Higher Moments of Correlators at Four Loop

A. Maier P. Maierhöfer P. Marquard

Institut für Theoretische Teilchenphysik
Universität Karlsruhe

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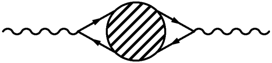
Outline

Quark Current Correlators

$$i \int \mathrm{d}x \, e^{iqx} \langle 0 | T j(x) j(0) | 0 \rangle$$



$$j \in \{\bar\psi\psi, i\bar\psi\gamma_5\psi, \bar\psi\gamma_\mu\psi, \bar\psi\gamma_\mu\gamma_5\psi\}$$

lattice   experiment

Low energy expansion $\Rightarrow$ Quark massses

Moments and Quark Masses

from experiment

- Measure

$$R(s) = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)}$$

- Use dispersion relation:

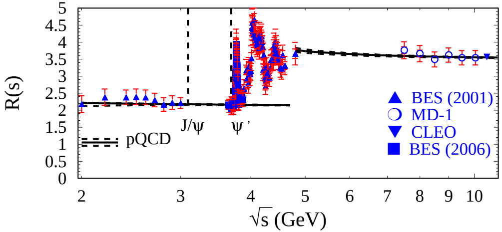
$$\Pi(q^2) = \frac{q^2}{12\pi^2} \int_0^\infty \mathrm{d}s \frac{R(s)}{s(s-q^2)}$$

- Taylor expand around $q^2 = 0$:

$$\frac{3Q^2}{16\pi^2} \sum_n C_n \left(\frac{q^2}{4m^2}\right)^n = \sum_n (q^2)^n \frac{1}{12\pi^2} \underbrace{\int \mathrm{d}s \frac{R(s)}{s^{n+1}}}_{\mathcal{M}_n^{exp}}$$

$$\boxed{m = \frac{1}{2}\left(\frac{9Q^2}{4}\frac{C_n}{\mathcal{M}_n^{exp}}\right)^{\frac{1}{2n}}}$$

Quark Mass Determination



Higher Moments $\Rightarrow$ More weight on threshold and resonances

$(\mathcal{M}_n^{exp} \sim \int ds \frac{R(s)}{s^{n+1}})$

Are higher moments better?

- ⊕ More sensitive to quark mass
- ⊕ Better data
- ⊖ Growing nonperturbative contributions
- ⊖ Increasing complexity of calculations

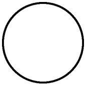
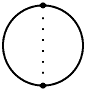
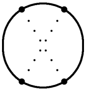
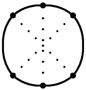
Calculating Moments

[Källén, Sabry '55; Chetyrkin, Kühn, Steinhauser '96; Chetyrkin, Kühn, Steinhauser '97; Kühn, Steinhauser, Sturm '06; Boughezal, Czakon, Schutzmeier '06; Sturm '08; Maier, Maierhöfer, Marquard '08]

$$\Pi(q^2) = \frac{3Q^2}{16\pi^2} \sum_n C_n \left(\frac{q^2}{4m^2} \right)^n$$

Reduce to scalar diagrams:

$$C_n = C_n^{(0)} + C_n^{(1)} \frac{\alpha_s}{\pi} + C_n^{(2)} \left(\frac{\alpha_s}{\pi}\right)^2 + {\color{red}C_n^{(3)}} \left(\frac{\alpha_s}{\pi}\right)^3 + \cdots$$

$=$  $+$  $+$  $+$  $+ \cdots$

Higher n means

- More additional propagator powers ("dots")
- More scalar products of loop momenta

$C_3^{(3)}$: 12 dots, 8 scalar products, $\sim 4\,000\,000$ diagrams

Reduction to Master Integrals

Scalar diagrams:

$$\int \frac{d^d k_1}{(2\pi)^d} \dots \frac{d^d k_l}{(2\pi)^d} \frac{1}{D_1^{a_1} \dots D_m^{a_m}}$$

Integration by parts (IBP) [Chetyrkin, Tkachov '81]:

$$\int \frac{d^d k_1}{(2\pi)^d} \dots \frac{d^d k_l}{(2\pi)^d} \frac{\partial}{\partial k_i^\mu} p_j^\mu \frac{1}{D_1^{a_1} \dots D_m^{a_m}} = 0$$

(p_j: External or loop momentum)

$\Rightarrow$ Linear system of equations for scalar integrals; can be solved via Gauss elimination [Laporta '00]

$\Rightarrow$ Solution in terms of small set of (known) master integrals

Reduction to Master Integrals

Problem: System of equations is huge!
(Naïvely: $\mathcal{O}(10^6)$ integrals $\times\, 16$ IBP relations)

Ideas:

- Use symmetries of diagrams
- Intelligent choice of input diagrams
- Special treatment of diagrams with self energies
- Different approach (e.g. Gröbner Bases [Smirnov, Smirnov '06])

Charm Quark Mass

[updated from Kühn, Steinhauser, Sturm '07]

$$C_3^{(3),v}\Big|_{n_f=4} = -2.839$$

	$m_c(3\,GeV)$	exp	α_s	μ	np	total	$\Delta C^{(3)}$	$m_c(m_c)$
1	0.986	0.009	0.009	0.002	0.001	0.013	—	1.286
2	0.976	0.006	0.014	0.005	0.000	0.016	—	1.277
3	0.978	0.005	0.015	0.007	0.002	0.017	—	1.278
4	1.012	0.003	0.008	0.030	0.007	0.032	0.016	1.309

- Everything in agreement
- Best value still from first moment

$$m_c(m_c) = 1.286(13)\ \text{GeV}$$

Bottom Quark Mass

[updated from Kühn, Steinhauser, Sturm '07]

$$C_3^{(3),v}\Big|_{n_f=5} = -1.174$$

	$m_b(10\,GeV)$	exp	α_s	μ	total	$\Delta C^{(3)}$	$m_b(m_b)$
1	3.593	0.020	0.007	0.002	0.021	—	4.149
2	3.607	0.014	0.012	0.003	0.019	—	4.162
3	3.617	0.010	0.014	0.006	0.019	—	4.172
4	3.631	0.008	0.015	0.021	0.026	0.012	4.185

- Still agreement, but increasing values of m_b (unknown systematic experimental error?)
- Second and third moment equally good, taking average

$$m_b(m_b) = 4.167(19)\ \text{GeV}$$

m_c and α_s from Lattice

[updated from HPQCD + Chetyrkin, Kühn, Steinhauser, Sturm '08]

$$C_4^{(3),p}\Big|_{n_f=4} = 13.328$$

	$m_c(3\text{ GeV})$	Δ
2	0.985	0.010
3	0.986	0.011
4	0.981	0.013
5	0.968	0.023

	$\alpha_s(3\text{ GeV})$	Δ
1	0.2523	0.0057
2/3	0.2486	0.0059
3/4	0.2368	0.0112
4/5	0.2207	0.0396

- Good agreement between different moments
- Good agreement between lattice and experiment

Even More Moments
together with Y. Kiyo

Moments $C_n^{(3)}$ with $n > 3$?

No Problem!

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... if we knew $\Pi(q^2)$ at four loops for arbitrary q^2 ...

Reconstruct $\Pi(q^2)$ approximately using information from

- low energy region
- threshold region
  [Beneke, Smirnov '97; Czarnecki, Melnikov '97; ...;Hoang, Teubner '98; Pineda, Signer '06; ...]
- high energy region
  [Chetyrkin, Harlander, Kühn '00; Baikov, Chetyrkin, Kühn '08]

and Taylor expand around $q^2 = 0$

Padé Approximation

[Broadhurst, Fleischer, Tarasov '93; Baikov, Broadhurst '95; Chetyrkin, Kühn, Steinhauser '96; Hoang, Mateu, Zerbarjad '08; Masjuan, Peris '08]

Approximation to $f(x)$

$$p_{n,m}(x) = \frac{a_0 + a_1 x^1 + \cdots + a_n x^n}{1 + b_1 x^1 + \cdots + b_m x^m}$$

Coefficients a_i, b_i from

$$f^{(k)}(x_j) = p^{(k)}_{n,m}(x_j)$$

($f^{(k)}(x_j) = k$th derivative of f at x_j)

Padé: Simple Example

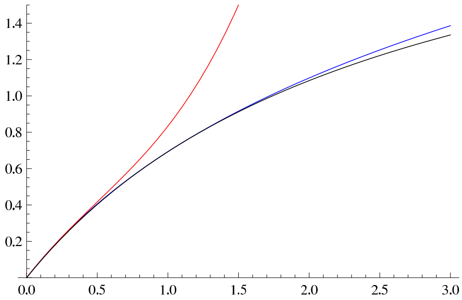
Example:

$$f(x) = \log(1+x)\,, \quad p_{1,1}(x) = \frac{a_0 + a_1 x}{1 + b_1 x}$$

Possible constraints:

$$p_{1,1}(0) = f(0)\,, \quad p_{1,1}(1) = f(1)\,, \quad p'_{1,1}(1) = f'(1)$$

$$\Rightarrow p_{1,1}(x) = \frac{0.96\,x}{1 - 0.39\,x}$$



Original function

Padé approximant

Taylor series

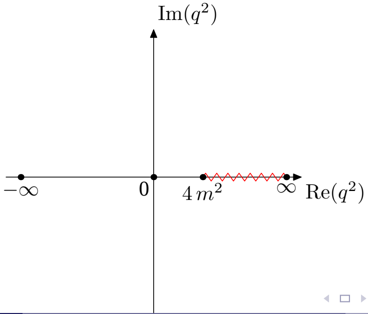
Approximating Correlators

Problems:

1. Logarithms in threshold and high energy expansions
   e.g. α_s^3 contribution to the vector correlator:

$$\Pi^{(3),v}(q^2) \xrightarrow{q^2 \to -\infty} = -6.172 - 0.06988 \log\left(-\frac{q^2}{m^2}\right) + 0.1211 \log^2\left(-\frac{q^2}{m^2}\right) - 0.03665 \log^3\left(-\frac{q^2}{m^2}\right) + \dots$$

2. Branch cut above threshold:



Application to Correlators
Charm Quark Vector Current

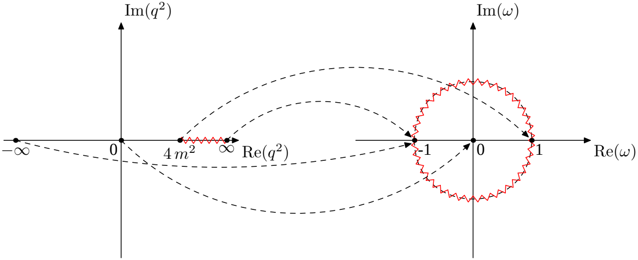
Solution:

1. Subtraction of an appropriate function (not unique):

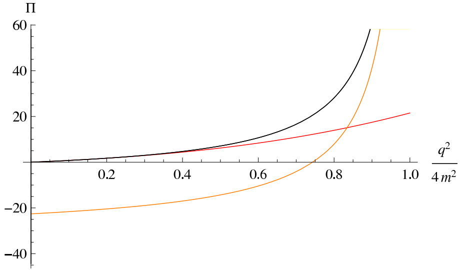
$$\Pi(q^2) = \Pi_{reg}(q^2) + \Pi_{log}(q^2)$$

2. Transformation:

$$\frac{q^2}{4m^2} = \frac{4\omega}{(1+\omega)^2}$$



Low Energy Behaviour
Charm Quark Vector Current preliminary



$$C_4^{(3),v}\Big|_{nf=4} = -3.347^{+0.027}_{-0.037}$$
$$C_5^{(3),v}\Big|_{nf=4} = -3.73^{+0.10}_{-0.10}$$
$$C_6^{(3),v}\Big|_{nf=4} = -3.72^{+0.22}_{-0.19}$$

Summary

- Low energy moments of quark correlators allow precise determinations of quark masses from experiment or lattice simulation
- First three (four) moments are calculated, extracted quantities are in good agreement
- Higher moments can be estimated from approximations to the correlators