Quark and Gluon Form Factor

to Three Loops

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Outline

I. Form factor

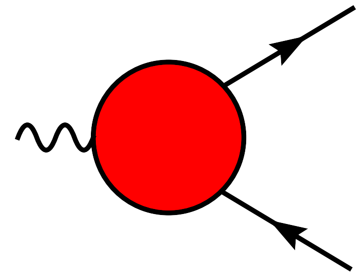
II. Calculation

III. Results

IV. Summary

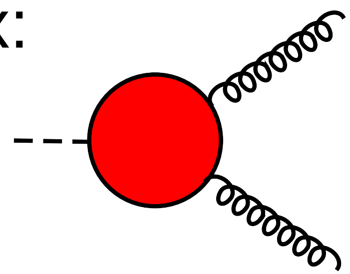
I. Form factor

Photon-quark vertex:



$\Rightarrow \quad \gamma^\mu F_q(q^2)$

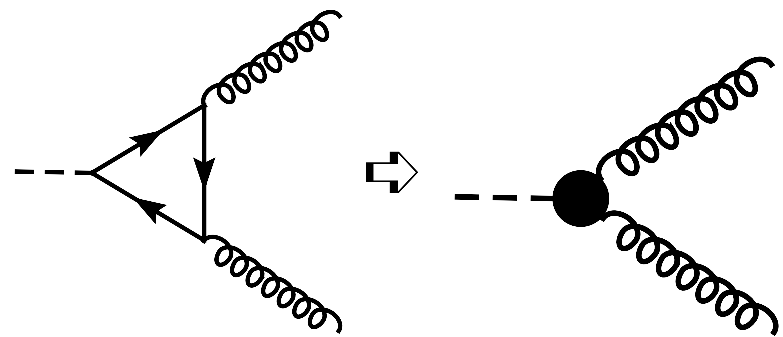
Higgs-gluon vertex:



$\Rightarrow \quad \left(g^{\mu\nu} q_1 q_2 - q_1^\nu q_2^\mu\right) F_g(q^2)$

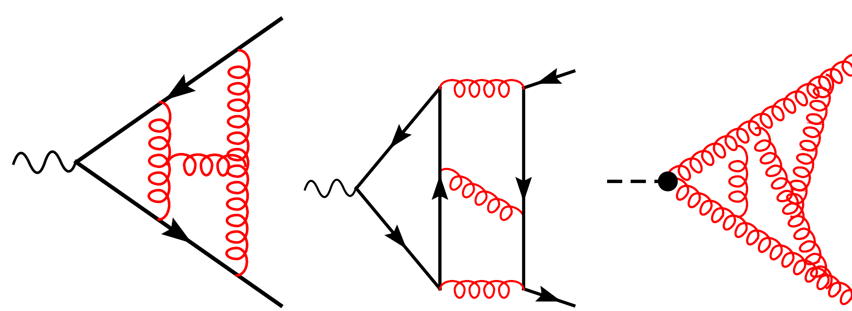
origin: construct effective theory for $m_t \to \infty$:

$\mathcal{L}^{\rm SM} \Rightarrow \mathcal{L}_{\rm eff} = C_1 \frac{H}{v} G_{\mu\nu} G^{\mu\nu}$ + ...



$$F = 1 + \frac{\alpha_s}{\pi} F^{(1)} + \left(\frac{\alpha_s}{\pi}\right)^2 F^{(2)} + \left(\frac{\alpha_s}{\pi}\right)^3 F^{(3)}$$

$$F_q^{(3)} = F_q^{(3),g} + F_q^{(3),n_f} + \sum_{q'} Q_{q'} F_q^{(3),sing}$$



Known results

- $F_q^{(2)}$ [(Gonsalves'83); Kramer,Lampe'86; Matsuura, van Neerven'87; Matsuura, van der Marck, van Neerven'88]
- $F_g^{(2)}$ [Harlander'00; Ravindran,Smith,van Neerven'04]
- $F_q^{(2)}$ and $F_g^{(2)}$ to $\mathcal{O}(\epsilon^2)$ [Gehrmann,Huber,Maitre'05]

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$$F_g^{(1)} = C_A\Big\{-\frac{2}{\epsilon^2}+\zeta_2+\epsilon\left(-2+\frac{14}{3}\zeta_3\right)+\epsilon^2\left(-6+\frac{47}{20}\zeta_2^{\,2}\right)+\epsilon^3\Big(-14+\zeta_2 -\frac{7}{3}\zeta_2\zeta_3+\frac{62}{5}\zeta_5\Big)+\epsilon^4\left(-30+3\zeta_2+\frac{14}{3}\zeta_3+\frac{949}{280}\zeta_2^{\,3}-\frac{49}{9}\zeta_3^{\,2}\right)\Big\}$$

Known results

- $F^{(2)}$

[(Gonsalves'83); Kramer,Lampe'86; Matsuura, van Neerven'87; Matsuura, van der Marck, van Neerven'88]

04]

5]

$$
\begin{aligned}
F_g^{(2)} \quad = \quad & C_A^2 \Bigg\{ \frac{2}{\epsilon^4} - \frac{11}{6\epsilon^3} + \frac{1}{\epsilon^2}\left(-\frac{67}{18} - \zeta_2\right) + \frac{1}{\epsilon}\left(\frac{68}{27} + \frac{11}{2}\zeta_2 - \frac{25}{3}\zeta_3\right) + \frac{5861}{162} + \frac{67}{6}\zeta_2 \\
& + \frac{11}{9}\zeta_3 - \frac{21}{5}\zeta_2^2 + \epsilon\left(\frac{158201}{972} + \frac{106}{9}\zeta_2 - \frac{1139}{27}\zeta_3 - \frac{77}{60}\zeta_2^2 + \frac{23}{3}\zeta_2\zeta_3 + \frac{71}{5}\zeta_5\right) \\
& + \epsilon^2\left(\frac{3484193}{5832} + \frac{481}{54}\zeta_2 - \frac{26218}{81}\zeta_3 - \frac{1943}{60}\zeta_2^2 - \frac{55}{3}\zeta_2\zeta_3 + \frac{341}{15}\zeta_5 + \frac{2313}{70}\zeta_2^3\right. \\
& \left. + \frac{901}{9}\zeta_3^2\right)\Bigg\} + C_A n_f \Bigg\{ \frac{1}{3\epsilon^3} + \frac{5}{9\epsilon^2} + \frac{1}{\epsilon}\left(-\frac{26}{27} - \zeta_2\right) - \frac{808}{81} - \frac{5}{3}\zeta_2 - \frac{74}{9}\zeta_3 \\
& + \epsilon\left(-\frac{23131}{486} - \frac{16}{9}\zeta_2 - \frac{604}{27}\zeta_3 - \frac{51}{10}\zeta_2^2\right) + \epsilon^2\left(-\frac{540805}{2916} + \frac{28}{27}\zeta_2 - \frac{3962}{81}\zeta_3\right. \\
& \left. - \frac{257}{18}\zeta_2^2 + \frac{50}{3}\zeta_2\zeta_3 - \frac{542}{15}\zeta_5\right)\Bigg\} + C_F n_f \Bigg\{ -\frac{1}{\epsilon} - \frac{67}{6} + 8\zeta_3 + \epsilon\left(-\frac{2027}{36}\right. \\
& \left. + \frac{7}{3}\zeta_2 + \frac{92}{3}\zeta_3 + \frac{16}{3}\zeta_2^2\right) + \epsilon^2\left(-\frac{47491}{216} + \frac{209}{18}\zeta_2 + \frac{1124}{9}\zeta_3 + \frac{184}{9}\zeta_2^2\right. \\
& \left. - \frac{40}{3}\zeta_2\zeta_3 + 32\zeta_5\right)\Bigg\}
\end{aligned}
$$

Known results

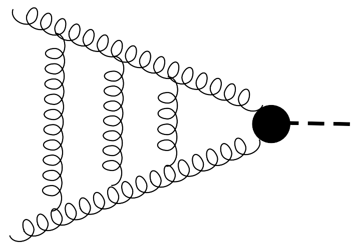
- $F_q^{(2)}$ [(Gonsalves'83); Kramer,Lampe'86; Matsuura, van Neerven'87; Matsuura, van der Marck, van Neerven'88]
- $F_g^{(2)}$ [Harlander'00; Ravindran,Smith,van Neerven'04]
- $F_q^{(2)}$ and $F_g^{(2)}$ to $\mathcal{O}(\epsilon^2)$ [Gehrmann,Huber,Maitre'05]
- $1/\epsilon$ poles for $F_q^{(3)}$ and $F_g^{(3)}$ [Moch,Vermaseren,Vogt'05]

$$F^{(3)} = \frac{\#}{\epsilon^6} + \frac{\#}{\epsilon^5} + \frac{\#}{\epsilon^4} + \frac{\#}{\epsilon^3} + \frac{\#}{\epsilon^2} + \frac{\#}{\epsilon^1} + \boxed{\text{??}}$$

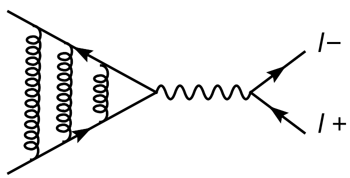
Applications

Virtual NNNLO corrections to

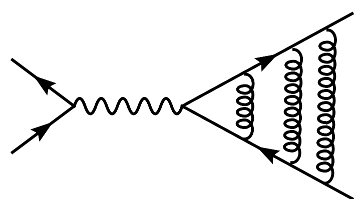
- $gg \to H$



- DY

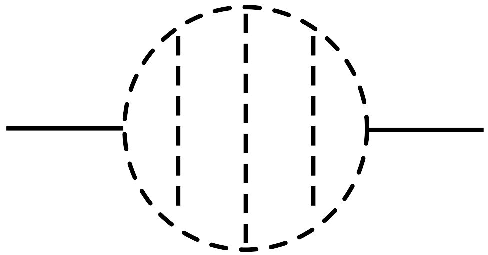


- $e^+e^- \to 2$ Jets



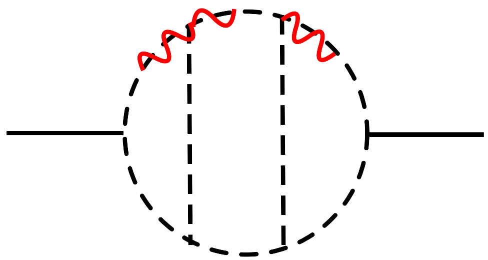
- soft-gluon resummation

Compare complexity



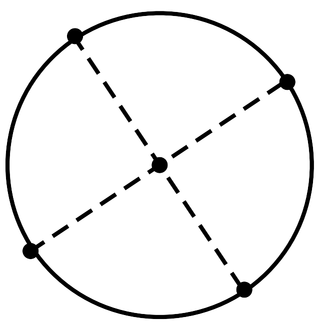
4 loops
2-point

$\longrightarrow A_1$



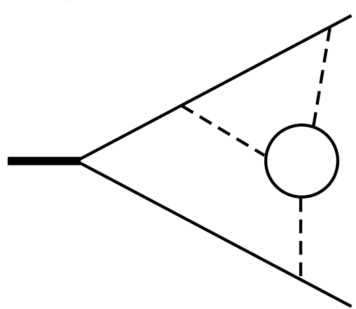
3 loops
2-point
with static lines

$\longrightarrow C_3$



4 loops
bubbles

$\longrightarrow A_1$



3-loop on-threshold
vertex $q^2 = 4m^2$
(not complete)

$\longrightarrow C_3, C_5$

II. The calculation

1. Reduction to MIs
2. Compute MIs

1. Reduction to MIs

Main approach:

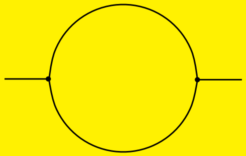
- equivalence for recurrence relations between N-loop 2-point and $(N-1)$-loop 3-point functions [Baikov,Smirnov'96]
- similar to `BAICER` [Baikov,Chetyrkin,Kühn'02...'08]
- `ParFORM`, `TFORM` [Tentyukov,Vermaseren,...'04...'09]
- "Baikov-method"

1. Reduction to MIs

- Baikov's method: $I = \sum$ "coef" $\times\ MI$ [Baikov'96,. . . ,Smirnov,MS'03]
- (nice) integral representation for "coef"

$$\sim \int \ldots \int \frac{\mathrm{d}x_1 \ldots \mathrm{d}x_N}{x_1^{n_1} \ldots x_N^{n_N}} \left[P(x_1,\ldots)\right]^{(d-h-1)/2}$$

- (simple) example:



$$F(n_1,n_2) = \int \frac{\mathrm{d}^d k}{(k^2)^{n_1}[(k-q)^2]^{n_2}} = c_1(n_1,n_2)\,F(1,1)$$

$$\Rightarrow P(x_1,x_2) = (q^2)^2 - 2q^2(x_1+x_2) + (x_1-x_2)^2$$

$$\Rightarrow c_1(n_1,n_2) = \frac{(q^2)^{(d-3)}}{(n_1-1)!}\left(\frac{\partial}{\partial x_1}\right)^{n_1-1} \frac{1}{(n_2-1)!}\left(\frac{\partial}{\partial x_2}\right)^{n_2-1} \left[P(x_1,x_2)\right]^{(d-3)/2}\Bigg|_{x_i=0}$$

- in general: compute for $d \to \infty$ ⇨ reconstruct "coef" [Baikov,Chetyrkin,Kühn'02...'08]

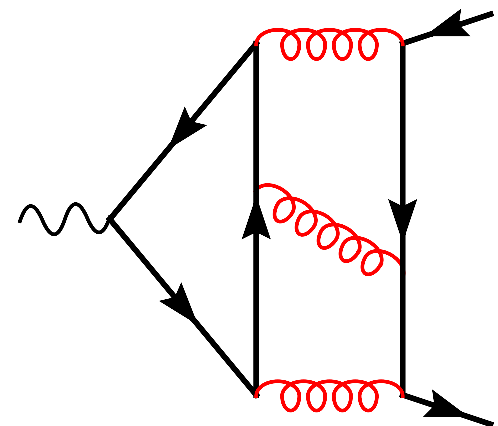
1. Reduction to MIs

Main approach:

- equivalence for recurrence relations between N-loop 2-point and $(N-1)$-loop 3-point functions [Baikov,Smirnov'96]
- similar to BAICER [Baikov,Chetyrkin,Kühn'02...'08]
- ParFORM, TFORM [Tentyukov,Vermaseren,...'04...'09]
- "Baikov-method"

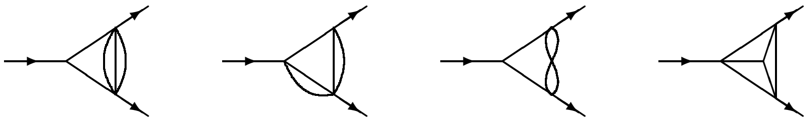
For singlet diagrams:

independent calculation with FIRE: Laporta $\oplus$ Gröbner [Smirnov'08]

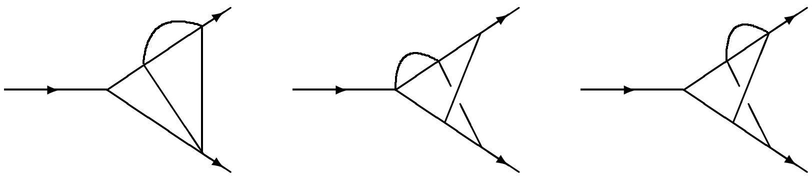


➪ 22 MIs

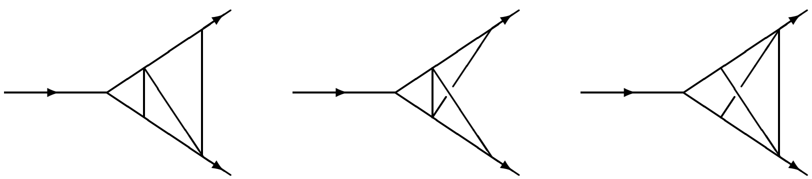
2. MIs



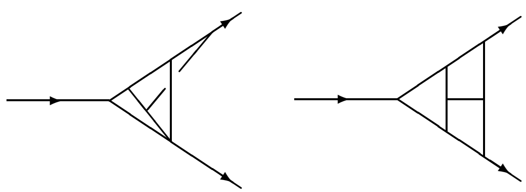
$A_{5,1}$ $A_{5,2}$ $A_{6,1}$ $A_{6,2}$



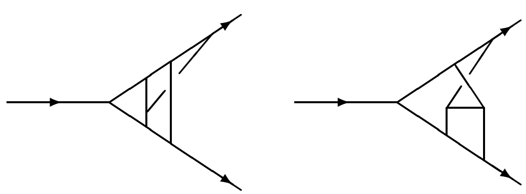
$A_{6,3}$ $A_{7,1}$ $A_{7,2}$



$A_{7,3}$ $A_{7,4}$ $A_{7,5}$



A_{8} $A_{9,1}$



$A_{9,2}$ $A_{9,4}$

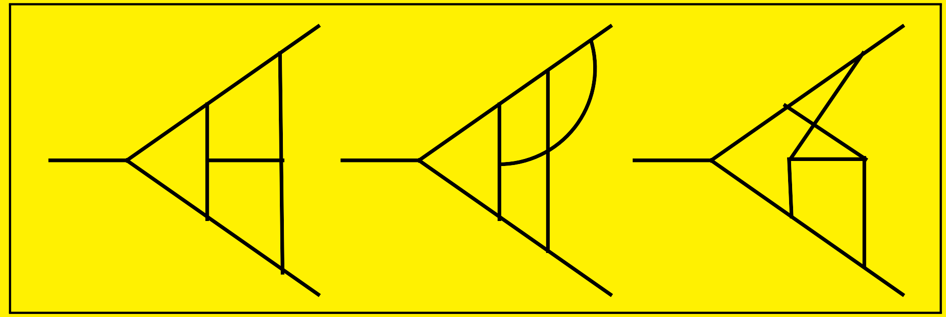
(+ 8 simple MIs)

2. MIs

$A_{5,1}$

- [Gehrmann,Heinrich,Huber,Studerus'06; Heinrich,Huber,Maitre'08]: most MIs

  not computed:



- missing MIs:

  extract by comparison with $F_q\Big|_{1/\epsilon-\text{poles}}$ [Moch,Vermaseren,Vogt'05]

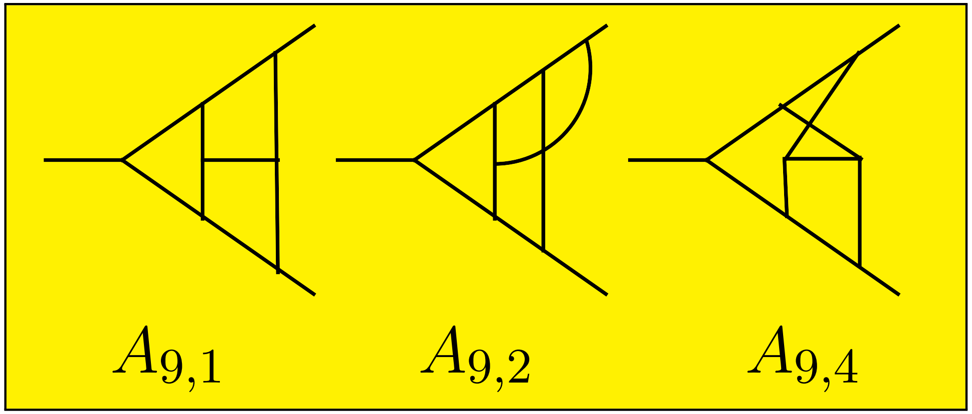
  $F_g\Big|_{1/\epsilon-\text{poles}}$: consistency check

- 3 remaining coefficients:
  `FIESTA` [Smirnov,Tentukov'08], Mellin Barnes, `MB` [Czakon'06]

$A_{9,2}$ $A_{9,4}$

(+ 8 simple MIs)

2. MIs (2)



[Baikov,Chetyrkin,Smirnov,Smirnov,Steinhauser'09]

$$A_{9,4} = -\frac{1}{9\epsilon^6} - \frac{8}{9\epsilon^5} + \frac{1}{\epsilon^4}\left(1 + \frac{43\zeta(2)}{18}\right) + \frac{1}{\epsilon^3}\left(\frac{14}{9} + \frac{106\zeta(2)}{9} + \frac{109\zeta(3)}{9}\right)$$
$$+ \frac{1}{\epsilon^2}\left(-17 - \frac{311\zeta(2)}{18} + \frac{608\zeta(3)}{9} - \frac{481\zeta(4)}{144}\right)$$
$$+ \frac{1}{\epsilon}\left(84 + \frac{11\zeta(2)}{3} - \frac{949\zeta(3)}{9} + \frac{425\zeta(4)}{6} + \frac{3463\zeta(5)}{45} - \frac{2975\zeta(2)\zeta(3)}{18}\right)$$
$$+ X_{9,4} + \mathcal{O}(\epsilon)\ .$$

independent (explicit) calculation: [Heinrich,Huber,Kosower,Smirnov'09]

($X_{9,1}$ analytically; less analytic information for $A_{9,4}$)

III. Results

- $F_q^{(3)}$ and $F_g^{(3)}$: agreement with poles [Moch,Vermaseren,Vogt'05]
- $F_q^{(3)}\Big|_{n_f-}$part [Moch,Vermaseren,Vogt'05]
- $F_q^{(3),sing}$ can be extracted from [Moch,Vermaseren,Vogt'04]

III. Results

$$
\left. F_q^{(3),g+n_f} \right|_{\rm fin} = C_F^3 \left(\frac{26871}{8} - \frac{95137\zeta(2)}{60} + \frac{5569\zeta(3)}{5} + \frac{95375\zeta(4)}{48} + \frac{30883\zeta(2)\zeta(3)}{15} - \frac{16642\zeta(5)}{5} + \frac{2669(\zeta(3))^2}{3} \right.
$$

$$
\left. + \frac{1961387\zeta(6)}{2880} - \frac{24X_{9,1}}{5} + \frac{24X_{9,2}}{5} + \frac{6X_{9,4}}{5} \right) + C_A C_F^2 \left(\frac{20003431}{29160} + \frac{4239679\zeta(2)}{1620} - \frac{121753\zeta(3)}{30} \right.
$$

$$
- \frac{11155817\zeta(4)}{4320} - \frac{92554\zeta(2)\zeta(3)}{45} + \frac{610462\zeta(5)}{225} - \frac{36743(\zeta(3))^2}{30} - \frac{1118529\zeta(6)}{640} + \frac{24X_{9,1}}{5}
$$

$$
\left. - \frac{16X_{9,2}}{5} - \frac{9X_{9,4}}{5} \right) + C_A^2 C_F \left(- \frac{88822328}{32805} - \frac{3486997\zeta(2)}{2916} + \frac{3062512\zeta(3)}{1215} + \frac{4042277\zeta(4)}{4320} \right.
$$

$$
\left. + \frac{5233\zeta(2)\zeta(3)}{12} - \frac{202279\zeta(5)}{450} + \frac{63043(\zeta(3))^2}{180} + \frac{4741699\zeta(6)}{11520} - X_{9,1} + \frac{2X_{9,2}}{5} + \frac{3X_{9,4}}{5} \right)
$$

$$
+ C_F^2 n_f T \left(- \frac{2732173}{1458} - \frac{45235\zeta(2)}{81} + \frac{102010\zeta(3)}{81} + \frac{40745\zeta(4)}{216} - \frac{686\zeta(3)\zeta(2)}{9} + \frac{556\zeta(5)}{45} \right)
$$

$$
+ C_A C_F n_f T \left(\frac{17120104}{6561} + \frac{442961\zeta(2)}{729} - \frac{90148\zeta(3)}{81} - \frac{5465\zeta(4)}{27} + \frac{736\zeta(3)\zeta(2)}{9} - \frac{416\zeta(5)}{3} \right)
$$

$$
+ C_F n_f^2 T^2 \left(- \frac{2710864}{6561} - \frac{248\zeta(2)}{3} + \frac{12784\zeta(3)}{243} - \frac{166\zeta(4)}{27} \right)
$$

$$
X_{9,1} \approx 1429(1)\,, X_{9,2} \approx 528.0(4)\,, X_{9,4} \approx -2085(5)
$$

[Baikov,Chetyrkin,Smirnov,Smirnov,Steinhauser'09]

III. Results

$$\left. F_q^{(3),sing}\right|_{\rm fin} = d^{abc}d^{abc}\left(\frac{2}{3}+\frac{5\zeta(2)}{3}+\frac{7\zeta(3)}{9}-\frac{\zeta(4)}{6}-\frac{40\zeta(5)}{9}\right)$$

$$\begin{aligned}\left. F_g^{(3)}\right|_{\rm fin} = {} & C_A^3\left(\frac{14423912}{6561}+\frac{384479\zeta(2)}{2916}-\frac{370649\zeta(3)}{486}+\frac{280069\zeta(4)}{864}+\frac{1821\zeta(2)\zeta(3)}{4}-\frac{66421\zeta(5)}{90}\right.\\ & \left.+\frac{545(\zeta(3))^2}{36}-\frac{167695\zeta(6)}{256}-X_{9,1}+2X_{9,2}\right)+C_A^2 n_f T\left(-\frac{10021313}{6561}-\frac{75736\zeta(2)}{729}-\frac{1508\zeta(3)}{27}\right.\\ & \left.+\frac{437\zeta(4)}{12}-\frac{878\zeta(3)\zeta(2)}{9}+\frac{6476\zeta(5)}{45}\right)+C_F C_A n_f T\left(-\frac{155629}{243}-\frac{82\zeta(2)}{3}+\frac{23584\zeta(3)}{81}-16\zeta(4)\right.\\ & \left.+96\zeta(3)\zeta(2)+\frac{64\zeta(5)}{9}\right)+C_F^2 n_f T\left(\frac{608}{9}+\frac{592\zeta(3)}{3}-320\zeta(5)\right)+C_F n_f^2 T^2\left(\frac{42248}{81}-\frac{64\zeta(2)}{3}\right.\\ & \left.-\frac{2816\zeta(3)}{9}-\frac{224\zeta(4)}{3}\right)+C_A n_f^2 T^2\left(\frac{2958218}{6561}+\frac{304\zeta(2)}{27}+\frac{47296\zeta(3)}{243}+\frac{1594\zeta(4)}{27}\right)\end{aligned}$$

$$X_{9,1}\approx 1429(1)\,,\,X_{9,2}\approx 528.0(4)\,,\,X_{9,4}\approx -2085(5)$$

[Baikov,Chetyrkin,Smirnov,Smirnov,Steinhauser'09]

III. Results

$$X_{9,1} \approx 1429(1)\,,\, X_{9,2} \approx 528.0(4)\,,\, X_{9,4} \approx -2085(5)$$

$$F_q^{(3),g+n_f}|_{\rm fin} \approx -13656.8 + 3062.1 n_f - 164.2 n_f^2 \pm 2.2\delta_{9,1} \pm 0.4\delta_{9,2} \pm 2.2\delta_{9,4}$$
$$F_q^{(3),sing}|_{\rm fin} \approx -5.944,$$
$$F_g^{(3)}|_{\rm fin} \approx 26102.7 - 8298.8 n_f + 585.3 n_f^2 \pm 27.0\delta_{9,1} \pm 21.6\delta_{9,2}$$

SYM ($C_A = C_F = 2T$, $n_f = 1$):

$$F_q^{(3),g+n_f}\Big|_{\rm fin} = C_A^3\left(\frac{389216}{243} - \frac{155935\zeta(2)}{972} - \frac{54703\zeta(3)}{162} + \frac{23897\zeta(4)}{72} + \frac{15875\zeta(2)\zeta(3)}{36} - \frac{11279\zeta(5)}{10} + \frac{545(\zeta(3))^2}{36} - \frac{167695\zeta(6)}{256} - X_{9,1} + 2X_{9,2}\right)$$

$$F_g^{(3)}\Big|_{\rm fin} = C_A^3\left(\frac{676219}{486} + \frac{61937\zeta(2)}{972} - \frac{93295\zeta(3)}{162} + \frac{95171\zeta(4)}{288} + \frac{16361\zeta(2)\zeta(3)}{36} - \frac{1645\zeta(5)}{2} + \frac{545(\zeta(3))^2}{36} - \frac{167695\zeta(6)}{256} - X_{9,1} + 2X_{9,2}\right)$$

IV. Summary

- 3-loop corrections for F_q and F_g
- 3 most complicated MIs
- $1^{\rm st}$ complete, non-trivial 3-loop vertex correction
- applications: NNNLO contribution to
  - $gg \to H$
  - DY
  - $e^+e^- \to 2$ Jets