$\mathcal{N} = 3$ SCFTs in four dimensions

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"Field theories"

One can define a quantum theory (and in particular, a quantum field theory) abstractly by giving a Hilbert space of states

$$|i\rangle \in \mathcal{H} \tag{1}$$

operators in this Hilbert space

$$\mathcal{O}\,|i\rangle = |j\rangle \tag{2}$$

and a prescription for computing inner products of states in the Hilbert space

$$\langle i|j\rangle = c_{ij} \in \mathbb{C}\,. \tag{3}$$

(Typically some additional conditions: unitarity, existence of one stress tensor, . . .)

Lagrangian field theories

A subclass of such theories are given by **Lagrangian** quantum field theories. They are theories in which $\mathcal{H}$, $\mathcal{O}$ and $\langle i|j\rangle$ can be defined in terms of a **path integral**. For instance, correlation functions of operators can be defined in terms of the partition function

$$Z_\Lambda[J(x)] = \int_\Lambda [D\Phi] \exp\left(-\int \mathrm{d}^4x\, \mathcal{L}_\Lambda(\Phi(x), \partial\Phi(x), \ldots, J(x))\right) \tag{4}$$

with $\mathcal{L}$ a function from $\Phi(x), J(x)$ to the real numbers. Here Φ stands for the set of fields in the theory (maps from spacetime[1] to $\mathbb{R}, \mathbb{C}$ or Grassmann numbers), and $J(x)$ is an "external source", useful when computing correlators.

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Two comments

- Not every QFT is necessarily Lagrangian.
- Being "non-Lagrangian" is often a time-dependent statement.

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Conformal (quantum) field theories

We are familiar with the previous, abstract presentation of QFTs in the case in which the system has **conformal symmetry**. This is a natural symmetry to consider: as we go to the deep IR the resulting theory may be non-trivial, and will be scale-invariant. It is conjectured that for unitary 4d QFTs scale invariance implies conformal invariance. That is, the theory is invariant under the group of transformations preserving the angle

$$\frac{x \cdot y}{\sqrt{(x \cdot x)(y \cdot y)}} \,. \tag{5}$$

This larger (than Lorentz) symmetry group constrains the properties of the theory in useful ways.

Universality

Particularly useful is the notion of **universality**: a given CFT may arise as the IR limit of a number of different QFTs.

In the literature on 4d $\mathcal{N} = 1$ field theories, such universality goes under the name "Seiberg duality".

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(But, as I will explain, one can show that there is no $\mathcal{N} = 3$ Lagrangian theory flowing to the $\mathcal{N} = 3$ SCFTs we construct. It is in this sense that the theories that we construct are known to be non-Lagrangian.)

"$\mathcal{N} = 3$"

In four dimensions, though, conformal invariance by itself is not very constraining. We obtain a more tractable class of theories if we also assume **supersymmetry**: a symmetry of the theory with generators Q such that

$$Q\,|a\rangle = |b\rangle \tag{6}$$

and spin$(|b\rangle) =$ spin$(|a\rangle) \pm \frac{1}{2}$.

The ways of constructing such Q compatible with Lorentz symmetry can be classified [Haag, Lopuszanski and Sohnius], with the result that the Q_i must be spinors of the Lorentz group Spin$(3,1)$.

We say that we have $\mathcal{N} = k$ supersymmetry if there are k distinct such minimal spinors. In 4d, a minimal spinor is Majorana $\cong_{4d}$ Weyl, with four components. So in 4d, we have $4k$ distinct supercharges.

Simplifications due to supersymmetry

Increasing $\mathcal{N}$ helps, since for supersymmetric theories

- Loops corrections to certain couplings tend to cancel.
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...but we cannot increase $\mathcal{N}$ indefinitely, since there are bounds on the existence of SCFTs with $\mathcal{N} > \mathcal{N}_{\text{max}}$.[Nahm '78], [Córdova, Dumitrescu, Intriligator '16].

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(Of course, increasing $\mathcal{N}$ makes the theories increasingly unrealistic, but let's not worry about that today.)

$\mathcal{N}=4$ SYM theory in four dimensions

This theory has 16 supercharges. No classification is known, but all $\mathcal{N}=4$ theories which are known are Lagrangian (Yang-Mills), and can be specified by giving a gauge group G, a complexified coupling $\tau=\theta+i/g^2$, and some extra discrete choices $\mathscr{L}$ for the dyonic and magnetic line operators [Aharony, Seiberg, Tachikawa]:

$$\mathcal{L}=\frac{1}{g^2}\operatorname{Tr}(F\wedge\star F)+\theta\operatorname{Tr}(F\wedge F)+\ldots \tag{7}$$

This description is nevertheless overcomplete, due to the existence of **Montonen-Olive duality**:

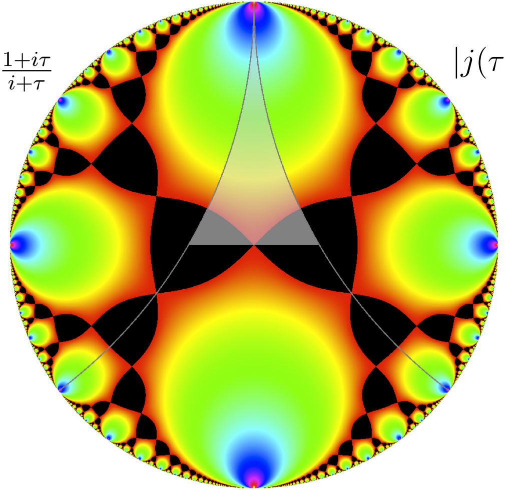
$$\mathcal{T}(G,\tau,\mathscr{L})=\mathcal{T}({}^{L}G,-\frac{1}{n_G\tau},\mathscr{L}') \tag{8}$$

and the classical $\tau\to\tau+1$ symmetry. ${}^{L}G$ is the Langlands dual to G:

$$\{U(N),SU(N),SO(2N),SO(2N+1),\ldots\}\xrightarrow{L}\{U(N),SU(N)/\mathbb{Z}_N,SO(2N),USp(2N),\ldots\} \tag{9}$$

Duality in $\mathcal{N} = 4$

$u = \frac{1+i\tau}{i+\tau}$



$|j(\tau)|$

In this representation $\tau \to -1/\tau$ is $u \to -u$.

$\mathcal{N}=3$ SCFTs

Many examples of $\mathcal{N}=0,1,2$ CFTs are known, both Lagrangian and non-Lagrangian. But no $\mathcal{N}=3$ SCFT (which was not $\mathcal{N}=4$) was known until our work.[2] In fact, they were widely thought not to exist! (This is also what I though until last year.)

[2] $\mathcal{N}=3$ supergravities were known for a long time, but no $\mathcal{N}=3$ examples without gravity were known, aside from the holographic dual of the $\mathcal{N}=6$ AdS_5 compactification in [Ferrara, Porrati, Zaffaroni '98].

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Theorem: Every non-gravitational CPT-invariant $\mathcal{N}=3$ Lagrangian is automatically $\mathcal{N}=4$.

Minimal $\mathcal{N}=3$ multiplet: $\{A_\mu(+1), 3\lambda(+\frac{1}{2}), 3\phi(0), \lambda(-\frac{1}{2})\}$. Its CPT-conjugate changes the helicities, completing the content into a $\mathcal{N}=4$ multiplet.

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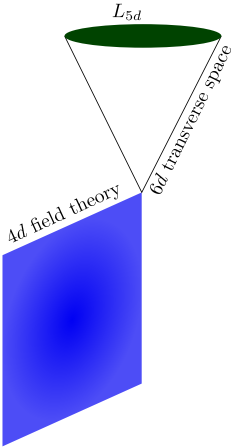
"Four dimensional"

We are after a theory in four dimensions which, if it exists, has no semi-classical limit compatible with the $\mathcal{N} = 3$ symmetry.

It turns out that the most robust way of constructing the 4d $\mathcal{N} = 3$ theories is by using string theory techniques in 10 and 11d.

In our first ("A_N") class of $\mathcal{N} = 3$ theories we will construct a string setting in 10d with a topological defect. On the core of this defect we will have a four dimensional theory coupled to 10d supergravity. In the IR the 10d supergravity decouples, leaving the 4d theory we are after.



IIB string theory

A good setting for our purposes is IIB string theory (described by type IIB supergravity at low energies). It contains certain supergravity defects ("D3-branes") where $\mathcal{N}=4$ four dimensional $U(N)$ SYM lives.

Furthermore, it has a scalar field τ_{10d}, whose restriction to the D3s gives the $\tau = \theta + i/g^2$ in the $\mathcal{N}=4$ Lagrangian.

Montonen-Olive duality on the low energy theory on the D3s

$$\mathcal{T}(U(N),\tau) = \mathcal{T}\left(U(N), -\frac{1}{\tau}\right) \tag{10}$$

extends to the full 10d string theory:

$$\text{IIB}(N\,\text{D3s}, \tau_{10d}) = \text{IIB}\left(N\,\text{D3s},\ -\frac{1}{\tau_{10d}}\right) \tag{11}$$

$\mathcal{N}=4$ quotient perspective

We want to reduce $\mathcal{N}=4 \to \mathcal{N}=3$. It turns out that this is possible, by taking an appropriate gauging of a symmetry of the $\mathcal{N}=4$ theory.

We start from the observation that for particular (self-dual) values of τ_{YM}, certain $\mathbb{Z}_k$ subgroups of the $SL(2,\mathbb{Z})$ **duality** become **symmetries**. For instance, when $\tau = i$ we have that S-duality

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{12}$$

becomes a symmetry of the theory. $(-i^{-1} = i.)$

We can then construct appropriate quotients

$$\mathcal{Q}_k = \frac{\mathcal{N}=4\ U(N)}{\mathbb{Z}_k^R \cdot \mathbb{Z}_k^{SL(2,\mathbb{Z})}}\,. \tag{13}$$

We choose $\mathbb{Z}_k^R$ appropriately to preserve 12 supercharges.

Supersymmetry

These theories preserve (just) 12 supercharges for $k > 2$.
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The transformation of the supercharge generators under a $SL(2,\mathbb{Z})$ transformation in the $\mathbb{Z}_k$ symmetry group is [Kapustin, Witten '06]

$$Q^A \to \gamma_k^{\frac{1}{2}} Q^A \qquad \text{with} \quad \gamma_k = \frac{|c\tau + d|}{c\tau + d} \,. \tag{14}$$

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Under a $U(1) \subset SU(4)_R$ rotation the supercharges transform as

$$(Q^1, Q^2, Q^3, Q^4) \to (\omega^{\frac{1}{2}} Q^1, \omega^{\frac{1}{2}} Q^2, \omega^{\frac{1}{2}} Q^3, \omega^{-\frac{3}{2}} Q^4)\,. \tag{15}$$

We take $\omega = \gamma_k^{-1}$, so Q^A with $A = 1, 2, 3$ survive the quotient. (For $\mathbb{Z}_4$: $g_{SL(2,\mathbb{Z})} = S$, $\tau = i$, so $\gamma_4 = -i$, while $\omega_4 = i$.) (Notice that for $k = 1, 2$ we preserve $\mathcal{N} = 4$.)

F-theory viewpoint: Probing rigid singularities

From a string theory point of view, we will be interested in understanding the four dimensional physics coming from (probe D3 branes on) F-theory compactifications in the presence of singularities that do no admit supersymmetric smoothings. I.e. they cannot be resolved or deformed into a smooth space without spending energy.

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- Complex codimension 4 Calabi-Yau singularity in a geometry with a F-theory limit. There are many such geometries, and we will only scratch the surface.
- Simplest case: $\mathbb{Z}_k$ orbifolds of $\mathbb{C}^3 \times T^2$, with non-trivial T^2 action and isolated fixed points.

(Such orbifolds have appeared for two-folds [Dasgupta, Mukhi '96] and threefolds [Witten '96], but these cases admit deformations.)

Generalizing the O3 plane

Calabi-Yau fourfolds of the form $(\mathbb{C}^3 \times T^2)/\mathbb{Z}_k$ can be classified completely: the orbifold actions preserving susy were classified in [Morrison, Stevens '84], [Anno '03], [Font, López '04]. We focus on the cases preserving at least 12 supercharges.

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In the F-theory limit, adding D3 brane probes:

- $k = 1$ gives IIB string theory $\rightarrow$ 4d $U(N)$ $\mathcal{N}=4$ SYM.
- $k = 2$ gives IIB w/ O3 plane $\rightarrow$ 4d $\mathcal{N}=4$ SYM w/ orthogonal or symplectic groups.

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- $k=3,4,6$ give IIB w/ exotic "OF3" plane $\rightarrow$ 4d $\mathcal{N}=3$ SCFTs.

Outline

1. Introduction
2. Revisiting the O3 plane
3. M5 brane realization of $\mathcal{N} = 3$ quotient

EYAWTK about the O3 plane

It will prove very illuminating to revisit the O3 plane (i.e. $(\mathbb{C}^3 \times T^2)/\mathbb{Z}_2$) from multiple viewpoints, since it is the simplest case of a complex codimension four singularity with a F-theory lift, and is relatively well understood.

- Worldsheet CFT.
- F/M-theory.
- Holographic picture.
- Field theory.

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- Worldsheet CFT.
- **F/M theory.**
- **Holographic picture.**
- **Field theory.**

Everything but the CFT approach potentially generalizes to $k = 3, 4, 6$.

Worldsheet description of the O3 plane

We start with IIB string theory on $\mathbb{R}^{10} = \mathbb{R}^4 \times \mathbb{C}^3$, and quotient by $\mathcal{I}(-1)^{F_L}\Omega$. Here $\mathcal{I}$ acts as reflection on the $\mathbb{C}^3$:

$$\mathcal{I}\colon (x,y,z) \to (-x,-y,-z) \tag{16}$$

while $(-1)^{F_L}\Omega$ acts on the worlsheet. Its induced effect on the spacetime fields is easily computed, for instance

$$(-1)^{F_L}\Omega\colon \begin{pmatrix} B_2 \\ C_2 \end{pmatrix} \to \begin{pmatrix} -B_2 \\ -C_2 \end{pmatrix} \tag{17}$$

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If we have a stack of N D3 branes we need to choose an action on the Chan-Paton factors, which will project $U(N)$ down to an orthogonal or symplectic group:

$\mathrm{O3}^-$ $\quad$ $\widetilde{\mathrm{O3}}^-$ $\quad$ $\mathrm{O3}^+$ $\quad$ $\widetilde{\mathrm{O3}}^+$

Last three are related by Montonen-Olive duality. [Witten '98])

F(M)-theory description of the O3 plane

IIB without orientifold is given by M-theory on T^2 in the $\mathrm{vol}(T^2) \to 0$ limit, we wish to quotient this by the lift of $\mathcal{I}(-1)^{F_L}\Omega$.

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The $(-1)^{F_L}\Omega$ action acts as

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which when rewritten in terms of C_3 implies that

$$(-1)^{F_L}\Omega \colon (p, q) \to (-p, -q) \tag{19}$$

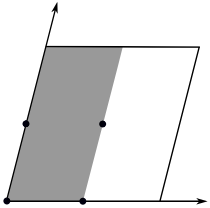
i.e. an inversion of the T^2: $u \to -u$. (Denoted by $-1 \in SL(2, \mathbb{Z})$)

F(M)-theory description of the O3 plane

Writing x, y, z, u for the $\mathbb{C}^3 \times T^2$ coordinates acted upon by the involution, we thus find

$$\mathcal{I}(-1)^{F_L}\Omega\colon (x, y, z, u) \to (-x, -y, -z, -u)$$

and the total geometry is $(\mathbb{C}^3 \times T^2)/\mathbb{Z}_2$. This has four fixed points at $(x, y, z, u) = (0, 0, 0, p)$, with p a fixed point of the T^2 under the $\mathbb{Z}_2$.



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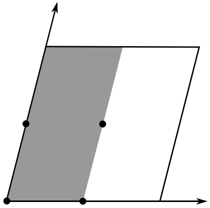

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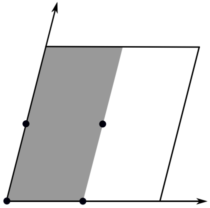

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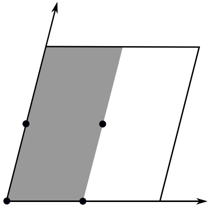

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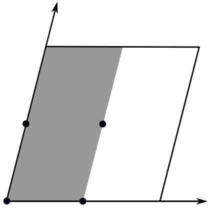

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- M2 branes probing $\mathbb{C}^4/\mathbb{Z}_k$: [Aharony, Bergman, Jafferis, Maldacena '08].
- Different O3 types: different discrete fluxes on the fixed points [Hanany, Kol '00].

F(IIB)-theory description of the O3 plane

A holography appetizer

In IIB string theory the $\mathbb{C}^3/\mathcal{I}$ orbifold is non-supersymmetric, while the O3 preserves 16 supercharges. I discuss the near horizon geometry, $AdS_5 \times (S^5/\mathbb{Z}_2)$, which naively is non-supersymmetric.

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So we do not have the vanilla orbifold, but in addition it has a non-trivial flat $SL(2,\mathbb{Z})$ duality bundle on top, acting with $-1 \in SL(2,\mathbb{Z})$ as we go round the non-trivial one-cycle in the $S^5/\mathbb{Z}_2$ horizon manifold. One can check that the $-1 \in SL(2,\mathbb{Z})$ acting on the sugra spinors restores susy as expected.

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The different kinds of orientifolds in this language are classified by discrete flux: $[H_3],[F_3] \in H^3(S^5/\mathbb{Z}_2, \widetilde{\mathbb{Z}}) = \mathbb{Z}_2$. [Witten '98]

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So we can understand the orientifold projection as a quotient by a particular symmetry of $\mathcal{N}=4$ $U(N)$ SYM: $U(N)/(\mathbb{Z}_2^R \cdot \mathbb{Z}_2^{SL(2,\mathbb{Z})})$. (In this language we also have a choice of Chan-Paton factors.)

Recap and strategy

We have discussed four ways of viewing the action of an O3 plane on a stack of D3 branes:

- Worldsheet CFT: a projection of the CFT by $\mathcal{I}(-1)^{F_L}\Omega$, with a choice of Chan-Paton factors.
- M-theory: M2 branes probing $(\mathbb{C}^3 \times T^2)/\mathbb{Z}_2$, with a choice of discrete torsion on the fixed points.
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Strategy for generalization

Quotient by other possible symmetries of $\mathbb{C}^3 \times T^2$, S^5 or $U(N)$.

The generalization of the CFT approach seems less obvious.

"OF3" planes ("S-folds") from M-theory

We start by considering the M-theory picture, given by $\mathbb{Z}_k$ $(k>2)$ quotients of $\mathbb{C}^3 \times T^2$ leaving isolated fixed points. It turns out that maximal supersymmetry ($\mathcal{N}=3$) is preserved for $k=3,4,6$, with action [Font, López '04]

$$(x, y, z, u) \rightarrow (\omega_k x, \omega_k^{-1} y, \omega_k z, \omega_k^{-1} u) \tag{21}$$

with $\omega_k = \exp(2\pi i/k)$. (These are known to be terminal Gorenstein [Morrison, Stevens '84].) We focus on these.

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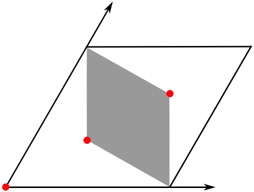
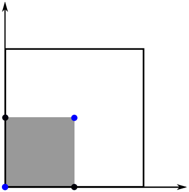
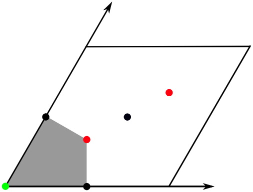
This action only maps the torus to itself for specific complex structures:



$\mathbb{Z}_3$: $\tau = e^{2\pi i/6}$
Three $\mathbb{C}/\mathbb{Z}_3$ points.



$\mathbb{Z}_4$: $\tau = i$
One $\mathbb{Z}_2$ and two $\mathbb{Z}_4$ points.



$\mathbb{Z}_6$: $\tau = e^{2\pi i/6}$
One $\mathbb{Z}_6$, one $\mathbb{Z}_2$ and one $\mathbb{Z}_3$ point.

Holographic perspective

There seems to be no obstruction to taking the F-theory limit, so we end up with a IIB background of the form $\mathbb{C}^3/\mathbb{Z}_k$. Putting D3 branes on the singularity, and taking the near horizon limit, this suggests a dual description for the field theories in terms of $AdS_5 \times (S^5/\mathbb{Z}_k)$, with a non-trivial flat $SL(2,\mathbb{Z})$ bundle. (A realization of the setup proposed in [Ferrara,Porrati,Zaffaroni '98].)

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Remarkably, the axio-dilaton τ is frozen to a $\mathcal{O}(1)$ value in these backgrounds. We learn that the theories on the branes no longer have the marginal deformation associated to changing the Yang-Mills coupling.

$\mathcal{N}=4$ quotient perspective

In terms purely of the theory on the probe branes, we start from the observation that for particular (self-dual) values of τ_{YM}, certain $\mathbb{Z}_k$ subgroups of the $SL(2,\mathbb{Z})$ become symmetries. For instance, when $\tau = i$ we have that S-duality

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{22}$$

becomes a symmetry of the theory. $(-i^{-1} = i.)$

We can then construct appropriate quotients

$$\mathcal{Q}_k = \frac{\mathcal{N}=4\ U(N)}{\mathbb{Z}_k^R \cdot \mathbb{Z}_k^{SL(2,\mathbb{Z})}}\,. \tag{23}$$

We choose $\mathbb{Z}_k^R$ to be the R-symmetry generator associated with the $\mathbb{Z}_k$ rotation in the transverse $\mathbb{R}^6$, in order to preserve susy.

Supersymmetry

We claim that these theories preserve (just) 12 supercharges for $k > 2$. We now show this in the $\mathcal{N} = 4$ SYM quotient perspective (the computation from the other viewpoints is essentially isomorphic). (Also in [Nishinaka, Tachikawa '16].)

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The 16 supercharges arrange into four spacetime spinors Q^A_α, a spinor of $SU(4)_R$. Under the $\mathbb{Z}_k$ rotation these transform as ($\omega_k = \exp(2\pi i/k)$)

$$(Q^1, Q^2, Q^3, Q^4) \to (\omega_k^{\frac{1}{2}} Q^1, \omega_k^{\frac{1}{2}} Q^2, \omega_k^{\frac{1}{2}} Q^3, \omega_k^{-\frac{3}{2}} Q^4)\,. \tag{24}$$

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The transformation of the supercharge generators under a $SL(2,\mathbb{Z})$ transformation is [Kapustin, Witten '06]

$$Q^A \to \gamma^{\frac{1}{2}} Q^A \qquad \text{with} \quad \gamma = \frac{|c\tau + d|}{c\tau + d}\,. \qquad (25)$$

For the theories we are constructing, we have $\gamma = \omega_k^{-1}$, so only Q^A with $A = 1, 2, 3$ survive the quotient. (For $\mathbb{Z}_4$: $g_{SL(2,\mathbb{Z})} = S$, $\tau = i$, so $\gamma = -i$, while $\omega_4 = i$.) (Notice that for $k = 1, 2$ we preserve $\mathcal{N} = 4$.)

Limitations of the IIB picture

The previous viewpoint is very illuminating, and is the cleanest setup to argue for the existence of these theories.

But there are some bothersome limitations:

- The field theory construction seems to apply also to E-type $\mathcal{N}=4$ theories.[3] But there is no known construction of E-type theories from D3s.
- Very little known about the analysis of D3s on these singularities (both α' and g_s of order 1).

[3] Non-simply laced cases are more subtle, and D-type less subtle, so I will discuss neither.

M5 brane realization

In recent years there have been very important advances in the study of four dimensional $\mathcal{N}=2$ SCFTs obtainable from compactifying six dimensional $(0,2)$ theories on Riemann surfaces. [Gaiotto '09], ...

I will now describe a realization of the OF3 $\mathcal{N}=3$ theories in these terms, which should be helpful in studying their properties, and allows us to construct new ("exceptional") examples of $\mathcal{N}=3$ theories.

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I will now describe a realization of the OF3 $\mathcal{N} = 3$ theories in these terms, which should be helpful in studying their properties, and allows us to construct new ("exceptional") examples of $\mathcal{N} = 3$ theories.

(Although for time reasons I will not explain the construction of the $\mathcal{N} = 3$ exceptional theories themselves.)

M5 brane realization of the $\mathcal{N}=4$ theory

Start with N D3s on $\underline{\mathbb{R}^4} \times S^1_T \times \mathbb{R}^5$. T-dualizing on S^1_T and lifting to M-theory gives N M5s on

$$\underline{\mathbb{R}^4 \times \tilde{S}^1_T \times S^1_M} \times \mathbb{R}^5\,, \tag{26}$$

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We want to track the action of the $\mathcal{N}=3$ $\mathbb{Z}_k$ quotient in this representation. I will do $\mathbb{Z}_4$ for simplicity, i.e. a quotient by

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{27}$$

times the corresponding $SU(4)_R$ rotation.

M5 brane realization of the $\mathbb{Z}_4$ $\mathcal{N}=3$ theory

S-duality generator

The easiest part is the S-duality part. T-dualizing on a circle:

$$r_{\text{IIA}} = r_{\text{IIB}}^{-1} \quad ; \quad g_{\text{IIA}} = \frac{g_{\text{IIB}}}{r_{\text{IIB}}} = g_{\text{IIB}} r_{\text{IIA}} \,. \tag{28}$$

The uplift to M-theory is then given by

$$R_M = g_{\text{IIA}}^{\frac{2}{3}} = \left(\frac{g_{\text{IIB}}}{r_{\text{IIB}}}\right)^{\frac{2}{3}} \quad ; \quad R_{\tilde{T}} = \frac{r_{\text{IIA}}}{\sqrt{R_M}} = \left(\frac{r_{\text{IIB}}}{g_{\text{IIB}}}\right)^{\frac{1}{3}} \frac{1}{r_{\text{IIB}}} = \frac{1}{g_{\text{IIB}}^{\frac{1}{3}} r_{\text{IIB}}^{\frac{2}{3}}}$$

so we have

$$\frac{R_M}{R_{\tilde{T}}} = g_{\text{IIB}} \tag{29}$$

which extends to the familiar

$$\tau_{\tilde{T}M} = C_0 + \frac{i}{g_s} \,. \tag{30}$$

M5 brane realization of the $\mathbb{Z}_4$ $\mathcal{N}=3$ theory

The compactification of S^1_T breaks $SO(6) \supset SO(5)$, and in particular the $\mathbb{Z}_4 \subset SO(6)$ we need is no longer a symmetry. We fix this by compactifying an extra circle, i.e. we start with N D3s on

$$\underline{\mathbb{R}^4} \times S^1_T \times S^1_E \times \mathbb{R}^4 \tag{31}$$

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If we choose $\tau_{TE} = i$ (which implies $R(S^1_T) = R(S^1_E)$) we get an enhancement to $SO(4) \times \mathbb{Z}_4 \not\subset SO(5)$, with generator

$$(x_E, x_T) \to (-x_T, x_E) \tag{32}$$

or in terms of the complex structure of the $S^1_T \times S^1_E$ torus

$$\tau_{TE} \to -\frac{1}{\tau_{TE}}\,. \tag{33}$$

M5 brane realization of the $\mathbb{Z}_4$ $\mathcal{N}=3$ theory

After T-duality on S^1_T the action is no longer geometric: it acts on $\tilde{S}^1_T \times S^1_E$ as

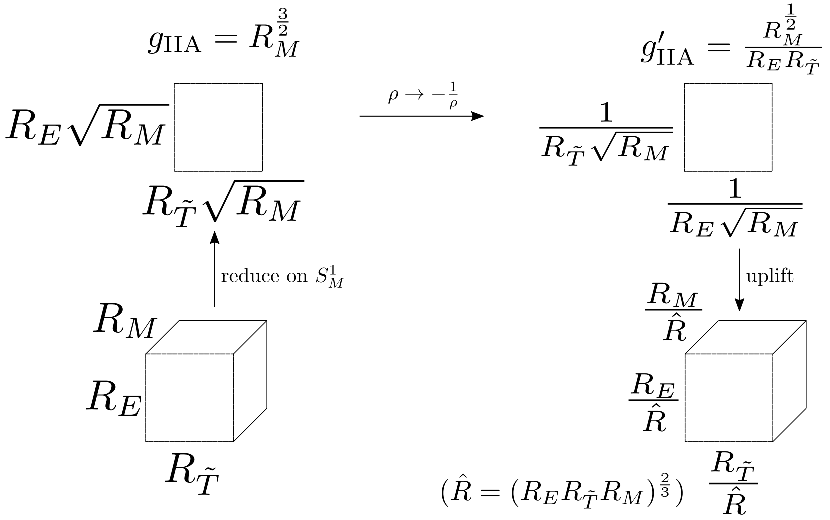
$$\mathbb{Z}_4^{\text{IIA}}\colon \rho_{\tilde{T}E} \to -\frac{1}{\rho_{\tilde{T}E}}\,. \tag{34}$$

with

$$\rho_{\tilde{T}E} = \int_{\tilde{T}E} B + i\,\mathrm{vol}(\tilde{T}E)\,. \tag{35}$$

We can view this duality generator as a T-duality action along both $S^1_{\tilde{T}}$ and S^1_E together with a $\frac{\pi}{2}$ rotation.

M5 brane realization of the $\mathbb{Z}_4$ $\mathcal{N}=3$ theory



M5 brane realization of the $\mathbb{Z}_4$ $\mathcal{N}=3$ theory

The uplift of the $\mathbb{Z}_4$ R-symmetry generator to M-theory is thus

$$\mathbb{Z}_4^R \colon \rho_M = \int_{T^3} C_3 + i R_E R_{\tilde{T}} R_M \to -\frac{1}{\rho_M}\,. \qquad (36)$$

Moduli at the fixed point

In order to have a theory invariant under $\mathbb{Z}_4^R$ we thus require

$$\rho_M = -\frac{1}{\rho_M} \iff \rho_M = i \iff R_E R_{\tilde{T}} R_M = 1\,. \tag{37}$$

On the other hand, invariance under S-duality requires

$$\tau_{\tilde{T}M} = -\frac{1}{\tau_{\tilde{T}M}} \iff \tau_{\tilde{T}M} = i \iff R_M = R_{\tilde{T}}\,. \tag{38}$$

Together these leave a single modulus unfixed:

$$R_M = R_{\tilde{T}} = R \quad ; \quad R_E = R^{-2}\,. \tag{39}$$

The four dimensional limit is obtained whenever $\rho_{TE} \to i\infty$, which maps to $R \to 0$.

The $\mathcal{N}=3$ quotient

In summary: the $\mathcal{N}=3$ theories coming from S-folds (generalized O3 planes) can be constructed by taking M5 branes on $\underline{\mathbb{R}^{1,3}\times(S^1_M\times S^1_{\tilde{T}}}\times S^1_E\times\mathbb{C}^2)/\mathbb{Z}_k$, with

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$\mathbb{Z}_k^R$ is an ordinary $\mathbb{Z}_k$ rotation on $\mathbb{C}^2$ generated by

$$\hat{\omega}_k^R=\begin{pmatrix}\omega_k^{-1} & 0\\ 0 & \omega_k\end{pmatrix} \tag{41}$$

with $(\omega_k)^k=1$. This is an element of the manifest $SO(4)_R\subset SO(5)_R$ in the ordinary construction.

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$\mathbb{Z}_k^\tau$ is an ordinary $SL(2,\mathbb{Z})$ transformation action of the T^2 formed by $\tilde{T}$ and the M-theory circle. I.e. the T^2 wrapped by the M5 stack, so this element maps to the corresponding $SL(2,\mathbb{Z})$ duality generator in $\mathcal{N}=4$ SYM.

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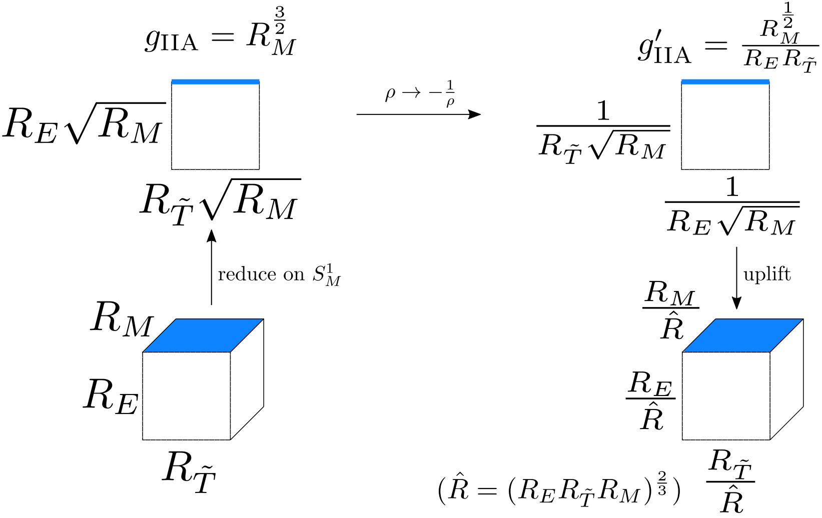
$\tilde{\mathbb{Z}}_k^R$ is the non-geometric element sending

$$\rho_M\rightarrow\frac{a\rho_M+b\rho_M}{c\rho_M+d\rho_M} \tag{41}$$

for the appropriate $SL(2,\mathbb{Z})$ transformation corresponding to the $\mathbb{Z}_k$ rotation. Recall that the $O(2,2;\mathbb{Z})$ T-duality group of type II on T^2 can be written as

$$O(2,2;\mathbb{Z})=(SL(2,\mathbb{Z})_\tau\times SL(2,\mathbb{Z})_\rho)\rtimes(\mathbb{Z}_2^{\tau\leftrightarrow\rho}\times\mathbb{Z}_2^{(\tau,\rho)\leftrightarrow(-\overline{\tau},-\overline{\rho})})\,. \tag{42}$$

Duality on the M5 brane



Duality on the M5 brane

This is a fairly exotic thing we are doing in terms of the M5:

- We are compactifying a transverse direction, introducing a breaking $SO(5) \times \mathbb{Z}_2 \to SO(4) \times \mathbb{Z}_2$. For certain values of this deformation of the M5 theory we have an enhancement $SO(4) \times \mathbb{Z}_2 \to SO(4) \times \mathbb{Z}_4$. (All this is a subgroup of the $SU(4)_R \to SO(6)_R$ arising in the IR fixed point.)

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- The $\mathbb{Z}_4$ acts in a very exotic way, mixing degrees of freedom along the M5 with transverse modes.

Duality on the M5 brane

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- We are compactifying a transverse direction, introducing a breaking $SO(5) \times \mathbb{Z}_2 \to SO(4) \times \mathbb{Z}_2$. For certain values of this deformation of the M5 theory we have an enhancement $SO(4) \times \mathbb{Z}_2 \to SO(4) \times \mathbb{Z}_4$. (All this is a subgroup of the $SU(4)_R \to SO(6)_R$ arising in the IR fixed point.)
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- But by construction, we know the symmetry to be there.

I don't have more to say about this at the moment, but it is clearly very interesting: figuring the precise action of this $\mathbb{Z}_4$ on the AGT [Alday, Gaiotto, Tachikawa '09] theory on the M5 would, to a large extent, solve these $\mathcal{N}=3$ theories.

Supersymmetry

One can compute the number of preserved supercharges directly in this picture. The full duality group of M-theory on T^3 is

$$SL(3,\mathbb{Z}) \times SL(2,\mathbb{Z})_{\rho_M} \tag{43}$$

of which we are taking elements in a $SL(2,\mathbb{Z})_\tau \times SL(2,\mathbb{Z})_{\rho_M}$ subgroup. Take the corresponding $U(1)$ rotation subgroups inside the respective tori. Then the supercharges preserved by the M5 transform as [Kumar, Vafa '96]:

$$(S_4^+, S_4^+)_{-\frac{1}{2},\frac{1}{2}} \oplus (S_4^+, S_4^-)_{-\frac{1}{2},-\frac{1}{2}} \oplus (S_4^-, S_4^+)_{\frac{1}{2},-\frac{1}{2}} \oplus (S_4^-, S_4^-)_{\frac{1}{2},\frac{1}{2}}$$

under

$$SO(1,3) \times SO(4)_R \times U(1)_\tau \times U(1)_{\rho_M} \,. \tag{44}$$

Supersymmetry

$\mathbb{Z}_k^R$ acts on the supercharges as

$$\mathbb{Z}_k^R \; : \; \begin{array}{ccc} (\bullet, S_4^+)_{p,q} & \to & (\bullet, S_4^+)_{p,q} \\ (\bullet, (+\frac{1}{2}, -\frac{1}{2}))_{p,q} & \to & e^{-2\pi i/k}(\bullet, (+\frac{1}{2}, -\frac{1}{2}))_{p,q} \\ (\bullet, (-\frac{1}{2}, +\frac{1}{2}))_{p,q} & \to & e^{2\pi i/k}(\bullet, (-\frac{1}{2}, +\frac{1}{2}))_{p,q} \, . \end{array} \tag{45}$$

Under the non-geometric action $\tilde{\mathbb{Z}}_k^R$ we find

$$\tilde{\mathbb{Z}}_k^R \colon (\bullet, \bullet)_{p,\pm\frac{1}{2}} \to e^{\pm\pi i/k}(\bullet, \bullet)_{p,\pm\frac{1}{2}} \tag{46}$$

where the bullets stand for omitted $S_4^\pm$ terms. Finally, under the rotation of the torus wrapped by the M5 branes we find that

$$\mathbb{Z}_k^\tau \colon (\bullet, \bullet)_{\pm\frac{1}{2},q} \to e^{\pm\pi i/k}(\bullet, \bullet)_{\pm\frac{1}{2},q} \tag{47}$$

Under the combined action $\mathbb{Z}_k = \mathbb{Z}_k^R \cdot \tilde{\mathbb{Z}}_k^R \cdot \mathbb{Z}_k^\tau$, only twelve supercharges remain invariant. So we have $\mathcal{N}=3$, as expected.

Conclusions (OF3 realization)

- We have constructed the first known examples of $\mathcal{N}=3$ SCFTs.
- We do so by a very natural F-theoretical generalization of the O3 plane, which freezes out the axio-dilaton, giving intrinsically strongly coupled backgrounds.
- The geometry involves rigid (neither deformable nor resolvable in a Calabi-Yau way) singularities.
- F-theoretical example of branes at singularities.
- The SCFTs we find have natural holographic descriptions as $AdS_5 \times X$, where X is a non-trivial smooth F-theory background with frozen axio-dilaton.
- The M-theory picture suggests that upon compactification on a circle we flow to $\mathcal{N} \geq 6$ ABJM theories.

Conclusions (M5 brane realization)

- I gave a different construction of the "classical" $\mathcal{N} = 3$ theories in terms of M5 branes wrapping a T^2 in M-theory. The M-theory background is a U-manifold.
- From the point of view of the M5 this seems like an interesting generalization of the usual case considered in the literature (M5 branes on Riemann surfaces). Something like an asymmetric orbifold version of [Alday, Gaiotto, Tachikawa '09]. Perhaps a useful viewpoint for probing $\mathcal{N} = 3$ theories in more detail.
- A simple generalization of the construction (combining with the U-manifold construction of 6d E-type $(0, 2)$ theories in M-theory) gives a new set of "exceptional" $\mathcal{N} = 3$ theories.
- More broadly, this is an interesting example of a SCFT which seems to be most naturally (only?) described in terms of non-geometric string backgrounds.

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- More broadly, this is an interesting example of a SCFT which seems to be most naturally (only?) described in terms of non-geometric string backgrounds.

Non-geometric engineering of QFTs?

Exceptional $\mathcal{N} = 3$ theories

The previous construction is based on the six-dimensional A-type $(0, 2)$ theory. The D-type theories can be constructed similarly. Can we generalize this to the E-type $(0, 2)$ theories?

From field theory at least it seems like it should be possible. The $\mathcal{N} = 4$ theories with E_6, E_7, E_8 gauge groups are self-dual under $SL(2, \mathbb{Z})$, so basically the same field theory intuition applies.

Non-simply laced cases are trickier:

- $B_n \leftrightarrow C_n$
- Duality for the $\mathcal{N} = 4$ G_2, F_4 theories acts on the moduli space too. [Argyres,Kapustin,Seiberg '06]

But it should be interesting to understand precisely what happens in these cases too. The construction here is a first step, which should then be decorated with branch cuts for outer automorphisms along the M5 worldvolume. [Vafa '97], [Tachikawa '11].

Exceptional $(0, 2)$ theories from M-theory

We need a way of realizing six-dimensional E-type $(0, 2)$ theories directly from M-theory. This is in fact not too hard, since we allow ourselves to consider U-manifolds.

We start with the well known construction of $(0, 2)$ E-type theories in IIB. These appear for a $\mathbb{C}^2/E_k$ singularity, for appropriate choices of B-field. We embed this into a Weierstrass fibration. I focus on the E_7 case. A Weierstrass fibration that does the job is:

$$y^2 = x^3 + t^3 x z^4 \tag{48}$$

with $[x : y : z]$ coordinates on $\mathbb{P}^{2,3,1}$, and t a coordinate on the $\mathbb{C}$ base. This has a $\mathbb{C}^2/E_7$ singularity at $t = 0$, constant $j(\tau) = i$, constant fiber volume, and a $\mathbb{Z}_4$ $SL(2, \mathbb{Z})$ monodromy

$$\mathcal{M} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \implies \tau \to -\frac{1}{\tau}\,. \tag{49}$$

Exceptional $(0,2)$ theories from M-theory

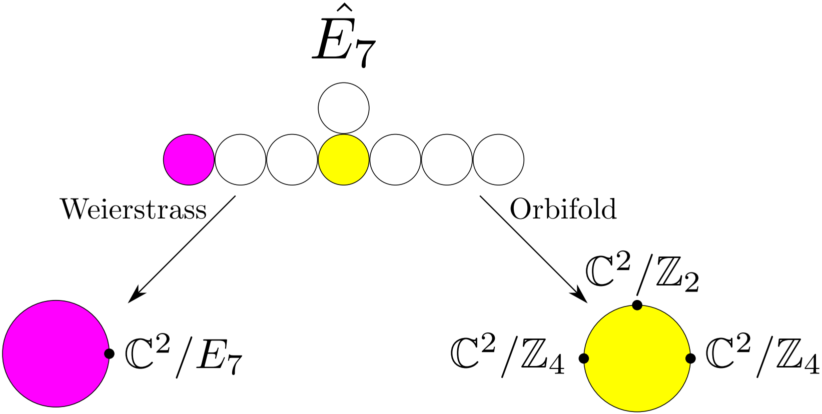
We can uplift this monodromy to M-theory, by T-dualizing along one of the directions of the T^2:

$$\begin{matrix} \text{IIB} \\ \tau \to -\frac{1}{\tau} \end{matrix} \quad \leftrightarrow \quad \begin{matrix} \text{IIA} \\ \rho_{\text{IIA}} \to -\frac{1}{\rho_{\text{IIA}}} \end{matrix} \quad \leftrightarrow \quad \begin{matrix} \text{M-theory} \\ \rho_{\text{M}} \to -\frac{1}{\rho_{\text{M}}} \end{matrix} \; . \tag{50}$$

The resulting space is a non-trivial U-manifold, and it is non-trivial (for us!) to compute the preserved supersymmetry, moduli spaces, etc.

Instead of attempting to do this, we use a trick: the original Weierstrass fibration is birationally equivalent (keeping $c_1(TX) = 0$) to IIB on $(\mathbb{C} \times T^2)/\mathbb{Z}_4$, where the $\mathbb{Z}_4$ acts as a simultaneous rotation on $\mathbb{C}$ and T^2.

The birational transformation



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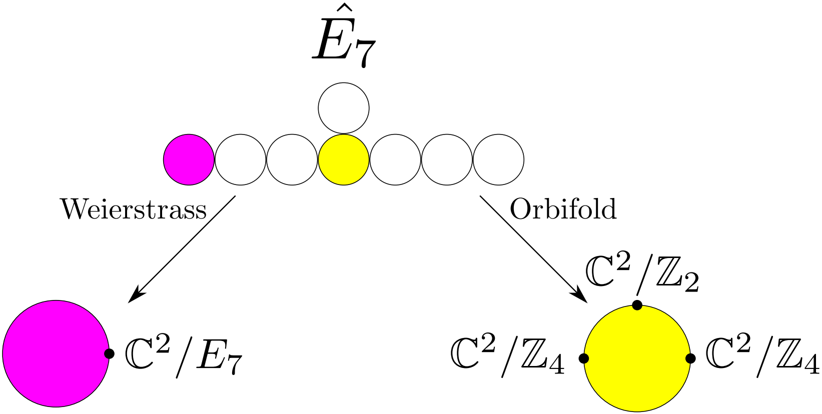


(This generalizes: $\mathbb{C}^2/\{D_4, E_6, E_7, E_8\}$ are birational to the $(\mathbb{C} \times T^2)/\mathbb{Z}_k$ quotients with $k \in \{2, 3, 4, 6\}$.)

Exceptional $(0,2)$ theories from M-theory

Explicit dualization to M-theory is now easy. We obtain M-theory on $(\mathbb{C}\times T^3)/\mathbb{Z}_k$, with the $\mathbb{Z}_k$ acting non-geometrically on the T^3.

Note that, as opposed to the $\mathcal{N}=3$ action we constructed before, this is a non-geometric action **transverse** to the $(0,2)$ theory.

Preserved supersymmetry

We have M-theory on $\mathbb{R}^{1,5}\times(\mathbb{C}\times T^3)$, so the supercharges transform as

$$\left(S_6^+,\frac{1}{2},\frac{1}{2},\mathbf{2}\right)\oplus\left(S_6^+,-\frac{1}{2},-\frac{1}{2},\mathbf{2}\right)\oplus\left(S_6^-,\frac{1}{2},-\frac{1}{2},\mathbf{2}\right)\oplus\left(S_6^-,-\frac{1}{2},\frac{1}{2},\mathbf{2}\right)$$

under $SO(1,5)\times U(1)_{\mathbb{C}}\times U(1)_\rho\times SU(2)$, where

- $S_6^\pm$ are the positive/negative chirality spinors in six dimensions.
- $U(1)_{\mathbb{C}}$ is the rotation group in $\mathbb{C}$.
- $U(1)_\rho\times SU(2)$ is the maximal compact subgroup of the (continuous version of the) duality group $SL(2,\mathbb{Z})_\rho\times SL(3,\mathbb{Z})$.

Preserved supersymmetry

Under the geometric rotation in $\mathbb{C}$:

$$\mathbb{Z}_p^{\mathbb{C}} \; : \; \begin{array}{rcl} \left(S_6^+, \pm\frac{1}{2}, \pm\frac{1}{2}, \mathbf{2}\right) & \rightarrow & e^{\pm i\pi/p}\left(S_6^+, \pm\frac{1}{2}, \pm\frac{1}{2}, \mathbf{2}\right) \\ \left(S_6^-, \pm\frac{1}{2}, \mp\frac{1}{2}, \mathbf{2}\right) & \rightarrow & e^{\pm i\pi/p}\left(S_6^-, \pm\frac{1}{2}, \mp\frac{1}{2}, \mathbf{2}\right) \, . \end{array} \tag{51}$$

On the other hand, under the non-geometric monodromy:

$$\mathbb{Z}_p^{\rho} \; : \; \begin{array}{rcl} \left(S_6^+, \pm\frac{1}{2}, \pm\frac{1}{2}, \mathbf{2}\right) & \rightarrow & e^{\pm i\pi/p}\left(S_6^+, \pm\frac{1}{2}, \pm\frac{1}{2}, \mathbf{2}\right) \\ \left(S_6^-, \pm\frac{1}{2}, \mp\frac{1}{2}, \mathbf{2}\right) & \rightarrow & e^{\mp i\pi/p}\left(S_6^-, \pm\frac{1}{2}, \mp\frac{1}{2}, \mathbf{2}\right) \, . \end{array} \tag{52}$$

Clearly S_6^- is preserved, while S_6^+ is projected out. I.e. as expected we end up with a $(0,2)$ theory in six dimensions.

Superconformal limit

The superconformal 6d theory is obtained, in the IIB frame, when the size of the fibered torus is sent to infinity. More precisely, $\rho_{T^2} \to \infty$ with τ_{T^2} and τ_{IIB} held constant.

In terms of the M-theory data, this corresponds (for E_7) to taking the limit $R \to 0$ in

$$R_A = R^{-2} c^{-1}\,, \qquad R_T = R\, c^{-1}\,, \qquad R_M = R\, c^2\,, \tag{53}$$

where $c = (g_{IIB})^{\frac{1}{3}}$ and R_A, R_T, R_M are the radii of $T^3 = S^1_A \times S^1_T \times S^1_M$ in the M-theory metric.

We again find a T^2 that contracts, and a normal direction that decompactifies. But notice that these are directions **normal** to the 6d theory.

Exceptional $\mathcal{N} = 3$ theories

To construct exceptional $\mathcal{N} = 3$ theories we combine both quotients,[4] i.e. we need to consider M-theory on

$$\mathbb{R}^4 \times (T^5 \times \mathbb{C})/(\mathbb{Z}_k \times \mathbb{Z}_p) \tag{54}$$

where the actions are non-geometric on the $T^5 = S^1_a \times S^1_b \times S^1_c \times S^1_d \times S^1_e$, and we will denote the different subtori as $T^2_{ab} = S^1_a \times S^1_b$, etc.

[4] Assuming that the birational transformation survives the $\mathcal{N} = 3$ quotient.

Exceptional $\mathcal{N}=3$ theories

E-type quotient: $\mathbb{R}^{1,3} \times T^2_{ab} \times (T^3_{cde} \times \mathbb{C})/\mathbb{Z}^E_p$, where

$$\mathbb{Z}^E_p = \mathbb{Z}^{\mathbb{C}}_p \cdot \mathbb{Z}^{\rho}_p \,, \tag{55}$$

S-fold quotient: $\mathbb{R}^{1,3} \times (T^3_{abc} \times T^2_{de} \times \mathbb{C})/\mathbb{Z}^S_k$, where

$$\mathbb{Z}^S_k = \mathbb{Z}^R_k \cdot \tilde{\mathbb{Z}}^R_k \cdot \mathbb{Z}^{\tau}_k \,. \tag{56}$$

Four dimensional limit

We can take a decoupling limit compatible with the orbifold actions. For instance, for $k = p = 4$

$$R_a = R_b = R_d = R_e = R \,, \qquad R_c = R^{-2} \,. \tag{57}$$

The four dimensional theory is reached when $R \to 0$, where the 6d $(0, 2)$ theory becomes a SCFT, and the T^2 where it lives contracts to zero size.

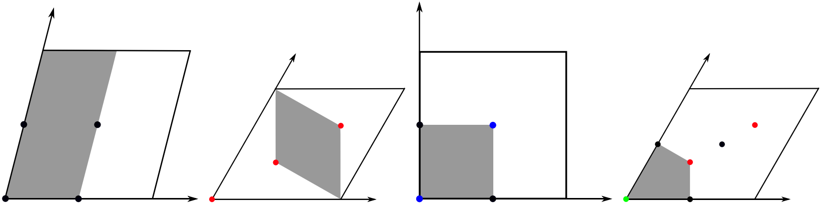
By an explicit embedding of the supersymmetry generators in the duality group $O(5, 5; \mathbb{Z})$, and using the formulas of [Kumar, Vafa '96], we can compute that the combined action $\mathbb{Z}_p \times \mathbb{Z}_k$ preserves 12 supercharges.

Some remarks

- As discussed in [Kumar, Vafa '96], if we compactify the configuration above on an extra S^1 (down to 3d), the U-duality action above can be conjugated to an ordinary T-duality. I.e. the 3d reduction of the new $\mathcal{N} = 3$ theories can be understood as coming from an asymmetric orbifold of IIA.
- While both $\mathbb{Z}_k$ and $\mathbb{Z}_p$ can separately be dualized into a geometric representation, I don't know of a way of doing so for the combined orbifold.

Potential OF3 planes

From the M-theory perspective we can classify all possible D3 charges for OF3 planes.



$\mathbb{Z}_2$: τ arb. — $\mathbb{Z}_2^{\oplus 4}$

$\mathbb{Z}_3$: $\tau = e^{2\pi i/3}$ — $\mathbb{Z}_3^{\oplus 3}$

$\mathbb{Z}_4$: $\tau = i$ — $\mathbb{Z}_2 \oplus \mathbb{Z}_4^{\oplus 2}$

$\mathbb{Z}_6$: $\tau = e^{2\pi i/3}$ — $\mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_6$

Around each $\mathbb{C}^4/\mathbb{Z}_k$ fixed point we can turn on a discrete F_4 flux valued in $H^4(S^7/\mathbb{Z}_k, \mathbb{Z}) = \mathbb{Z}_k$.

Potential OF3 planes

From here we can compute the M2 charge around each fixed point. If the torsion is trivial this comes just from curvature [Bergman, Hirano '09]

$$Q(\mathsf{OM}_{k,0}) = -\frac{\chi(\mathbb{C}^4/\mathbb{Z}_k)}{24} = -\frac{1}{24}\left(k - \frac{1}{k}\right) . \tag{58}$$

The contribution from a $p \in H^4(S^7/bZ_k, \mathbb{Z})$ flux gives an additional term [Aharony, Hashimoto, Hirano, Ouyang '09]

$$Q(\mathsf{OM}_{k,p}) = Q(\mathsf{OM}_{k,0}) + \frac{p(k-p)}{2k}. \tag{59}$$

Potential OF3 planes

Orientifold	Charges
OF_2	$-\frac{1}{4}, 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}$
OF_3	$-\frac{1}{3}, 0, \frac{1}{3}, \frac{2}{3}$
OF_4	$-\frac{3}{8}, -\frac{1}{8}, 0, \frac{1}{8}, \frac{1}{4}, \frac{3}{8}, \frac{1}{2}, \frac{5}{8}, \frac{3}{4}, \frac{7}{8}$
OF_6	$-\frac{5}{12}, -\frac{1}{6}, -\frac{1}{12}, 0, \frac{1}{6}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{7}{12}, \frac{2}{3}, \frac{5}{6}, \frac{11}{12}$

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But notice!

Not all of these M-theory settings lift to non-trivial orientifolds in IIB! [Aharony, Tachikawa '16]

Classification results

The proper classification was achieved by [Aharony, Tachikawa '16].

$$H^3(S^5/\mathbb{Z}_k, (\mathbb{Z} \oplus \mathbb{Z})_\rho) = \begin{cases} \mathbb{Z}_2 \oplus \mathbb{Z}_2 & (k = 2) \\ \mathbb{Z}_3 & (k = 3) \\ \mathbb{Z}_2 & (k = 4) \\ \mathbb{Z}_1 & (k = 6) \end{cases} \tag{60}$$

or alternatively, directly seeing which fluxes lift in F-theory to non-shift orientifolds.