3–Loop Moments of the Heavy Flavor Contributions to $F_2(x, Q^2)$ for $Q^2 \gg m^2$

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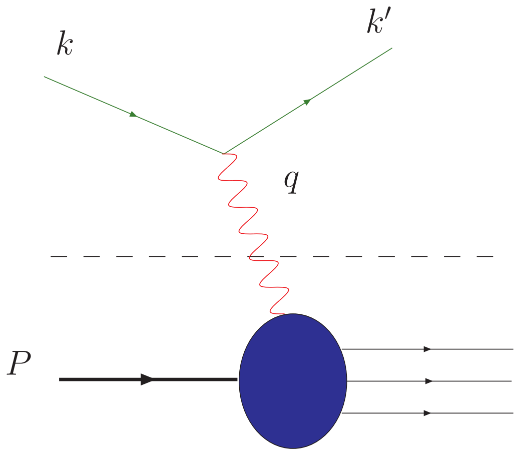
- Introduction and Theory Status
- The Method
- Renormalization
- Asymptotic 3 Loop Results (Fixed Moments) & Anomalous Dimensions
- Conclusions

1. Introduction

Deep–Inelastic Scattering (DIS):



$\longrightarrow \quad L_{\mu\nu}$

$\longrightarrow \quad W_{\mu\nu}$

$$Q^2 := -q^2, \quad x := \frac{Q^2}{2pq} \quad \text{Bjorken–x}$$

$$\nu := \frac{Pq}{M} \ ,$$

$$\frac{d\sigma}{dQ^2\,dx} \sim W_{\mu\nu} L^{\mu\nu}$$

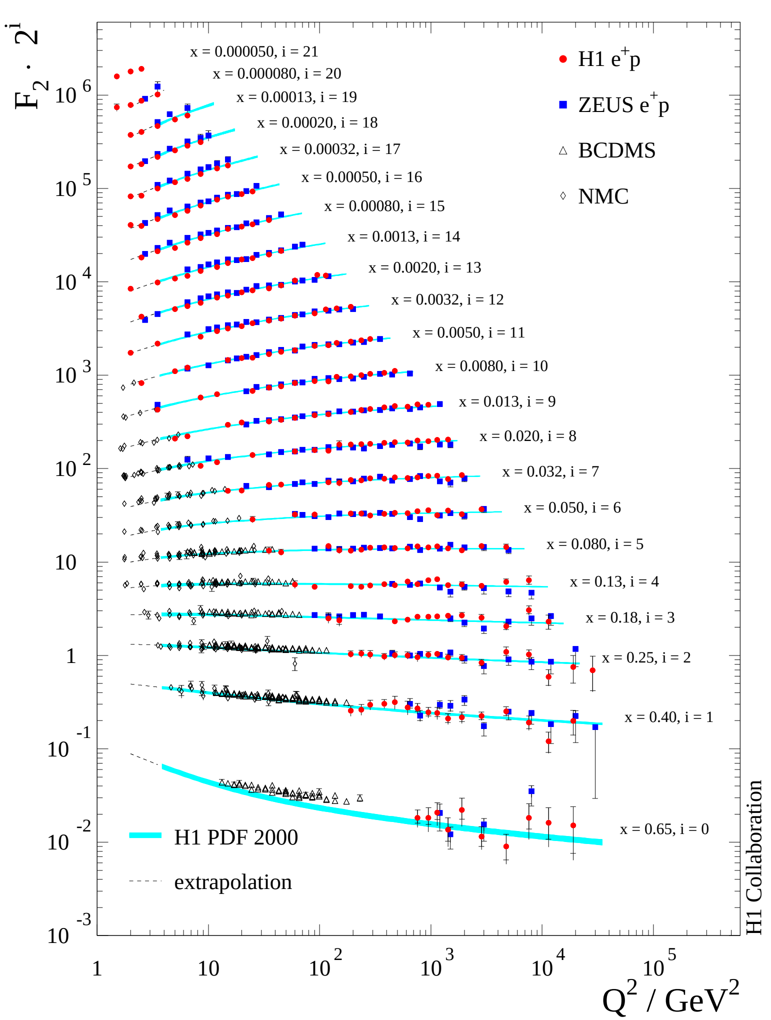
$$W_{\mu\nu}(q,P,s) = \frac{1}{4\pi}\int d^4\xi \exp(iq\xi)\langle P,s \mid [J_\mu^{em}(\xi), J_\nu^{em}(0)] \mid P,s\rangle$$

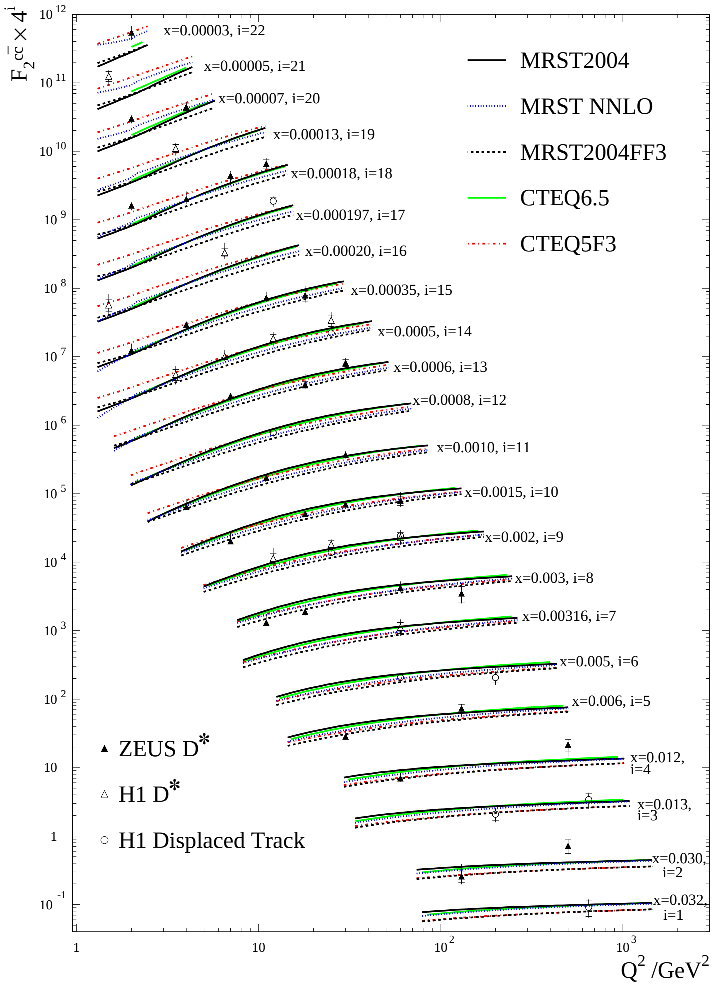
$$\text{unpol.}\left\{ \ = \frac{1}{2x}\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) F_L(x,Q^2) + \frac{2x}{Q^2}\left(P_\mu P_\nu + \frac{q_\mu P_\nu + q_\nu P_\mu}{2x} - \frac{Q^2}{4x^2} g_{\mu\nu}\right) F_2(x,Q^2)\right.$$

$$\text{pol.}\left\{ \quad - \frac{M}{2Pq}\varepsilon_{\mu\nu\alpha\beta} q^\alpha \left[s^\beta g_1(x,Q^2) + \left(s^\beta - \frac{sq}{Pq}p^\beta\right) g_2(x,Q^2)\right] \ .\right.$$

Structure Functions: $F_{2,L}$, $\quad g_{1,2}$

contain light and heavy quark contributions.





[Thompson, 2007]

- High statistics for F_2 and $F_2^{c\bar{c}}$ $\Longrightarrow$ Accuracy will increase in the future.
- Different scaling violations.

Leading Order : $F_{2,L}(x, Q^2)$ [Witten, 1976; Babcock, Sivers, 1978; Shifman, Vainshtein, Zakharov, 1978; Leveille, Weiler, 1979; Glück, Reya, 1979; Glück, Hoffmann, Reya, 1982.]

Leading Order : $g_1(x, Q^2)$ [Watson, 1982; Glück, Reya, Vogelsang, 1991; Vogelsang, 1991]

Leading Order : $g_2(x, Q^2)$ [Blümlein, Ravindran, van Neerven, 2003]

Soft resummation: $F_{2,L}(x, Q^2)$ [Laenen & Moch, 1998; Alekhin & Moch, 2008]

Next-to-Leading Order : $F_{2,L}(x, Q^2)$ [Laenen, Riemersma, Smith, van Neerven, 1993, 1995]
asymptotic: [Buza, Matiounine, Smith, Migneron, van Neerven, 1996; Bierenbaum, Blümlein, S.K., 2007]
Mellin–space expressions: [Alekhin, Blümlein, 2003.].
Next-to-Leading Order : $g_1(x, Q^2)$ asymptotic: [Buza, Matiounine, Smith, Migneron, van Neerven, 1997; Bierenbaum, Blümlein, S.K., 2009]

$\Longrightarrow$ 3-Loop corrections needed to line up with the accuracy reached for light flavor contributions.

2. The Method

- massless RGE and light–cone expansion in Bjorken–limit $\{Q^2,\ \nu\} \ \rightarrow \ \infty,\ x$ fixed:

$$\lim_{\xi^2\rightarrow 0}\left[J(\xi), J(0)\right] \propto \sum_{i,N,\tau} c_{i,\tau}^N(\xi^2,\mu^2)\xi_{\mu_1}...\xi_{\mu_N} O_{i,\tau}^{\mu_1\cdots\mu_m}(0,\mu^2)\ .$$

- Operators: flavor non-singlet (≤ 3), pure–singlet and gluon; consider leading twist.
- RGE for collinear singularities: mass factorization of the structure functions into Wilson coefficients and parton densities:

$$F_i(x,Q^2) = \sum_j \underbrace{C_i^j\left(x,\frac{Q^2}{\mu^2}\right)}_{\text{perturbative}} \otimes \underbrace{f_j(x,\mu^2)}_{\text{non-perturbative}}$$

- Light-flavor Wilson coefficients: process dependent ($O(a_s^3)$: [Moch, Vermaseren, Vogt, 2005.])

$$C_{(2,L);i}^{\text{fl}}\left(\frac{Q^2}{\mu^2}\right) = \delta_{i,q} + \sum_{l=1}^{\infty} a_s^l C_{(2,L),i}^{\text{fl},(l)}, \qquad i = q, g$$

- Heavy quark contributions given by heavy quark Wilson coefficients, $H_{(2,L),i}^{\text{S,NS}}\left(\frac{Q^2}{\mu^2},\frac{m^2}{Q^2}\right)$.

- In the limit $Q^2 \gg m_h^2$ $[Q^2 \approx 10\, m^2$ for $F_2, g_1]$:
massive RGE, derivative $m^2\partial/\partial m^2$ acts on Wilson coefficients only: all terms but power corrections calculable through partonic operator matrix elements, $\langle i|A_l|j\rangle$, which are process independent objects!

$$H_{(2,L),i}^{\rm S,NS}\left(\frac{Q^2}{\mu^2},\frac{m^2}{\mu^2}\right) = \underbrace{A_{k,i}^{\rm S,NS}\left(\frac{m^2}{\mu^2}\right)}_{\text{massive OMEs}} \otimes \underbrace{C_{(2,L),k}^{\rm S,NS}\left(\frac{Q^2}{\mu^2}\right)}_{\text{light–parton–Wilson coefficients}} .$$

- holds for polarized and unpolarized case. OMEs obey expansion

$$A_{k,i}^{\rm S,NS}\left(\frac{m^2}{\mu^2}\right) = \langle i|O_k^{\rm S,NS}|i\rangle = \delta_{k,i} + \sum_{l=1}^{\infty} a_s^l A_{k,i}^{{\rm S,NS},(l)}\left(\frac{m^2}{\mu^2}\right), \qquad i = q, g$$

[Buza, Matiounine, Migneron, Smith, van Neerven, 1996; Buza, Matiounine, Smith, van Neerven, 1997.]

- Heavy OMEs occur as well as transition functions to define the VFNS starting from a fixed flavor number scheme(FFNS).

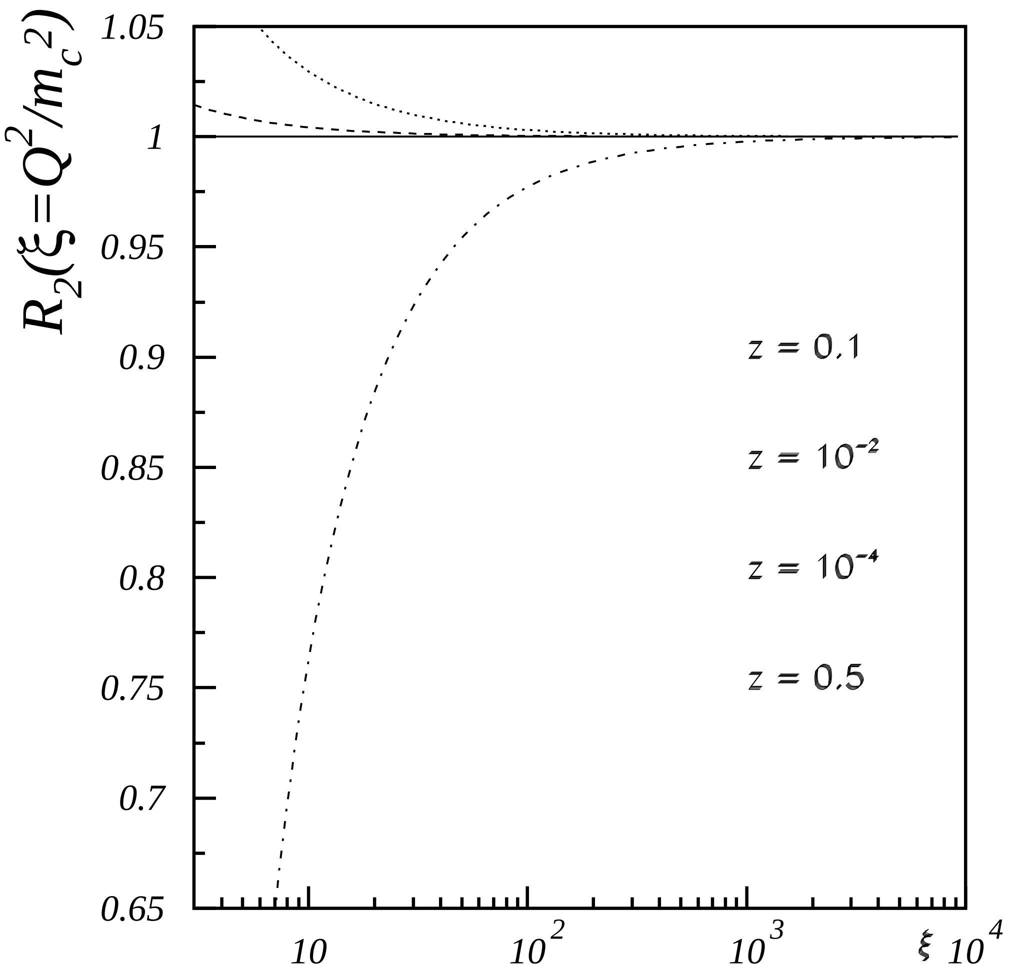
[Buza, Matiounine, Smith, van Neerven, 1998; Chuvakin, Smith, van Neerven, 1998.]

- Comparison for LO:

$$R_2\left(\xi \equiv \frac{Q^2}{m^2}\right) \quad \equiv \quad \frac{H^{(1)}_{2,g}}{H^{(1)}_{2,g,(asym)}} \; .$$

- Comparison to exact order $O(a_s^2)$ result: asymptotic formulae valid for $Q^2 \gtrsim 20$ $(\mathrm{GeV}/c)^2$ in case of $F_2^{c\bar{c}}(x, Q^2)$ and $Q^2 \gtrsim 800$ $(\mathrm{GeV}/c)^2$ for $F_L^{c\bar{c}}(x, Q^2)$

- Drawbacks:
  - Power corrections $(Q^2/m^2)^k$ cannot be calculated using this method.
  - The case of two different heavy quark masses is still too complicated $\Longrightarrow$ 2 scale problem to be treated semi-analytically.
  - Only inclusive quantities can be calculated $\Longrightarrow$ structure functions.



3. Renormalization

- Mass renormalization (on–mass shell scheme)
- Charge renormalization: MOM scheme for the gluon propagator.
  MOM scheme $\rightarrow$ $\overline{\text{MS}}$ scheme:

$$a_s^{\text{MOM}} = a_s^{\overline{\text{MS}}} - \beta_{0,Q} \ln\left(\frac{m^2}{\mu^2}\right) {a_s^{\overline{\text{MS}}}}^2 + \left[\beta_{0,Q}^2 \ln^2\left(\frac{m^2}{\mu^2}\right) - \beta_{1,Q} \ln\left(\frac{m^2}{\mu^2}\right) - \beta_{1,Q}^{(1)}\right] {a_s^{\overline{\text{MS}}}}^3 . \quad (1)$$

- Renormalization of ultraviolet singularities
  $\Longrightarrow$ are absorbed into Z–factors given in terms of anomalous dimensions γ_{ij}.
- Factorization of collinear singularities
  $\Longrightarrow$ are factored into Γ–factors Γ_{NS}, $\Gamma_{ij,S}$ and $\Gamma_{qq,PS}$.
  Γ–matrices apply to parts of the diagrams with massless lines only .

Generic formula for operator renormalization and mass factorization:

$$A_{ij} = Z_{il}^{-1} \hat{A}_{lm} \Gamma_{mj}^{-1}$$

$\Longrightarrow$ $O(\varepsilon)$–terms of the 2–loop OMEs are needed for renormalization at 3–loops.

4. Fixed moments at 3–Loop

$F_L^{Q\overline{Q}}(x, Q^2)$ @ 3-loop: [Blümlein, De Freitas, S.K., van Neerven, Nucl. Phys. B755 (2006) 272.]

Contributing OMEs:

Singlet	A_{Qg}	$A_{qg,Q}$	$A_{gg,Q}$	$A_{gq,Q}$	} mixing
Pure–Singlet		$A_{Qq}^{\rm PS}$	$A_{qq,Q}^{\rm PS}$		
Non–Singlet		$A_{qq,Q}^{\rm NS,+}$	$A_{qq,Q}^{\rm NS,-}$		

- All 2–loop $O(\varepsilon)$–terms in the unpolarized case are known.
- Unpolarized anomalous dimensions are known up to $O(a_s^3)$ [Moch, Vermaseren, Vogt, 2004.]
  $\Longrightarrow$ All terms needed for the renormalization of unpolarized 3–Loop heavy OMEs are present.
  $\Longrightarrow$ The calculation provides first independent checks on $\gamma_{qg}^{(2)}$, $\gamma_{qq}^{(2),\rm PS}$ and on respective color projections of $\gamma_{qq}^{(2),\rm NS\pm}$, $\gamma_{gg}^{(2)}$ and $\gamma_{gq}^{(2)}$.
- The calculation proceeds in the same way in the polarized case.
- Number of Diagrams to be calculated:
  $A_{Q(q)q}^{(3),\rm PS}$: 132, $A_{qq}^{(3),\rm NS}$: 128, $A_{gq}^{(3)}$: 89, $A_{Q(q)g}^{(3)}$: 1498, $A_{gg,Q}^{(3)}$: 865.

Fixed moments using MATAD

- three–loop "self-energy" type diagrams with an operator insertion
- Extension: additional scale compared to massive propagators: Mellin variable N
- Genuine tensor integrals due to

$$\Delta^{\mu_1}...\Delta^{\mu_n} \langle p|O_{\mu_1...\mu_n}|p\rangle = \Delta^{\mu_1}...\Delta^{\mu_n} \langle p|S\, \bar{\Psi}\gamma_{\mu_1} D_{\mu_2}...D_{\mu_n}\Psi|p\rangle = A(N)\cdot(\Delta p)^N$$

$$D_\mu = \partial_\mu - i g t_a A_\mu^a \quad , \qquad \Delta^2 = 0.$$

- Construction of a projector to obtain the desired moment in N [undo Δ-contraction]
- 3–loop OMEs are generated with QGRAF [Nogueira, 1993.]
- Color factors are calculated with [van Ritbergen, Schellekens, Vermaseren, 1998.]
- Translation to suitable input for MATAD [Steinhauser, 2001.]

Tests performed:

- 2–loop calculations for $N = 2, ..., 12$ were repeated in general gauge
  $\longrightarrow$ agreement with our previous calculation.
- Several non–trivial scalar 3–loop diagrams were calculated using Feynman–parameters for all N
  $\longrightarrow$ agreement with MATAD.

General structure of the result: the PS –case

$$
\begin{aligned}
A_{Qq}^{(3),\mathrm{PS}} + A_{qq,Q}^{(3),\mathrm{PS}} = & \frac{\hat{\gamma}_{qg}^{(0)}\gamma_{gq}^{(0)}}{48}\Biggl\{\gamma_{gg}^{(0)} - \gamma_{qq}^{(0)} + 4(n_f+1)\beta_{0,Q} + 6\beta_0\Biggr\}\ln^3\left(\frac{m^2}{\mu^2}\right) \\
& + \Biggl\{\frac{\hat{\gamma}_{qq}^{(1),\mathrm{PS}}}{2}\Bigl((n_f+1)\beta_{0,Q} - \beta_0\Bigr) + \frac{\hat{\gamma}_{qg}^{(0)}}{8}\Bigl((n_f+1)\hat{\gamma}_{gq}^{(1)} - \gamma_{gq}^{(1)}\Bigr) - \frac{\gamma_{gq}^{(0)}\hat{\gamma}_{qg}^{(1)}}{8}\Biggr\}\ln^2\left(\frac{m^2}{\mu^2}\right) \\
& + \Biggl\{\frac{\hat{\gamma}_{qq}^{(2),\mathrm{PS}}}{2} - \zeta_2\frac{\hat{\gamma}_{qg}^{(0)}\gamma_{gq}^{(0)}}{16}\Bigl(\gamma_{gg}^{(0)} - \gamma_{qq}^{(0)} + 4(n_f+1)\beta_{0,Q} + 6\beta_0\Bigr) - 2a_{Qq}^{(2),\mathrm{PS}}\beta_0 \\
& + \frac{n_f+1}{2}\hat{\gamma}_{qg}^{(0)}a_{gq,Q}^{(2)} - \frac{\gamma_{gq}^{(0)}}{2}a_{Qg}^{(2)}\Biggr\}\ln\left(\frac{m^2}{\mu^2}\right) + \zeta_3\frac{\gamma_{gq}^{(0)}\hat{\gamma}_{qg}^{(0)}}{48}\Bigl(\gamma_{gg}^{(0)} - \gamma_{qq}^{(0)} + 4n_f\beta_{0,Q} + 6\beta_0\Bigr) \\
& + \frac{\zeta_2}{16}\Bigl(-4n_f\beta_{0,Q}\hat{\gamma}_{qq}^{(1),\mathrm{PS}} + \hat{\gamma}_{qg}^{(0)}\gamma_{gq}^{(1)}\Bigr) + 4(\beta_0 + \beta_{0,Q})\overline{a}_{Qq}^{(2),\mathrm{PS}} + \gamma_{gq}^{(0)}\overline{a}_{Qg}^{(2)} - (n_f+1)\hat{\gamma}_{qg}^{(0)}\overline{a}_{gq,Q}^{(2)} \\
& + C_F\Bigl(-(4 + \frac{3}{4}\zeta_2)\hat{\gamma}_{qg}^{(0)}\gamma_{gq}^{(0)} - 4\hat{\gamma}_{qq}^{(1),\mathrm{PS}} + 12a_{Qq}^{(2),\mathrm{PS}}\Bigr) + a_{Qq}^{(3),\mathrm{PS}} + a_{qq,Q}^{(3),\mathrm{PS}} \ .
\end{aligned}
$$

- n_f–dependence non–trivial. Take all quantities at n_f flavors and adopt notation

$$
\hat{\gamma}_{ij} \equiv \gamma_{ij}(n_f+1) - \gamma_{ij}(n_f) \ , \quad \beta_{0,Q} \equiv \beta_0(n_f+1) - \beta_0(n_f) \ .
$$

- There are similar formulas for the remaining OMEs.

5. Results

- Using MATAD, we calculated the terms ($\approx$ 250 days of computer time)

$A_{Q(q)q}^{(3),\mathsf{PS}}, A_{gq,Q}^{(3)}: \quad (2,4,..,12); \quad A_{qq,Q}^{(3),\mathsf{NS}\pm}: \quad (2,3,..,12); \quad A_{Q(q)g}^{(3)}, A_{gg,Q}^{(3)}: \quad (2,4,..,10);$

and find agreement of the pole terms with the predictions obtained from renormalization.

- An additional check is provided by the sum rules for $N=2$

$$A_{qq,Q}^{(3),\mathsf{NS}} + A_{qq,Q}^{(3),\mathsf{PS}} + A_{Qq}^{(3),\mathsf{PS}} + A_{gq,Q}^{(3)} = 0~, \quad A_{Qg}^{(3)} + A_{qg,Q}^{(3)} + A_{gg,Q}^{(3)} = 0~.$$

- The moments $N=2,4$ were calculated in general R_ξ–gauge and we explicitly observe gauge–invariance for each OME at each loop order.

- All terms proportional to ζ_2 cancel in the renormalized result for the terms $A_{xq}^{(3)}$. If one chooses the $\overline{\mathrm{MS}}$–scheme for mass renormalization, they cancel in the terms $A_{xg}^{(3)}$ as well.

- We observe the number

$$\mathsf{B4} \quad = \quad -4\zeta_2 \ln^2 2 + \frac{2}{3}\ln^4 2 - \frac{13}{2}\zeta_4 + 16\mathsf{Li}_4\left(\frac{1}{2}\right) = -8\sigma_{-3,-1} + \frac{11}{2}\zeta_4~.$$

Result for the renormalized PS–term for $N = 4$.

$$
\begin{aligned}
&A_{Qq}^{(3),\mathrm{PS}} + A_{qq,Q}^{(3),\mathrm{PS}}\Big|_{N=4} = \Big\{-\frac{484}{2025}C_F{T_F}^2(2n_f+1) + \frac{4598}{3375}C_FC_AT_F - \frac{18997}{40500}{C_F}^2T_F\Big\}\ln^3\Big(\frac{m^2}{\mu^2}\Big) \\
&+\Big\{-\frac{16}{125}C_F{T_F}^2 + \frac{36751}{202500}C_FC_AT_F - \frac{697631}{405000}{C_F}^2T_F\Big\}\ln^2\Big(\frac{m^2}{\mu^2}\Big) + \Big\{-\frac{2131169}{303750}C_F{T_F}^2n_f \\
&-\frac{427141}{121500}C_F{T_F}^2 + \Big(-\frac{484}{75}\zeta_3 + \frac{24888821}{2700000}\Big)C_FC_AT_F + \Big(\frac{484}{75}\zeta_3 + \frac{63582197}{16200000}\Big){C_F}^2T_F\Big\}\ln\Big(\frac{m^2}{\mu^2}\Big) \\
&+\Big(\frac{7744}{2025}\zeta_3 - \frac{143929913}{27337500}\Big)C_F{T_F}^2n_f + \Big(-\frac{13552}{2025}\zeta_3 + \frac{218235943}{54675000}\Big)C_F{T_F}^2 + \Big(\frac{242}{225}\mathrm{B4} - \frac{242}{25}\zeta_4 \\
&+\frac{86833}{13500}\zeta_3 + \frac{4628174}{1265625}\Big)C_FC_AT_F + \Big(-\frac{484}{225}\mathrm{B4} + \frac{242}{25}\zeta_4 + \frac{298363}{20250}\zeta_3 - \frac{57518389433}{2187000000}\Big){C_F}^2T_F\ .
\end{aligned}
$$

The constant terms: $\mathrm{N} = 10 \quad \hat{\hat{a}}_{Qg}^{(3)} + \hat{\hat{a}}_{qg,Q}^{(3)}$:

$$
\begin{aligned}
\hat{\hat{a}}_{Qg}^{(3)}\Big|_{N=10} = \; & T_F \left(\frac{m^2}{\mu^2}\right)^{3\varepsilon/2} \Biggl\{ n_f T_F \Biggl(C_A \Biggl[-\frac{1505896}{245025}\zeta_3 + \frac{189965849}{188669250}\zeta_2 + \frac{297277185134077151}{1553283748170000} \Biggr] \\
& + C_F \Biggl[\frac{62292104}{13476375}\zeta_3 - \frac{49652772817}{93391278750}\zeta_2 - \frac{1178560772273339822317}{107642563748181000000} \Biggr] \Biggr) + C_A^2 \Biggl[-\frac{563692}{81675}\mathsf{B}_4 \\
& + \frac{483988}{9075}\zeta_4 - \frac{103652031822049723}{415451499724800}\zeta_3 - \frac{20114890664357}{581101290000}\zeta_2 \\
& + \frac{6830363463566924692253659}{685850575063965696000000} \Biggr] + C_A C_F \Biggl[\frac{1286792}{81675}\mathsf{B}_4 - \frac{643396}{9075}\zeta_4 \\
& - \frac{761897167477437907}{332361199779840000}\zeta_3 + \frac{15455008277}{660342375}\zeta_2 + \frac{872201479486471797889957487}{29928025093700321280000000} \Biggr] \\
& + C_F^2 \Biggl[-\frac{11808}{3025}\mathsf{B}_4 + \frac{53136}{3025}\zeta_4 + \frac{9636017147214304991}{7122025709568000}\zeta_3 + \frac{14699237127551}{15689734830000}\zeta_2 \\
& - \frac{247930147349635960148869654541}{148143724213816590336000000} \Biggr] + T_F C_A \Biggl[\frac{4206955789}{377338500}\zeta_2 + \frac{123553074914173}{5755172290560}\zeta_3 \\
& + \frac{2323118975810619964 5229}{6333973564804300800000} \Biggr] + T_F C_F \Biggl[-\frac{502987059528463}{113048027136000}\zeta_3 + \frac{24683221051}{46695639375}\zeta_2 \\
& - \frac{18319931182630444611912149}{1410892611560158003200000} \Biggr] - \frac{896}{1485} T_F^2 \zeta_3 \Biggr\} .
\end{aligned}
$$

$$
\begin{aligned}
\hat{\hat{a}}_{qg,Q}^{(3)}\Big|_{N=10} = \; & n_f T_F^2 \left(\frac{m^2}{\mu^2}\right)^{3\varepsilon/2} \Biggl\{ C_A \Biggl[-\frac{1505896}{245025}\zeta_3 + \frac{1109186999}{377338500}\zeta_2 + \frac{6542127929072987}{191763425700000} \Biggr] \\
& + C_F \Biggl[\frac{62292104}{13476375}\zeta_3 - \frac{83961181063}{93391278750}\zeta_2 - \frac{3538138549664428890 41}{2152851274963620000 00} \Biggr] \Biggr) \Biggr\}
\end{aligned}
$$

- We obtain e.g. for the moments of the $\hat{\gamma}_{qg}^{(2)}$ anomalous dimension

N	$\hat{\gamma}_{qg}^{(2)}/T_F$
2	$(1+2n_f)T_F\left(\frac{8464}{243}C_A - \frac{1384}{243}C_F\right) + \frac{\zeta_3}{3}\left(-416C_AC_F + 288C_A^2 + 128C_F^2\right) - \frac{7178}{81}C_A^2 + \frac{556}{9}C_AC_F - \frac{8620}{243}C_F^2$
4	$(1+2n_f)T_F\left(\frac{4481539}{303750}C_A + \frac{9613841}{3037500}C_F\right) + \frac{\zeta_3}{25}\left(2832C_A^2 - 3876C_AC_F + 1044C_F^2\right)$
	$-\frac{295110931}{3037500}C_A^2 + \frac{278546497}{2025000}C_AC_F - \frac{757117001}{12150000}C_F^2$
6	$(1+2n_f)T_F\left(\frac{86617163}{11668860}C_A + \frac{1539874183}{340341750}C_F\right) + \frac{\zeta_3}{735}\left(69864C_A^2 - 94664C_AC_F + 24800C_F^2\right)$
	$-\frac{58595443051}{653456160}C_A^2 + \frac{1199181909343}{8168202000}C_AC_F - \frac{2933980223981}{40841010000}C_F^2$
8	$(1+2n_f)T_F\left(\frac{10379424541}{2755620000}C_A + \frac{7903297846481}{1620304560000}C_F\right) + \zeta_3\left(\frac{128042}{1575}C_A^2 - \frac{515201}{4725}C_AC_F + \frac{749}{27}C_F^2\right)$
	$-\frac{24648658224523}{289340100000}C_A^2 + \frac{4896295442015177}{32406091200000}C_AC_F - \frac{4374484944665803}{56710659600000}C_F^2$
10	$(1+2n_f)T_F\left(\frac{1669885489}{988267500}C_A + \frac{1584713325754369}{323600780868750}C_F\right) + \zeta_3\left(\frac{1935952}{27225}C_A^2 - \frac{2573584}{27225}C_AC_F + \frac{70848}{3025}C_F^2\right)$
	$-\frac{21025430857658971}{255684567600000}C_A^2 + \frac{926990216580622991}{6040547909550000}C_AC_F - \frac{1091980048536213833}{13591232796487500}C_F^2$

- Agreement for the terms $\propto T_F$ of the anomalous dimensions $\gamma_{ij}^{(2),\mathsf{NS}^{\pm},\ \mathsf{S},\ \mathsf{PS}}$ with
  [Larin, Nogueira, Ritbergen, Vermaseren, 1997; Moch, Vermaseren, Vogt, 2004.]

- How far can we go ? $N = 14$ in some cases; generally: $N = 10 \Longrightarrow$ Phenomenology

- Unfortunately not enough to perform the automatic
  fixed moments $\rightarrow$ all moments turn. [Blümlein, Kauers, S.K., Schneider, 0902.4091 [hep-ph]].

- Recently with B. Tödtli: Calculation of moments $N = 1, ..., 11$ of the
  transversity heavy OMEs $A_{qq}^{h,(2,3)}$
  $\Longrightarrow$ Agreement with anomalous dimensions $\gamma_{qq}^{h,(1,2)}$ from
  [Kumano, 1997; 2–Loop: Hayashigaki, Kanazawa, Koike, 1997; Vogelsang, 1998; 3–Loop, N$\leq$ 8: Gracey, 2006]

6. Conclusions

- The heavy flavor contributions to the structure function F_2 are rather large in the region of lower values of x.
- QCD precision analyzes therefore require the description of the heavy quark contributions to 3–loop order.
- Complete analytic results are known in the region $Q^2 \gg m^2$ at NLO for $F_{2,L}(x, Q^2), g_{1,2}(x, Q^2)$. They are expressed in terms of massive operator matrix elements and the corresponding massless Wilson coefficients.
- $F_L(x, Q^2)$ is known to NNLO for $Q^2 \gg m^2$.
- The calculation of fixed moments of the massive operator matrix elements at $O(a_s^3)$ has been finished for $N = 10, 12$, which will be followed by some first phenomenological parameterization.
- We also calculate the matrix elements necessary to transform from the FFNS to the VFNS.

We obtain for the moments of the PS, and $\hat{\gamma}_{gq}^{(2)}$ anomalous dimensions

N	$\hat{\gamma}_{qq}^{(2),\mathsf{PS}}/T_F/C_F$
2	$-\frac{5024}{243}T_F(1+2n_f)+\frac{256}{3}\Big(C_F-C_A\Big)\zeta_3+\frac{10136}{243}C_A-\frac{14728}{243}C_F$
4	$-\frac{618673}{151875}T_F(1+2n_f)+\frac{968}{75}\Big(C_F-C_A\Big)\zeta_3+\frac{2485097}{506250}C_A-\frac{2217031}{675000}C_F$
6	$-\frac{126223052}{72930375}T_F(1+2n_f)+\frac{3872}{735}\Big(C_F-C_A\Big)\zeta_3+\frac{1988624681}{4084101000}C_A+\frac{11602048711}{10210252500}C_F$
8	$-\frac{13131081443}{13502538000}T_F(1+2n_f)+\frac{2738}{945}\Big(C_F-C_A\Big)\zeta_3-\frac{343248329803}{648121824000}C_A+\frac{39929737384469}{22684263840000}C_F$
10	$-\frac{265847305072}{420260754375}T_F(1+2n_f)+\frac{50176}{27225}\Big(C_F-C_A\Big)\zeta_3-\frac{1028766412107043}{1294403123475000}C_A+\frac{839864254987192}{485401171303125}C_F$
12	$-\frac{2566080055386457}{5703275664286200}T_F(1+2n_f)+\frac{49928}{39039}\Big(C_F-C_A\Big)\zeta_3-\frac{69697489543846494691}{83039693672007072000}C_A$ $+\frac{86033255402443256197}{54806197823524667520}C_F$

N	$\hat{\gamma}_{gq}^{(2)}/T_F/C_F$
2	$\frac{2272}{81}T_F(1+2n_f)+\frac{512}{3}\Big(C_A-C_F\Big)\zeta_3+\frac{88}{9}C_A+\frac{28376}{243}C_F$
4	$\frac{109462}{10125}T_F(1+2n_f)+\frac{704}{15}\Big(C_A-C_F\Big)\zeta_3-\frac{799}{12150}C_A+\frac{14606684}{759375}C_F$
6	$\frac{22667672}{3472875}T_F(1+2n_f)+\frac{2816}{105}\Big(C_A-C_F\Big)\zeta_3-\frac{253841107}{145860750}C_A+\frac{20157323311}{2552563125}C_F$
8	$\frac{339184373}{75014100}T_F(1+2n_f)+\frac{1184}{63}\Big(C_A-C_F\Big)\zeta_3-\frac{3105820553}{1687817250}C_A+\frac{8498139408671}{2268426384000}C_F$
10	$\frac{1218139408}{363862125}T_F(1+2n_f)+\frac{7168}{495}\Big(C_A-C_F\Big)\zeta_3-\frac{18846629176433}{11767301122500}\Big)C_A+\frac{529979902254031}{323600780868750}C_F$
12	$\frac{13454024393417}{5222779912350}T_F(1+2n_f)+\frac{5056}{429}\Big(C_A-C_F\Big)\zeta_3-\frac{641904930781397 89}{488852199795960 00}C_A+\frac{1401404001326440151}{3495293228541114000}C_F$

The moments of the NS$^{\pm}$ $\hat{\gamma}_{qq}^{(2)}$ anomalous dimensions

N	$\hat{\gamma}_{qq}^{(2),\mathsf{NS},+}/T_F/C_F$
2	$-\frac{1792}{243}T_F(1+2n_f)+\frac{256}{3}\Big(C_F-C_A\Big)\zeta_3-\frac{12512}{243}C_A-\frac{13648}{243}C_F$
4	$-\frac{384277}{30375}T_F(1+2n_f)+\frac{2512}{15}\Big(C_F-C_A\Big)\zeta_3-\frac{8802581}{121500}C_A-\frac{165237563}{1215000}C_F$
6	$-\frac{160695142}{10418625}T_F(1+2n_f)+\frac{22688}{105}\Big(C_F-C_A\Big)\zeta_3-\frac{13978373}{171500}C_A-\frac{44644018231}{243101250}C_F$
8	$-\frac{38920977797}{2250423000}T_F(1+2n_f)+\frac{79064}{315}\Big(C_F-C_A\Big)\zeta_3-\frac{1578915745223}{18003384000}C_A-\frac{91675209372043}{420078960000}C_F$
10	$-\frac{27995901056887}{1497656506500}T_F(1+2n_f)+\frac{192880}{693}\Big(C_F-C_A\Big)\zeta_3-\frac{9007773127403}{97250422500}C_A-\frac{75522073210471127}{307518802668000}C_F$
12	$-\frac{65155853387858071}{3290351344780500}T_F(1+2n_f)+\frac{13549568}{45045}\Big(C_F-C_A\Big)\zeta_3-\frac{25478252190337435009}{263228107582440000}C_A-\frac{35346062280941906036867}{131745667845011220000}C_F$

N	$\hat{\gamma}_{qq}^{(2),\mathsf{NS},-}/T_F/C_F$
3	$-\frac{2569}{243}T_F(1+2n_f)+\frac{400}{3}\Big(C_F-C_A\Big)\zeta_3-\frac{62249}{972}C_A-\frac{203627}{1944}C_F$
5	$-\frac{431242}{30375}T_F(1+2n_f)+\frac{2912}{15}\Big(C_F-C_A\Big)\zeta_3-\frac{38587}{500}C_A-\frac{5494973}{33750}C_F$
7	$-\frac{1369936511}{83349000}T_F(1+2n_f)+\frac{8216}{35}\Big(C_F-C_A\Big)\zeta_3-\frac{2257057261}{26671680}C_A-\frac{3150205788689}{15558480000}C_F$
9	$-\frac{20297329837}{1125211500}T_F(1+2n_f)+\frac{16720}{63}\Big(C_F-C_A\Big)\zeta_3-\frac{126810403414}{1406514375}C_A-\frac{1630263834317}{7001316000}C_F$
11	$-\frac{28869611542843}{1497656506500}T_F(1+2n_f)+\frac{1005056}{3465}\Big(C_F-C_A\Big)\zeta_3-\frac{1031510572686647}{10892047320000}C_A-\frac{1188145134622636787}{4612782040020000}C_F$

The moments of the $\hat{\gamma}_{gg}^{(2)}$ anomalous dimension

N	$\hat{\gamma}_{gg}^{(2)}/T_F$
2	$(1+2n_f)T_F\left(-\frac{8464}{243}C_A+\frac{1384}{243}C_F\right)+\frac{\zeta_3}{3}\left(-288C_A^2+416C_AC_F-128C_F^2\right)+\frac{7178}{81}C_A^2-\frac{556}{9}C_AC_F+\frac{8620}{243}C_F^2$
4	$(1+2n_f)T_F\left(-\frac{757861}{30375}C_A-\frac{979774}{151875}C_F\right)+\frac{\zeta_3}{25}\left(-6264C_A^2+6528C_AC_F-264C_F^2\right)$ $+\frac{53797499}{607500}C_A^2-\frac{235535117}{1012500}C_AC_F+\frac{2557151}{759375}C_F^2$
6	$(1+2n_f)T_F\left(-\frac{52781896}{2083725}C_A-\frac{560828662}{72930375}C_F\right)+\zeta_3\left(-\frac{75168}{245}C_A^2+\frac{229024}{735}C_AC_F-\frac{704}{147}C_F^2\right)$ $+\frac{9763460989}{116688600}C_A^2-\frac{9691228129}{32672808}C_AC_F-\frac{11024749151}{10210252500}C_F^2$
8	$(1+2n_f)T_F\left(-\frac{420970849}{16074450}C_A-\frac{6990254812}{843908625}C_F\right)+\zeta_3\left(-\frac{325174}{945}C_A^2+\frac{327764}{945}C_AC_F-\frac{74}{27}C_F^2\right)$ $+\frac{2080130771161}{25719120000}C_A^2-\frac{220111823810087}{648121824000}C_AC_F-\frac{14058417959723}{5671065960000}C_F^2$
10	$(1+2n_f)T_F\left(-\frac{2752314359}{101881395}C_A-\frac{3631303571944}{420260754375}C_F\right)+\zeta_3\left(-\frac{70985968}{190575}C_A^2+\frac{71324656}{190575}C_AC_F-\frac{5376}{3025}C_F^2\right)$ $+\frac{43228502203851731}{549140719050000}C_A^2-\frac{337408133551712319}{90608218643250000}C_FC_A-\frac{3009386129483453}{970802342606250}C_F^2$