# Running coupling constant on the lattice: $\phi$ˆ4 theory revisited

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# $\varphi^4$ action

lattice functional integral $\rightarrow$ yesterday's lecture, now with

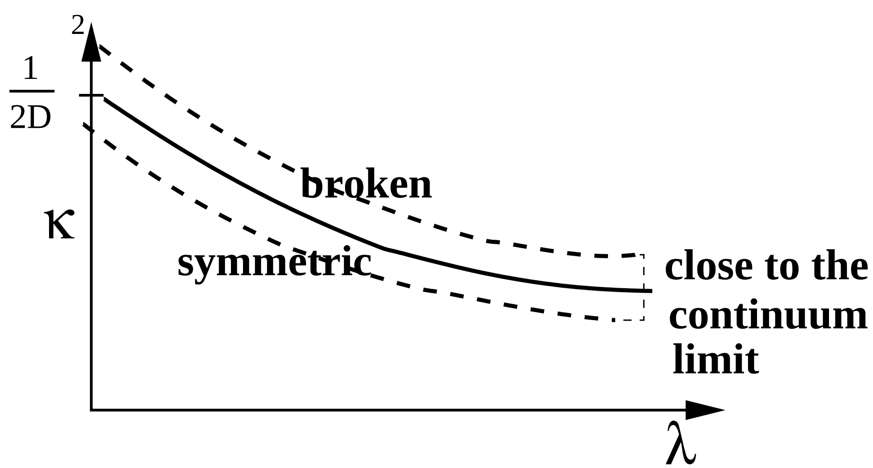
$$S[\varphi] = a^D \sum_x \left\{ \frac{1}{2}(\partial_\mu \varphi)^2 + \frac{m_0^2}{2}\varphi^2 + \frac{g_0}{4!}\varphi^4 \right\} \quad (\partial_\mu = n.n.\,\text{difference})$$

completely equivalent (field rescaled, dimensionless):

$$S[\varphi] = \underbrace{\sum_x \left\{\varphi(x)^2 + \lambda[\varphi(x)^2 - 1]^2\right\}}_{\text{on}-\text{site}} - 2\kappa \underbrace{\sum_{\langle x\, y\rangle} \varphi(x)\varphi(y)}_{\text{neighbors}}$$

with

$$g_0 = \frac{6\lambda}{\kappa^2}, \quad m_0^2\, a^2 = (1 - 2\lambda)/\kappa - 2D$$




continuum limit: $\kappa \rightarrow \kappa_c(\lambda)$
at fixed $\lambda$

Ising limit: $\lambda = \infty$
$\int\, d\varphi(x) \ldots \rightarrow \sum_{\varphi(x)=\pm 1}$

# Renormalized coupling, triviality

symmetric phase: $\kappa \nearrow \kappa_c(\lambda)$
renormalization conditions, lattice units $a=1$ (except sometimes):

$$\tilde{G}(p) = \sum_x \mathrm{e}^{-ip\cdot x} \langle \varphi(x)\varphi(0)\rangle, \quad \tilde{G}(0) = \frac{Z}{m^2}, \quad \tilde{G}(p_*) = \frac{Z}{\hat{p}_*{}^2 + m^2}$$

$$g_R = -\, m^D \frac{\sum_{x,y,z} \{\langle \varphi(x)\varphi(y)\varphi(z)\varphi(0)\rangle - \langle \varphi(x)\varphi(y)\rangle\langle \varphi(z)\varphi(0)\rangle - 2\,\mathrm{more}\}}{\left\{\sum_x \langle \varphi(x)\varphi(0)\rangle\right\}^2}$$

$\rightsquigarrow [Z(\kappa,\lambda)], am(\kappa,\lambda), g_R(\kappa,\lambda)$

- $g_R \geqslant 0$ universal ($Z$ cancel), dimensionless, zero if Gaussian
- definition in infinite or finite $V$ [then $p_* = (2\pi/T, \ldots)$]

triviality means: $\lim_{\kappa\to\kappa_c} g_R = 0$ at any $\lambda$

- then: effective theory: only $0 < am \ll 1$ possible, interaction limited
- $D=3$: nontrivial, $D=5$: rigorously trivial, $D=4$: only numerical

# What PT says in $D = 4$

$\lambda$ fixed, arbitrary, $\kappa \nearrow \kappa_c \longleftrightarrow am \searrow 0$, Callan Symanzik:

$$a\frac{\partial}{\partial a} g_R = \beta(g_R, am) = \beta(g_R, 0) + \mathrm{O}(a^2)$$

$$\beta(g_R, 0) = b_1 g_R^2 + b_2 g_R^3 + \ldots, \quad b_1 = \frac{3}{(4\pi)^2}, \quad b_2 = -\frac{17/3}{(4\pi)^4}, \quad b_{\geqslant 3}\text{: scheme dep.}$$

- $b_1 > 0 \Rightarrow$ triviality holds, once PT is applicable: $a\frac{\partial}{\partial a} g_R^{-1} = -b_1$
  $g_R^{-1} \simeq -b_1 \ln(a/\Lambda_{\mathrm{triv}})$ [ $\Lambda_{\mathrm{triv}} =$ integration constant]
- whether PT ever applies and the value of $\Lambda_{\mathrm{triv}}$ are NP questions
- Symanzik, Brezin, Le Guillou, Zinn-Justin, Lüscher, Weisz

# The numerical challenge ... and progress

...or why do I talk about this now...

$$\chi_4 = \sum_{x,y,z} \langle \varphi(x)\varphi(y)\varphi(z)\varphi(0)\rangle_{\rm con} = V^{-1}\{\langle M^4\rangle - 3\langle M^2\rangle^2\}, \quad M = \sum_x \varphi(x)$$

standard Monte Carlo:

- $\langle M^4\rangle$ and $\langle M^2\rangle$ each with some x% error
- large cancellation $|\chi_4| \ll V^{-1}\langle M^4\rangle$
- big relative errors in $\chi_4$ and hence in $g_R$

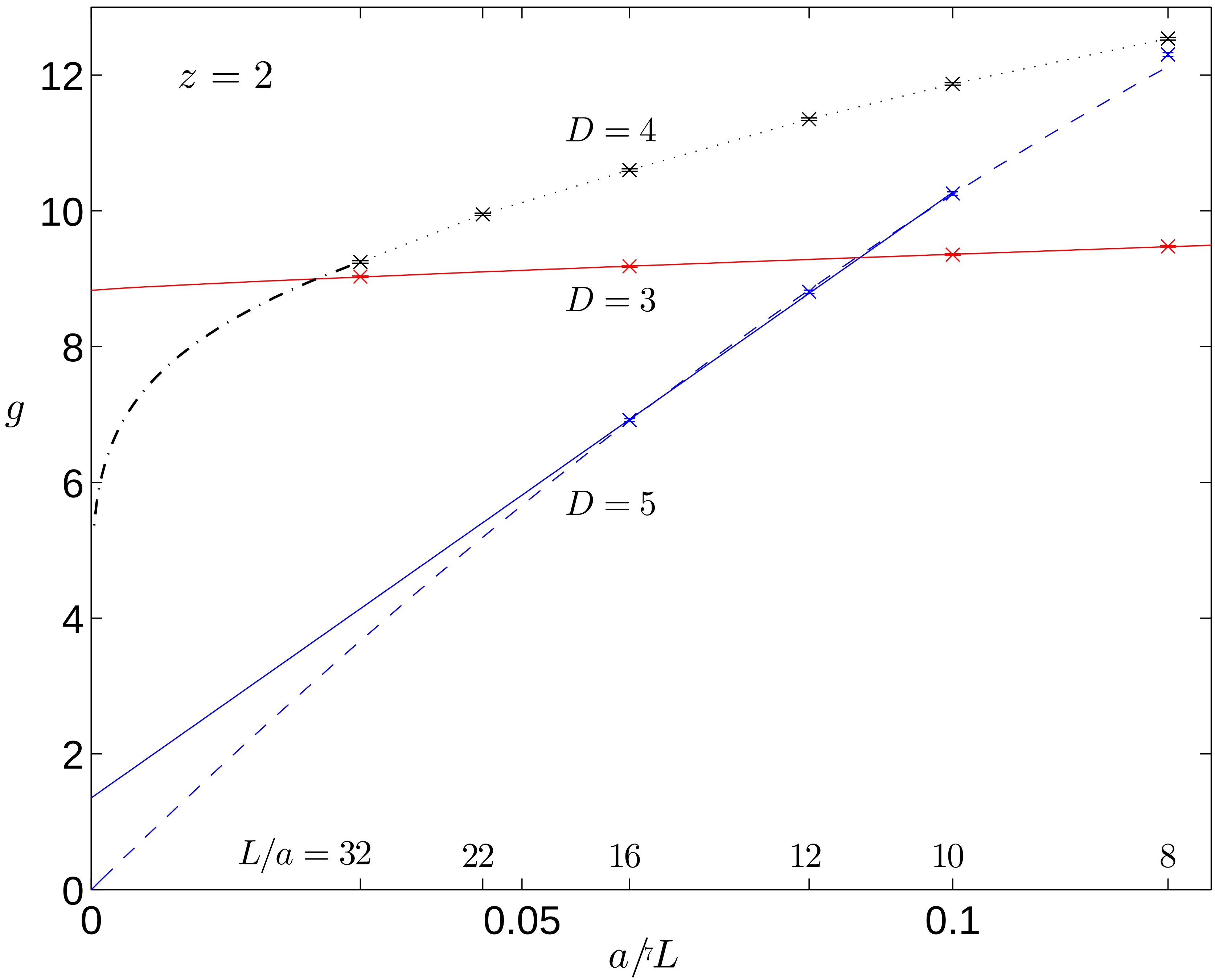
A new method (some details at the end, maybe....):

- lattice QFT has a strong coupling expansion (here in $\kappa$)
- on a finite lattice: analytic in $\kappa \Rightarrow$ arbitrary precision at large order
- orders like $10^5$ needed: cannot compute graphs systematically, but
- we can Monte Carlo sample them ('thermodynamics of graphs')
- using rigourous graph-theoretical identities of M. Aizenman, we find
- a positive small variance estimator for $g_R$ itself on the graphs

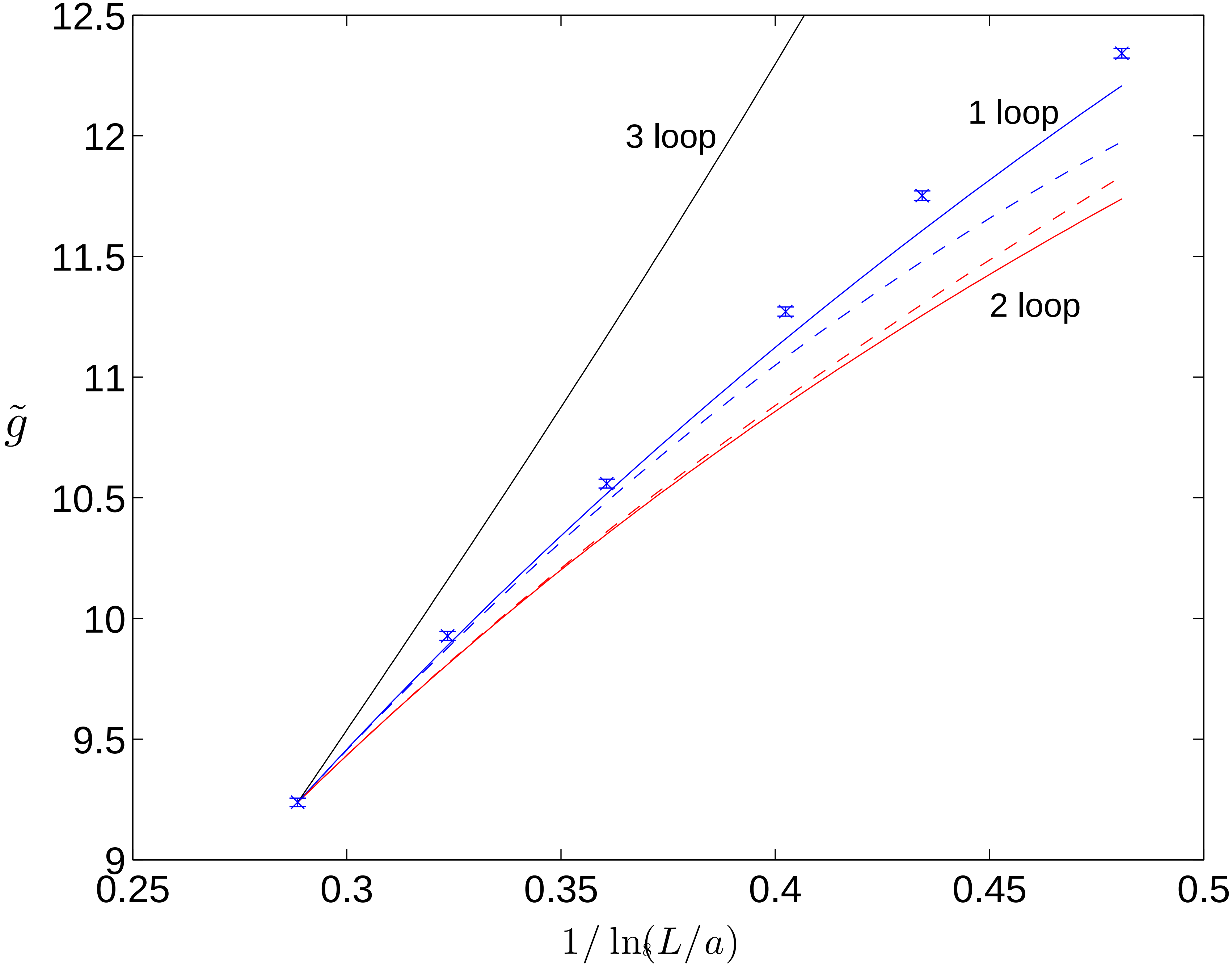
# Finite size renormalization scheme

- the computer limits $L/a$ [O(100) for $D=4$, 2009]
- if we insist on $z=mL\gg 1$ we get not so small $am$
- less close to the critical point, less variation of $am$
- less universality
- triviality is a UV question and can be addressed at $z=\mathrm{O}(1)$
- partially similar to Schrödinger functional in QCD....

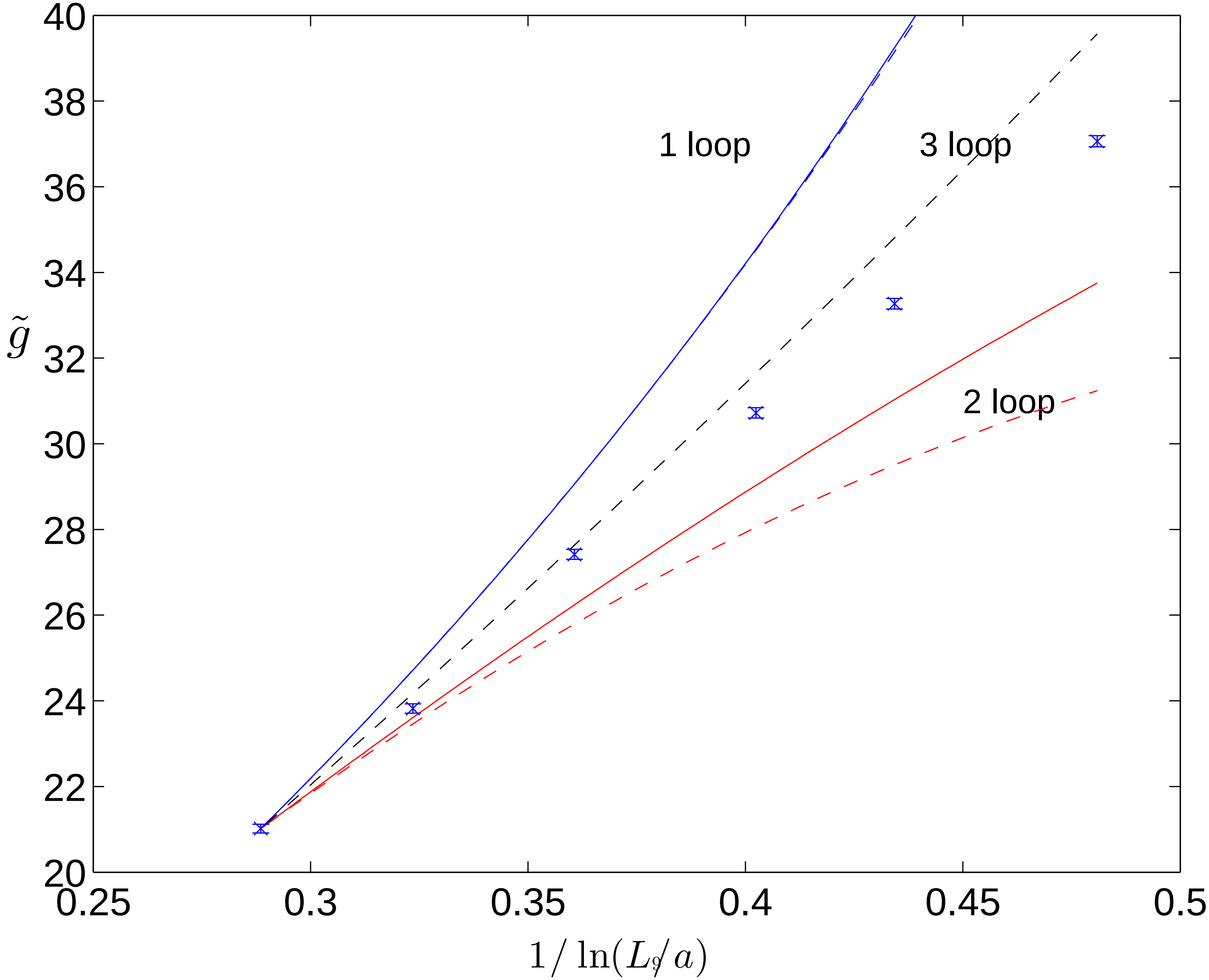
# Data for $D = 3, 4, 5$ and $z = mL = 2$



# Data for $D = 4$ and $z = mL = 2$ and the PT-RG



# Data for $D = 4$ and $z = mL = 4$ and the PT-RG



# The new simulation technique

Example: Ising 2 point function (it is *not* restricted to but simplest for...)

$$Z(u,v) = 2^{-V} \sum_{\varphi=\pm 1} \mathrm{e}^{\beta \sum_{l=\langle x\,y \rangle} \varphi(x)\varphi(y)} \varphi(u)\varphi(v)$$

$$\langle \varphi(u)\varphi(v) \rangle = \frac{Z(u,v)}{Z(x,x)}$$

$$\mathrm{e}^{\beta\varphi(x)\varphi(y)} = \sum_{k(l)=0}^{\infty} \frac{\beta^{k(l)}}{k(l)!} \left[\varphi(x)\varphi(y)\right]^{k(l)} \quad \text{on each link}$$

$$Z(u,v) = \sum_{\{k(l)\}} \prod_{l} \left[\frac{\beta^{k(l)}}{k(l)!}\right] \Delta(u,v;k) \quad (\varphi(x)\,\text{have been summed})$$

$$\Delta(u,v;k) = \begin{cases} 1 & \text{if} \quad \text{odd} \sum k(l)\,\text{around}\, u, v, \text{even around all other}\, x \\ 0 & \quad\text{else} \end{cases}$$

simulated the following ensemble

$$\mathcal{Z} = \sum_{u,v,k(l)} \prod_l \left[\frac{\beta^{k(l)}}{k(l)!}\right] \Delta(u,v;k)$$

by moves of $u$ [or $v$] together with one attached link to preserve $\Delta = 1$

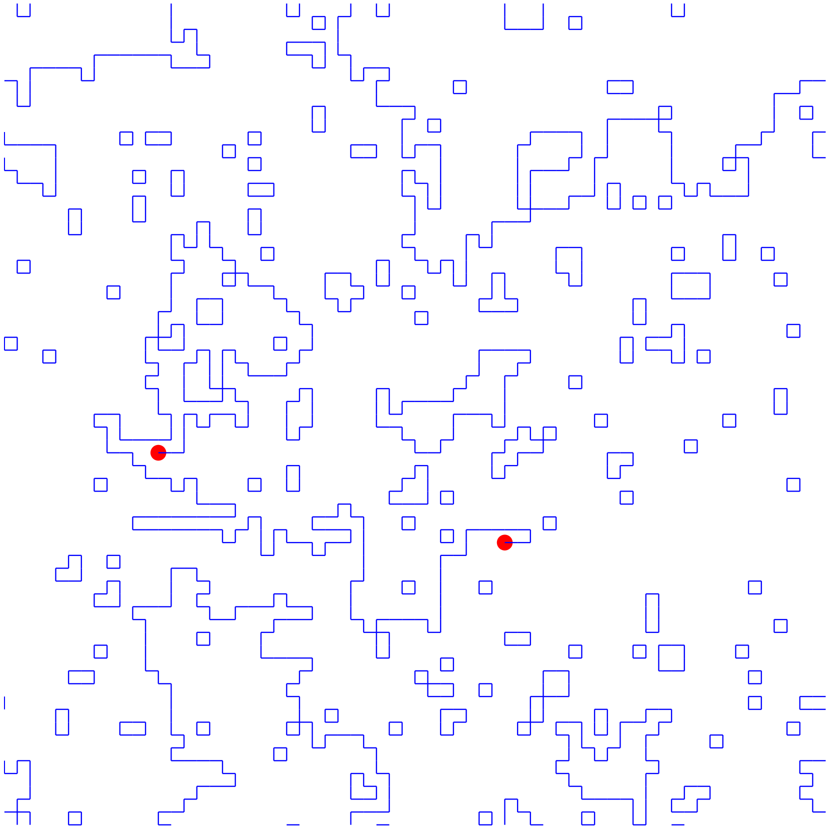
- Prokof'ev Svistunov worm algorithm
- can deform one strong coupling graph ( $\leftrightarrow \beta$ powers) to another
- In the $\mathcal{Z}$ ensemble:

$$\langle \varphi(x)\varphi(0)\rangle = \frac{\langle\langle \delta_{x,u-v}\rangle\rangle}{\langle\langle \delta_{u,v}\rangle\rangle}$$

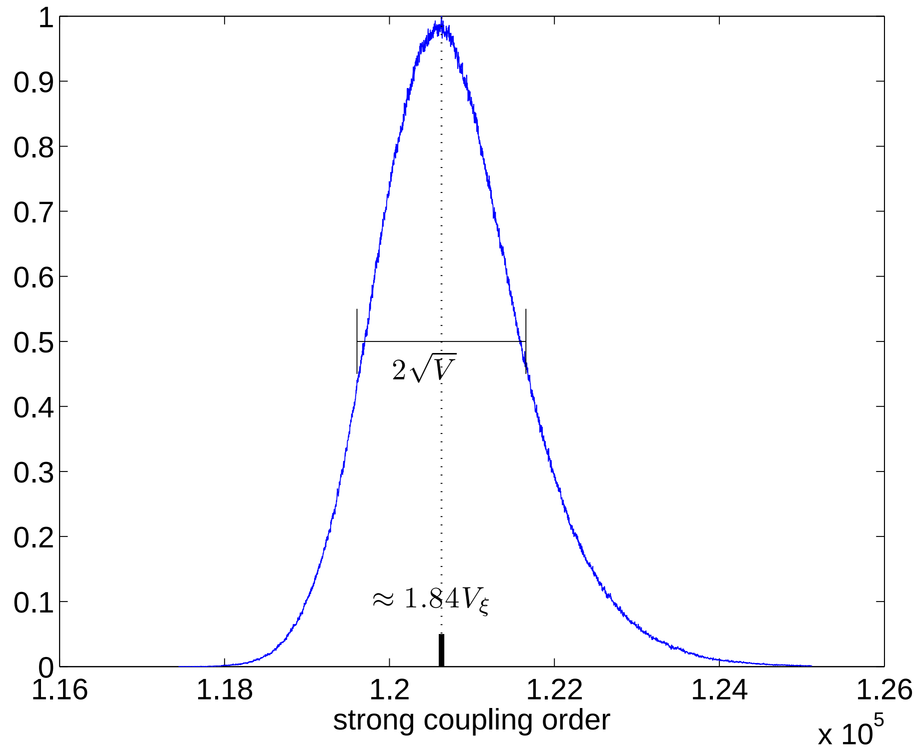
Aizenman identities (related to Symanzik polymer expansion of QFT...)

- run two un-coupled replica: $u, v, k(l), u', v', k'(l)$
- define: $(k, k') \rightarrow b$, $b(l) = \theta[k(l) + k'(l)] \in \{0,1\}$

renormalized $g = 2z^D \times$ probability that $u, v, u', v'$ are in the same bond-percolation cluster defined by $b(l)$



line $\leftrightarrow k(l) = \text{odd}$, periodic b.c.




sampled order in $\beta$, $32^4$, $z = 2$

# Conclusions

- interesting new simulation method
  - reformulation of the problem
  - NOT just a new algorithm for the same old path-integral
- PT triviality in $D=4$ marginally compatible with data at $z=2$
  - more PT orders presumably completely useless...
- PT triviality in $D=4$ more as expected at $z=4$
- large but finite $L$ (relative to $m^{-1}$) $\leftrightarrow$ scattering in $\infty$ volume with momenta $p \sim 2\pi L^{-1}$ (Lüscher) (note: only asymptotic).
- $z=2 \rightarrow p \sim \pi m$ hard to describe by PT at maximal interaction?
- other proposal: "epsilon regime":
  constant mode may need special treatment, rest PT