Introduction to lattice field theory

..not really (1 hr!), but a few points hopefully of interest here: 12 th SFB meeting 23.3.2009

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Rôle of the functional integral

$$\langle\varphi(x_1)\varphi(x_2)\cdots\varphi(x_n)\rangle = \frac{1}{Z}\int D\varphi\,\mathrm{e}^{-S[\varphi]}\varphi(x_1)\varphi(x_2)\cdots\varphi(x_n)$$

Formal:

- infinite Euclidean space time (Wick $\to$ Minkowski)
- Feynman rules from $S[\varphi]$ $\longrightarrow$ all orders PT
  - depends on split $S[\varphi] = S_0[\varphi] + g_0 S_I[\varphi]$, S_0 Gaussian
  - regularization ($D \neq 4$, μ,...), renormalization ($g_0 \to g_R, \ldots$), limit ($D \to 4, \ldots$)
- exists (renormalizability, finite correlations): known to all orders PT
- similar for gauge fields, fermions, ..., standard model
- $\Rightarrow$ coefficients of [expected] asymptotic expansion in g_R:
- $|\text{truth} - \sum_{k=0}^{l_{\text{sweat}}} c_k (g_R)^k| \propto (g_R)^{l_{\text{sweat}}+1}$ as $g_R \to 0$
  - a priori limited precision and limited sweat
- finite g_R: no convergence, c_k useful only up to some order
- $\int D\varphi \ldots$ has never been defined, PT of what? NP contributions?

$$\langle\varphi(x_1)\varphi(x_2)\cdots\varphi(x_n)\rangle = \frac{1}{Z}\int D\varphi\, \mathrm{e}^{-S[\varphi]}\varphi(x_1)\varphi(x_2)\cdots\varphi(x_n)$$

Lattice: define $\int D\varphi \ldots$ as a limit of finite dimensional integrals

- as Feynman did for QM $\equiv$ QFT with $D=1$
- finite volume V and finite spacing a
- evaluate exactly (if possible) or numerically or PT-expand
- in the end $a\to 0$ (and usually also $L\to\infty$) is necessary and difficult

lattice:	$x=(x_0,x_1,x_2,x_3)=a n_\mu \in a\mathbb{Z}^4$	$[x]=[a]=\mathrm{cm}$
torus:	$\varphi(x)$ T-periodic in x_0, L-periodic in $x_{1,2,3}$	$T/a, L/a$ integer

integrate over independent fields in volume $V=TL^3$

$$\int D\varphi \ldots \equiv \underbrace{\prod_x \int_{-\infty}^{\infty} d\varphi(x)}_{\frac{V}{a^4}-\mathrm{fold}} \ldots$$

- $S[\varphi]$ discretized. $\partial_\mu \to$ difference quotient, e.g. nearest neighbour
- continuum limit $=$ critical point: universality

Scales on the lattice

- a: UV-cutoff; L, T: IR-cutoff; phys. scale: mass $m \sim$ decay 2-point fct.
- it is usually assumed(?) that limits $a \to 0$ and $T, L \to \infty$ commute
- sometimes: $am \to 0$ at $mL = \#, mT = \#$ fixed: finite size scaling limit
- exists, universal under the same renormalization (counter terms)
- therefore: finite size effects = predictions of the continuum theory
  - realizable: finite T, Casimir situations, ?
- special case: mT finite, $mL = \infty$: finite temperature in ∞ volume

- limit $L \to \infty$ reached up to $\exp(-mL) \Rightarrow mL \approx 5$ usually okay
- limit $a \to 0$ reached up to $(am)^n, n = 1 \,\mathrm{or}\, 2 \Rightarrow$ more problematic

Discretization of gauge fields

gauge-covariant derivative (continuum):

$$D_\mu \psi = \partial_\mu \psi + i A_\mu \psi = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left[\underbrace{\mathrm{e}^{i\varepsilon A_\mu}}_{p.\mathrm{trans.}} \psi(x + \varepsilon \hat{\mu}) - \psi(x) \right]$$

has curvature

$$\left(\mathrm{e}^{i\varepsilon A_\mu(x)} \mathrm{e}^{i\delta A_\nu(x+\varepsilon\hat{\mu})} - \mathrm{e}^{i\delta A_\nu(x)} \mathrm{e}^{i\varepsilon A_\mu(x+\delta\hat{\nu})}\right)\psi \simeq \left(\mathrm{e}^{i\varepsilon\delta F_{\mu\nu}(x)} - 1\right)\psi$$

gauge invariant action

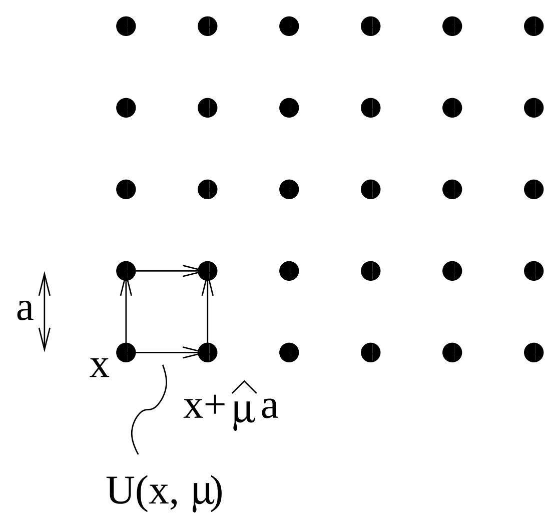
$$S[A_\mu] = \int d^4x \, \frac{1}{2} \, \mathrm{tr}\left(F_{\mu\nu} F_{\mu\nu}\right)$$

Lattice:

- smallest parallel transport: $x \leftarrow x + a\hat{\mu}$, nearst neighbor
- $\mathrm{e}^{i\,a\,A_\mu} \rightarrow U(x,\mu)$, algebra $\rightarrow$ in group SU(3), living on links
- smallest measure of curvature, Wilson action:

$$R(x,\mu,\nu) = U(x,\mu)U(x+a\hat{\mu},\nu) - U(x,\nu)U(x+a\hat{\nu},\mu) \not\equiv 1$$

$$S[U] = \frac{1}{g_0^2} \underbrace{\sum_{x,\mu<\nu}}_{\text{plaquettes}} \mathrm{tr}(RR^\dagger), \quad Z = \underbrace{\prod_{x,\mu}}_{\text{links}} \int dU\, \mathrm{e}^{-S[U]}$$



- continuum limit at $g_0^2 \to 0 \leftrightarrow$ asymptotic freedom
- confinement of static quarks $\leftrightarrow$ area decay of Wilson loop observable
- string tension vanishes in PT, trivial to show analytically in the strong coupling expansion valid for large lattice spacing ($\neq$continuum limit)
- very precise numerical 'proof' close to the continuum (Yang Mills)

Discretization of fermions (quarks)

continuum, $T = L = \infty$, Grassmann:

$$S[\overline{\psi}, \psi] = \int d^4x\, \overline{\psi}\, (\not{D} + m)\psi, \quad \not{D} = \gamma_\mu D_\mu, \quad \{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}, \quad \gamma_\mu = (\gamma_\mu)^\dagger$$

free, $D_\mu = \partial_\mu$:

$$\langle \psi(x)\overline{\psi}(0)\rangle = \int \frac{d^4p}{(2\pi)^4}\, \mathrm{e}^{ip\cdot x} \frac{-i\not{p} + m}{p^2 + m^2}$$

pole $\leftrightarrow$ particle: $(-iE, \vec{p})^2 = -m^2 \ \to E^2 = \vec{p}^{\,2} + m^2$

lattice, naive, $T = L = \infty$:

$$S[\overline{\psi}, \psi] = a^4 \sum_x \overline{\psi} (\gamma_\mu \tilde{\partial}_\mu + m)\psi, \quad \tilde{\partial}_\mu \psi = \frac{1}{2a}[\psi(x + a\hat{\mu}) - \psi(x - a\hat{\mu})]$$

- $\tilde{\partial}_\mu$ symmetric $\rightarrow \tilde{\not\partial}$ antihermitian

$$\langle \psi(x)\overline{\psi}(0)\rangle = \int_{-\pi/a}^{\pi/a} \frac{d^4p}{(2\pi)^4}\, \mathrm{e}^{ip\cdot x} \frac{-i\mathring{\not p} + m}{\mathring{p}^{\,2} + m^2} \quad \text{note: momenta cut off!}$$

$$\mathring{p}_\mu = \frac{1}{a}\sin(ap_\mu) = p_\mu(1 - a^2p_\mu^2/6 + \ldots.)$$

pole: $\sin^2(i\underbrace{aE}_{\text{small}}) + \sum_k \sin^2(\underbrace{ap_k}_{\text{small}}) = -a^2m^2 \Rightarrow$

$$E^2 = (\vec{p}^{\,2} + m^2)[1 - a^2(\vec{p}^{\,2} + m^2)/3] - a^2/3 \sum_k p_k^4 + \mathrm{O}(a^4)$$

but also (doubler):

$$\sin^2(i\underbrace{aE}_{\text{small}} \pm \pi) + \sum_k \sin^2(\underbrace{ap_k}_{\text{small}} \pm \pi) = -a^2m^2$$

$\Longrightarrow$ 16 fermions appear in loops

Wilson fermions

modify the action:

$$S[\overline{\psi}\,,\psi]=a^4\sum_x\,\overline{\psi}\,(\gamma_\mu\tilde{\partial}_\mu+m-a\frac{r}{2}\underbrace{\partial_\mu\partial^*_\mu}_{\text{Laplacian}})\psi,$$

$$\langle\psi(x)\overline{\psi}\,(0)\rangle=\int_{-\pi/a}^{\pi/a}\frac{d^4p}{(2\pi)^4}\,\mathrm{e}^{ip\cdot x}\frac{-i\mathring{\not{p}}+M(p)}{\mathring{p}^{\,2}+M^2(p)}$$

$$M(p)=m+a\frac{r}{2}\hat{p}^2,\quad \hat{p}_\mu=\frac{2}{a}\sin(ap_\mu/2)=\begin{cases}p_\mu+\mathrm{O}(a^2) & \text{for}\quad ap_\mu\approx 0\\ 2/a & \text{for}\quad ap_\mu\approx\pi\end{cases}$$

- at physical momenta: $\mathrm{O}(a)$ modification
- at doubler momenta: $\mathrm{O}(a^{-1})$ mass
- $\{\gamma_5,M(p)\}\neq 0$ even at $m=0$ $\Rightarrow$ chiral symmetry broken by cutoff
- recovered in the continuum limit up to $\mathrm{O}(a)$ violations of Ward identities if we tune $am=am_c(g_0)\neq 0$
- value $r\neq 0$ irrelevant for continuum limit (but enters in counterterms)
- $r=1$ standard

Worked example, trivial but explicit...

...for $a \to 0$, $T, L \to \infty$ and all that... free Wilson fermion, first a, L, T finite, periodic with L, antiperiodic with T ($1/T =$ temperature)

$$\breve{\psi}(x_0, \vec{p}) = a^3 \sum_{\vec{x}} \mathrm{e}^{-i\vec{p}\cdot\vec{x}} \psi(x), \quad \breve{\bar{\psi}}(x_0, \vec{p}) = ..., \quad \text{time/momentum}$$

$$\langle \breve{\psi}(t, -\vec{p}) \breve{\bar{\psi}}(0, \vec{p}) \rangle \sim \mathrm{tr}[\underbrace{\mathrm{e}^{-(T-t)\mathbb{H}}}_{\to |0\rangle\langle 0|} c(-\vec{p}) \mathrm{e}^{-t\mathbb{H}} c^\dagger(\vec{p})] \sim \exp[-E(\vec{p})t]$$

necessary: $\vec{p} \in \frac{2\pi}{L}\mathbb{Z}^3$

$$G(t, \vec{p}) = L^{-3} \langle \breve{\psi}(t, -\vec{p}) \breve{\bar{\psi}}(0, \vec{p}) \rangle = \frac{1}{T} \sum_{p_0} \mathrm{e}^{ip_0 t} \frac{-i\mathring{\not{p}} + M(p)}{\mathring{p}^{\,2} + M^2(p)}$$

$p_0 = (2\pi/T)n_0, \quad n_0 = 1/2, 3/2, ..., T/a - 1/2$, Matsubara

exact summation (e.g. contour integral), scalar: $\mathrm{tr}\, G(t,\vec{p}\,) = 2\dfrac{M - a\hat{\omega}^2/2}{(1+aM)\mathring{\omega}}\, \sinh(\omega(T/2-t))/\cosh(\omega T/2) \quad (0<t<T)$

- $\omega = \sqrt{m^2+\vec{p}^{\,2}} + \mathrm{O}(a), \quad \hat{\omega}\,, \mathring{\omega} = \omega + \mathrm{O}(a^2)$
- $M = m + \frac{a}{2}\hat{\vec{p}}^{\,2} = m + \mathrm{O}(a)$

send $a \to 0$:

$$\mathrm{tr}\, G(t,\vec{p}\,) = 2\frac{m}{\omega}\, \sinh(\omega(T/2-t))/\cosh(\omega T/2), \quad \omega = \sqrt{m^2+\vec{p}^{\,2}}$$

send $T \to \infty$ [and $a \to 0$]

$$\mathrm{tr}\, G(t,\vec{p}\,) = 2\frac{M - a\hat{\omega}^2/2}{(1+aM)\mathring{\omega}}\, \exp(-\omega t)\Big[\to 2\frac{m}{\omega}\exp(-\omega t)\Big]$$

- $a \to 0$ convergence linear ($\to$ improvement, see below)
- $T \to \infty$ convergence exponential
- the limits commute indeed

more lessons to be learnt:

$$\text{for } \vec{p}=0: \qquad \omega = m_R = a^{-1}\ln(1+am) = m - \frac{1}{2}am^2 + \mathrm{O}(a^2)$$

- m_R is the pole-mass $\leftrightarrow \exp(-m_R t)$ and differs from bare m at $\mathrm{O}(a)$
- if we eliminate $m \to m_R$, then also for $\vec{p} \neq 0$:

$$\omega = \sqrt{m_R^2 + \vec{p}^{\,2}} + \mathrm{O}(a^2)$$

- the ugly prefactor:

$$[1+am] \times \frac{M - a\hat{\omega}^2/2}{(1+aM)\mathring{\omega}} = \frac{m_R}{\sqrt{m_R^2+\vec{p}^{\,2}}} + \mathrm{O}(a^2)$$

overall upshot:

$$-Z^{-1}L^{-3}\langle \breve{\bar{\psi}}(0,\vec{p})\breve{\psi}(t,-\vec{p})\rangle = \frac{m_R}{\omega}\sinh(\omega(T/2-t))/\cosh(\omega T/2) + \mathrm{O}(a^2)$$

$$Z^{-1} = 1 + am = \text{wavefunction `renormalization'}$$

Symanzik O(a) improvement

we have just encountered an example of it:

renormalization	improvement
action: all terms up to $D = 4$ compatible with symmetries	action: all terms up to $D = 5$ compatible with symmetries
eliminate divergencies from relations between physical quantities	eliminate O(a) terms from relations between physical quantities

we had the free theory only!

- $m \to m_R$ or $m\,\overline{\psi}\psi \to m(1+\frac{1}{2}am)\,\overline{\psi}\psi$, then $m_{\mathrm{pole}} = m + \mathrm{O}(a^2)$
- $\psi \to Z^{-1/2}\psi$
- also in interacting QCD [scope of a proof: all orders PT]
- renormalization- and improvement constants are needed NP (at least in principle)
- they can (and should!) be determined with finite $T, L \to$ SF
- equally nontrivial as renormalizability

Staggered fermions with and without rooting

'naive' fermions:

$$S[\overline{\psi}\,,\psi] = a^4 \sum_x \overline{\psi}\,(\gamma_\mu \tilde{D}_\mu + m)\psi \qquad \to 16 \text{ species}$$

$$(\tilde{D}_\mu \psi)(x) = \frac{1}{2}\big[U(x,\mu)\psi(x+\hat{\mu}) - U^\dagger(x-\hat{\mu}\,,\mu)\psi(x-\hat{\mu})\big] \quad (a=1 \text{ here})$$

$$S = S_m + \frac{1}{2}\sum_{x\,,\mu} \{\overline{\psi}\,(x)\gamma_\mu U(x,\mu)\psi(x+\hat{\mu}) - \overline{\psi}\,(x+\hat{\mu})\gamma_\mu U^\dagger(x,\mu)\psi(x)\}$$

'spin-diagonalization', $x = (x_0, x_1, x_2, x_3)$, $\quad x_\mu \equiv x_\mu / a$ integer

$$\psi(x) \to \gamma_0^{x_0}\gamma_1^{x_1}\gamma_2^{x_2}\gamma_3^{x_3}\psi(x), \qquad \overline{\psi}\,(x) \to \overline{\psi}\,(x)\gamma_3^{x_3}\gamma_2^{x_2}\gamma_1^{x_1}\gamma_0^{x_0}$$

the $\Rightarrow$

$$\overline{\psi}\,(x)\gamma_\mu U\psi(x+\hat{\mu}) \to \eta_\mu(x)\overline{\psi}\,(x)U\psi(x+\hat{\mu})$$

$$\eta_0(x) = 1, \quad \eta_1(x) = (-1)^{x_0}, \quad \eta_2(x) = (-1)^{x_0+x_1}, \quad \eta_3(x) = (-1)^{x_0+x_1+x_2}$$

- now $\eta_\mu(x)\overline{\psi}\,(x)U\psi(x+\hat{\mu})$ = same term for all 4 spinor components
- erase 3/4 of them: 16 species $\to$ 4 species ($4 \to 2$ in $D=2$)

- field $U(x,\mu)$ translation invariance $x \to x + a\hat{\mu}$
- field $\psi(x)$ translations $x \to x + 2a\hat{\mu}$ because of 'staggered' $\eta_\mu(x)$
- $\psi, \overline{\psi}$ have only one component per site
- 4 spinor fields 'spread out' over the 16 sites of a $4D$ hypercube
- translations $x \to x + a\hat{\mu} \leftrightarrow$ discrete subgroup of (broken) $\mathrm{SU}(4)_{\mathrm{taste}}$

Transformation, $\mathrm{U}(1) \times \mathrm{U}(1)$:

$$\psi(x) \to \mathrm{e}^{i\alpha + i\beta\eta_5(x)}\psi(x), \qquad \overline{\psi}(x) \to \mathrm{e}^{-i\alpha + i\beta\eta_5(x)}\overline{\psi}(x)$$

$$\eta_5(x) = (-1)^{x_0 + x_1 + x_2 + x_3}$$

- α: fermion number
- β: only a symmetry for $m = 0$ $[\overline{\psi}\psi(x) \to \mathrm{e}^{i2\beta\eta_5(x)}\overline{\psi}\psi(x)]$
- 'remnant chiral symmetry'
- sufficient to protect (distinguish by symmetry incl. cutoff) $m = 0$
- remember: Wilson $m_c(g_0)$ fine tuning for chiral point

idea A: use taste as flavor

- 4 'naturally degenerate' flavours not ideal
  - 1977 Banks et al.: Hamiltonian, only 2 'Kogut-Susskind' tastes
- mixing of broken taste and space $\Rightarrow$ big mess to build operators to 'tickle' well-defined quantum numbers (clear for Wilson)

more recent idea B, rooting:

$$\int D\psi D\overline{\psi}\, \mathrm{e}^{-\sum_x \overline{\psi} M[U]\psi} = \det M[U] \to (\det\mathrm{M}[U])^{1/4}$$

$\to$ one flavor, independent m, problem:

- predictive power of QFT, universality of critical points, artefacts only $\propto (am_{\mathrm{phys.}})^2$, depend on:
- local action/Hamiltonian (strict or exponentially decaying coefficients), $a =$ 'size' of the bare action
- although in simulations we (almost) always use det $M[U]$ (nonlocal $F[U]$) the existence of the local-action integral is relevant

$$(\det\mathrm{M}[U])^{1/4} = \int D\cancel{\psi} D\overline{\psi}\, \mathrm{e}^{-\sum_x \cancel{\overline{\psi}} X[U]\psi}$$

Twisted mass fermions

Wilson:

$$D_W(m) = (\gamma_\mu \tilde{D}_\mu + m - \frac{a}{2} D_\mu D^*_\mu), \quad m > m_c(g_0)$$

Although the Grassmann integral (and det) on any finite lattice

$$\int D\psi D\overline{\psi}\, \mathrm{e}^{-\sum_x \overline{\psi} D_W \psi}$$

is a polynomial in the elements of the matrix D_W, standard methods spend all their time with *inverting* D_W in the *fluctuating* $U(x, \mu)$

In the continuum $\not{D} + m$ has eigenvalues $i\lambda + m$, never 0 if $m > 0$,

- nothing strictly protects D_W from zero EVs at $m > m_c$
- this is an artefact in the sense that for $\lim_{a\to 0}$ at $m_\pi > 0$ this problem eventually goes away ('measure zero' (?) = does not occur in simulations of $\leqslant 10$ yrs duration?..)
- problem in practice

one way out:

- consider a degenerate doublet, set $m = m_c$ (simplified..)
- but introduce a different mass term:

$$D_W(m_c)\,1_{\text{iso}} + i\mu\gamma_5\,\tau_3 \quad \Rightarrow \det(D_W(m_c)1 + i\mu\gamma_5\tau_3) = \det(D_W^\dagger D_W + \mu^2)$$

- in the continuum $i\mu\gamma_5\,\tau_3$ could be transformed into a normal mass term
- not on the lattice $\Rightarrow$ different regularization, should have the same continuum limit
- different counter term structure, seems advantageous to study certain 4-fermion composite operators in QCD
- simplifications in Symanzik O(a) improvement
- cutoff breaks Isospin, Goldstones split up (a bit like taste..)

Ginsparg-Wilson-Neuberger fermions

No go theorem:

- we cannot have $\{\gamma_5, \not{D}_{\text{discretized}}\} = 0$ without doubling under reasonable assuptions

the long forgotten Ginsparg Wilson relation had replaced this by

$$\{\gamma_5, \not{D}\} = a\not{D}\gamma_5\not{D} \quad \Rightarrow \quad \{\gamma_5, \not{D}^{-1}(x,y)\} = \underbrace{a\gamma_5 a^{-4}\delta_{x,y}}_{\text{contact term}}$$

Neuberger realization:

$$\not{D} = a^{-1}\{1 - A(A^\dagger A)^{-1/2}\}, \quad A = 1 - a D_W(-s)$$

- obeys GW (simple algebra)
- local (not ultralocal, though)
- spectrum on a circle, with a massterm added it is protected
- the mere application requires $(A^\dagger A)^{-1/2}$ $\rightarrow$ ultraexpensive in HMC...

Lüscher modified chiral symmetry

$$\delta\psi = \varepsilon\gamma_5(1 - \frac{1}{2}a\not{D})\psi, \quad \delta\overline{\psi} = \varepsilon\overline{\psi}(1 - \frac{1}{2}a\not{D})\gamma_5$$

- is an exact symmetry of $a^4\sum_x \overline{\psi}\,\not{D}\psi$
- if $\not{D}$ obeys Ginsparg Wilson

there are more fermion discretizations

- domain wall (chiral fermions from $5D$)
- various of the above together with various kinds of
- smearing [plug links averaged in a gauge-covariant way into the fermion matrix]

critical remark:
there are few studies significantly varying the cutoff a [and nothing else at the same time] to check, if we are talking about continuum physics...

Concluions

- let's have coffee........