put in the grid for one branching

$$\widetilde{g}(x,t) = \widetilde{g}(x,t_0)\Delta_g(t) + \frac{\alpha_s}{2\pi}\frac{1}{t_1}\left(\widetilde{g}\left(x_{i-1},t_0\right)\Delta_g(t_1,t_0)\int_x^{z_{max}}\mathrm{d}zz\sum_j\left(\widehat{P}_{jg}(z)\right)\right) weight_z$$