Who's afraid of terminal singularities?

- P. Arras, A. Grassi, T.W. (arXiv: 1612.05646)
- P. Arras, A. Grassi, TW, with an appendix by V. Srinivas (to appear)

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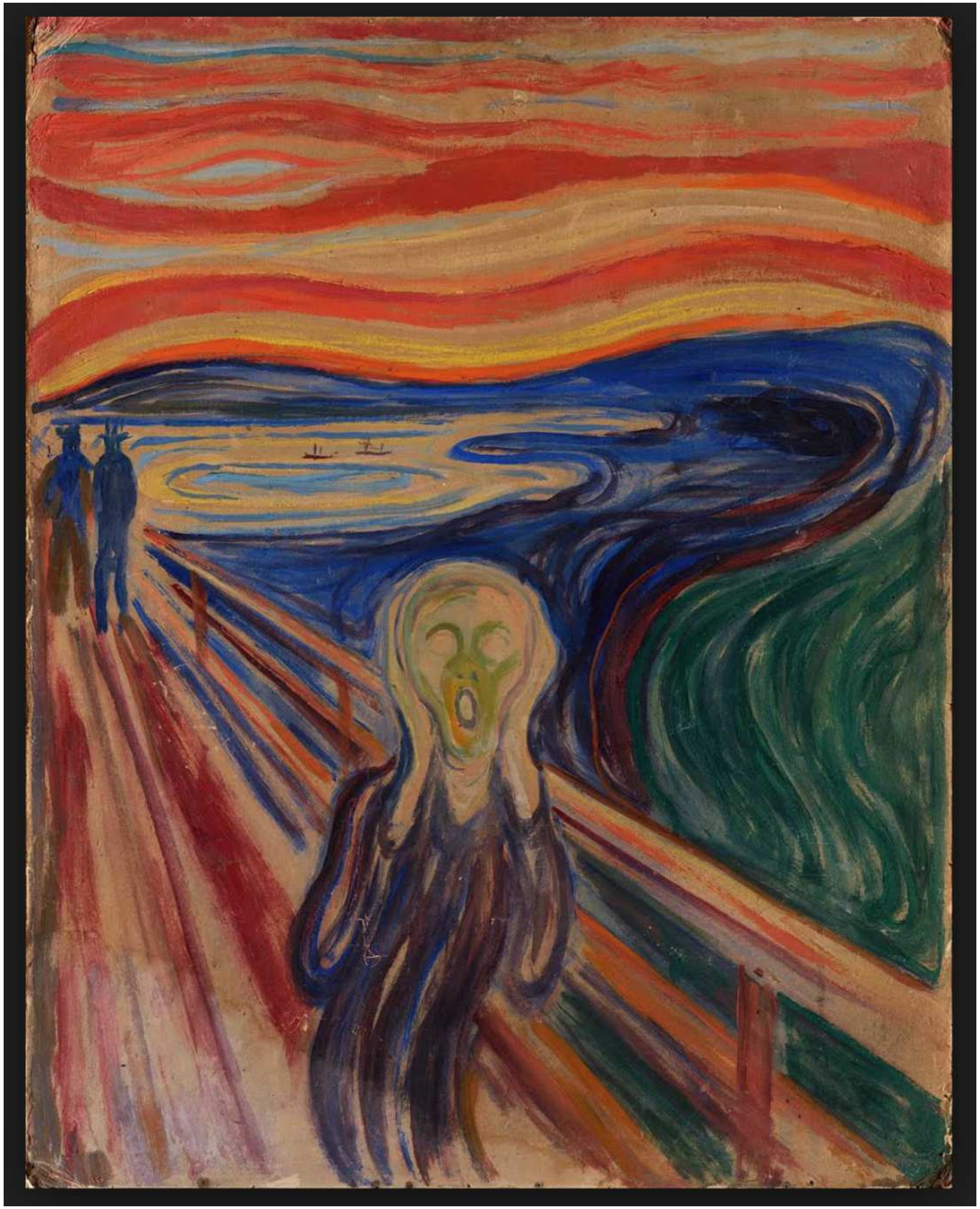


The ubiquity of singularities

The relevant geometric spaces for physics very often contain singularities:

- Singularities are ubiquitous.
- Singularities can form dynamically even if absent initially.

String theory is the ideal framework to study physics on singular spaces:

- Massless BPS states from wrapped branes or strings along vanishing cycles 'resolve' the singularity. [Strominger'95], ...
- Singularities indicate extra gauge symmetry and/or massless matter.

Typical procedure:

- Given singular CY n-fold X_n, study its resolution $\hat{X}_n \to X_n$.
- If $\hat{X}_n$ is also CY, study it via SUGRA and then take 'adiabatic limit'.

Some math background (I)

A singular Calabi-Yau n-fold X_n can always be resolved: [Hironaka'64]

$$\rho : \hat{X}_n \to X_n \qquad \text{s.t.} \qquad \hat{X}_n \quad \text{is smooth}$$

$\hat{X}_n$ and X_n agree on dense open sets, but not on exceptional locus Ex

- big resolution: Ex contains divisors E_i
- small resolution: Ex contains no divisors E_i

$$K_{\hat{X}_n} = K_{X_n} + \sum_i a_i E_i \qquad a_i : \text{discrepancies}$$

- all $a_i \geq 0$: canonical singularity
- all $a_i > 0$: terminal singularity
- all $a_i = 0$: crepant resolvable singularity: $\hat{X}_n$ is Calabi-Yau

Outline

Question of this talk:

What if no crepant=CY resolution exists?

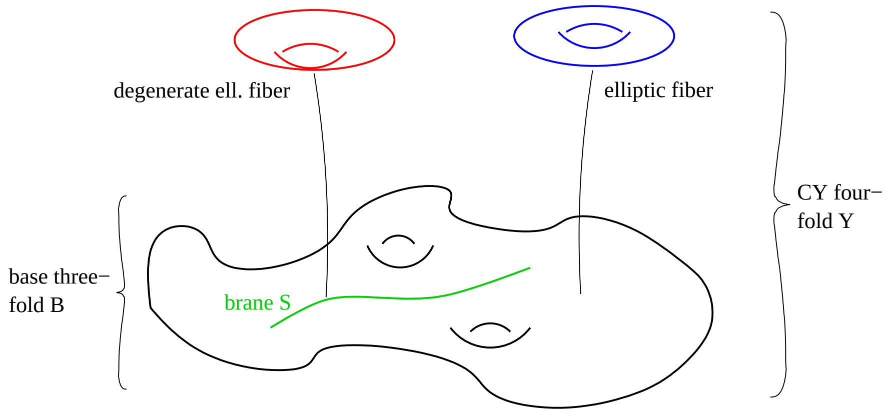
This is a question about **enlarging the moduli space of admissible string compactifications**.

We will study this in the context of F-theory compactifications but conclusions hold more generally.

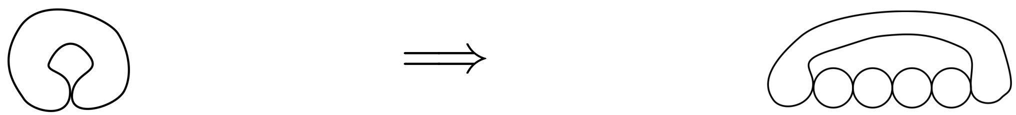
- Focus on isolated $\mathbb{Q}$-factorial terminal singularities in codimension-two of elliptic CY 3-fold.
  (F-theory to 6 dimensions - but physics more general!)
- Give physical meaning of absence of CY resolution.
- Discuss mathematics of singular Y_3 to get quantitative understanding.
- Recent appearance of same type of singularities in [Morrison,Park,Taylor'16]

Singularities in F-theory

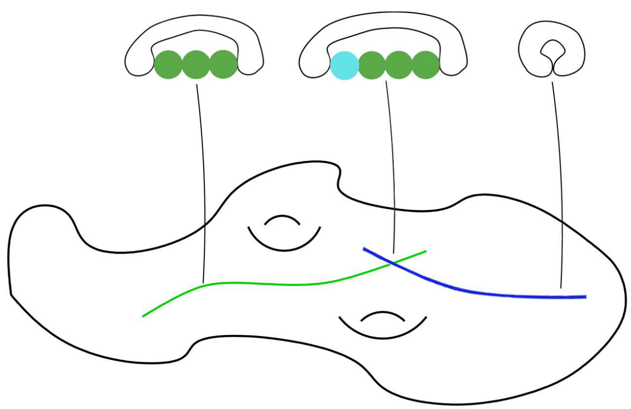
- elliptic fibre $\leftrightarrow$ τ variation
  singular loci $\leftrightarrow$ 7-branes



- **Standard lore:**
  resolve singular point in fibre by tree of $\mathbb{P}^1_i$ $\quad i = 1, \ldots, \mathrm{rk}(\mathfrak{g})$



- gauge field:
  $C_3 = A^i \wedge [E_i] + \ldots +$ M2-branes
- localised matter:
  wrapped M2-branes in codim. 2



Some math background (II)

Consider elliptically fibered Calabi-Yau n-fold Y_n:

- The singularities in codimension-one always allow for a crepant resolution $\hat{Y}_n \to Y_n$.
- All non-crepant resolvable singularities come from codimension-two or higher.

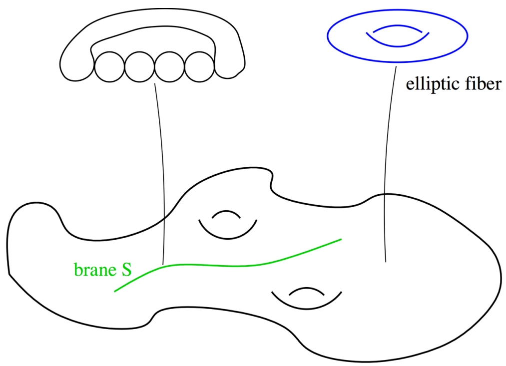
In this talk we focus on **terminal sing. in codimension-two** which are $\mathbb{Q}$**-factorial**:

- Every Weil divisor is also $\mathbb{Q}$-Cartier.
- This implies that no small resolution exists (which would be automatically Calabi-Yau) and also no big crepant resolution.

What are the consequences for the physics and for the mathematics?

F-theory and crepant resolutions

6d gauge field $\mathbb{A}^i$ $\xrightarrow[\text{reduction}]{S^1}$ 5d vector multiplet (A^i, ξ^i)



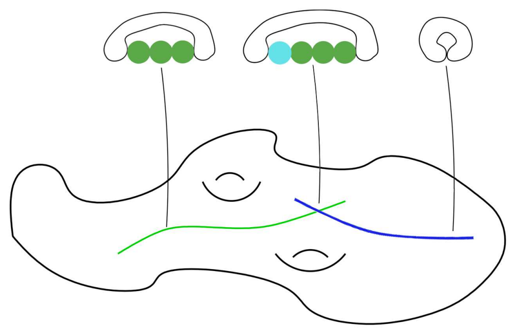
- $A^i \leftrightarrow C_3 = A^i \wedge [E_i] + \ldots$

  E_i: resolution divisors

- $\xi^i \leftrightarrow J = \xi^i\,[E_i] + \ldots$

  $\langle \xi^i \rangle \leftrightarrow$ volume of fibral $\mathbb{P}^1$s

6d charged hypermultiplet $\xrightarrow[\text{reduction}]{S^1}$ 5d KK zero mode ψ_0 + KK tower ψ_n



- State in repr. R with weight $\lambda_{m\,i}$:

  $\lambda_{m\,i} \sim \int_{C_m} [E_i]$

- Mass along Coulomb branch:

  $m_0 = \sum_i \lambda_{m\,i}\,\xi^i \sim \mathrm{vol}(C_m)$

Geometry of the Coulomb branch

F/M-theory provides an intriguing 1-1 correspondence between

web of Calabi-Yau resolutions $\longleftrightarrow$ **Coulomb branches in 5d/3d/1d**

huge literature, including e.g.

[Witten'96] [Intriligator,Morrison,Seiberg'97] [Aharony,Hanany,IntriligatormSeiberg,Strassler'97] [Grimm,Hayashi'11] [Bonetti,Grimm'11] [Intriligator,Jockers,Mayr,Morrison,Plesser'12] [Hayashi,Lawrie,Schafer-Nameki'13] [Hayashi,Lawrie,Morrison,Schafer-Nameki'14] [Schafer-Nameki,TW'16] [Lawrie,Schafer-Nameki,TW'16]

Question of this talk:
What happens if the singular Y cannot be resolved into a Calabi-Yau space $\hat{Y}$ (i.e. when the singularity is non-crepant-resolvable)?

Non-crepant singularities

Suppose we have massless localised matter which is not charged under any massless gauge field in 6d F-theory

1. **Physical interpretation in 5d M-theory effective action:**
   - KK zero-mode remains massless everywhere on the Coulomb branch (no supersymmetric mass along Coulomb branch)
2. **Geometric interpretation for elliptic fibration:**
   - Massless localised matter requires a vanishing $\mathbb{P}^1$ in fibre with wrapped M2-branes
   - There exists no direction in Kähler moduli space such that vol$(\mathbb{P}^1) > 0$ without destroying the Calabi-Yau condition

Non-crepant-resolvable singularities in codim. 2 $\longleftrightarrow$ **Localised uncharged matter in codim. 2**

Examples in literature

1) I_1-Tate-model [Braun,Collinucci,Valandro'14]

- only I_1 singularities in codimension-one $\Delta = z\Delta'$
- $\mathbb{Z}_1$ symmetry - i.e. massive U(1) broken by instantons to $\mathbb{Z}_1 = \emptyset$

  cf. [Grimm,Kerstan,Palti,TW'11] [Martucci,TW'15]
- enhanced to I_2 in codimension-two: non-resolvable conifold singularity!
  $\leftrightarrow$ massless matter

2) $\mathbb{Z}_k$ Weierstrass models

- Weierstrass model associated with a torus-fibration with k-section has non-crepant resolvable conifold singularities in codim. 2 [Braun,Morrison'14]
- In Weierstrass model: location of matter charged (only) under $\mathbb{Z}_k$ in M-theory [Morrison,Taylor'14] [Mayrhofer,Palti,Till,TW'14] [Cvetič,Klevers,Poretschkin'15]; [Anderson,Grimm,Etxebarria,Keitel'14][Klevers,Mayorga,Oehlmann,Piragua,Reuter'14]

Terminal type II model

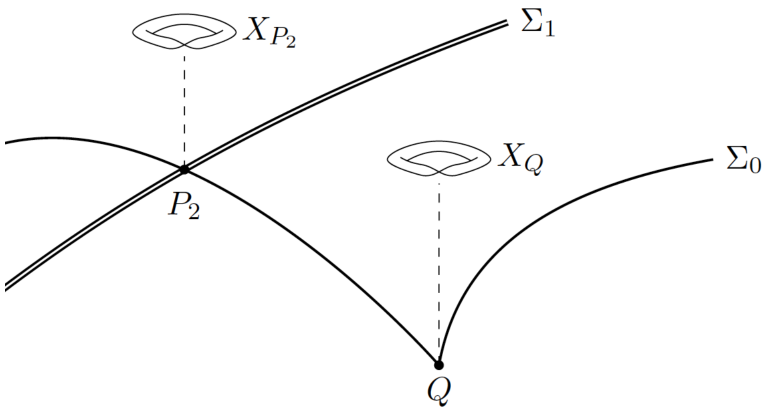
Weierstrass model Y_3 over base $B_2 = \mathbb{P}^2$ (for simplicity) [Arras,Grassi,TW'16]

$$y^2 = x^3 + f(u_i)\,x\,z^4 + g(u_i)\,z^6$$

- $[u_0 : u_1 : u_2]$ homogeneous coordinates on $\mathbb{P}^2$
- $f = f_{12}(u_i), \qquad g = g_{18}(u_i)$

Consider following model:

- $f = u_1\, f_{11}(u_i), \qquad g = u_1\, g_{17}(u_i)$
- $\Delta = 27g^2 + 4f^3 = u_1^2\,(27g_{17}^2 + 4f_{11}^3 u_1) = (\Sigma_1)^2\,\Sigma_0$
- Σ_1: type II singularity (cusp): Y_3 smooth - no gauge group !
- Σ_0: type I_1 singularity (node): Y_3 smooth!
- $\Sigma_1 \cap \Sigma_0$: type III singularity: Y_3 singular!



Terminal type II model

Local form of isolated type III singularity: [Arras,Grassi,TW'16]

$$z_0^a + (z_1^2 + z_2^2 + z_3^2) = 0 \qquad \text{for} \quad a = 3$$

Theorem [Flenner'81][Caibar'03]:

An isolated singularity in Y_3 of above type for general $a \in 2\mathbb{Z} + 1$ is

1. **terminal** (in particular non-crepant)

   if $\hat{Y}_3 \to Y_3$ is a resolution, then $K_{\hat{Y}_3} = K_{Y_3} + \sum_k a_k E_k \qquad a_k > 0$

2. **$\mathbb{Q}$-factorial** every Weil divisor is also $\mathbb{Q}$ Cartier

Further examples: [Arras,Grassi,TW'16]; see also [Morrison,Taylor,Park'16]

$\mathrm{I}_1 \to \mathrm{I}_2$	$\mathbb{Q}$-factorial terminal with $a = 2$
$\mathrm{II} \to \mathrm{III}$ or IV	$\mathbb{Q}$-factorial terminal with $a = 3$
$\mathrm{III} \to \mathrm{I}_0^*$	crepant **or**
gauge group $SU(2)$!	$\mathbb{Q}$-factorial terminal with $a = 2, 3, 5, \ldots$

Challenges with singularities

- Notion of (co)homology ambiguous:
  Intersection homology may not coincide with 'usual' (singular) homology.
- For usual homology, mixed Hodge structure may not be available.
- For usual homology, Poincaré duality need not hold.

Special care in computation in particular of complex str. defs!

Theorem: [Nakayama,Steenbrink'95],[Arras,Grassi,Srinivas,TW to appear]

Consider a Calabi-Yau 3-fold Y_3 with isolated $\mathbb{Q}$-factorial terminal singularities at points P. Then Y_3 admits a smoothing (deformation) to a smooth Calabi-Yau 3-fold X_t, and the space of complex structure deformations of Y_3 has dimension

$$\mathrm{CxDef}(Y_3) = \mathrm{CxDef}(X_t) = h^{2,1}(X_t) = \frac{1}{2}b_3(Y_3) - 1 + \frac{1}{2}\sum_P \underbrace{m_P}_{\text{Milnor number}}$$

$\mathrm{CxDef}(Y_3)$ and Milnor number

$$\mathrm{CxDef}(Y_3) = \mathrm{CxDef}(X_t) = h^{2,1}(X_t) = \frac{1}{2} b_3(Y_3) - 1 + \frac{1}{2} \sum_P \underbrace{m_P}_{\text{Milnor number}}$$

m_P: **Milnor number** of the singularity [Milnor '68]

- Deformation pastes in bouquet of 3-spheres
- $m_P \leftrightarrow$ number of independent 3-spheres

Facts:

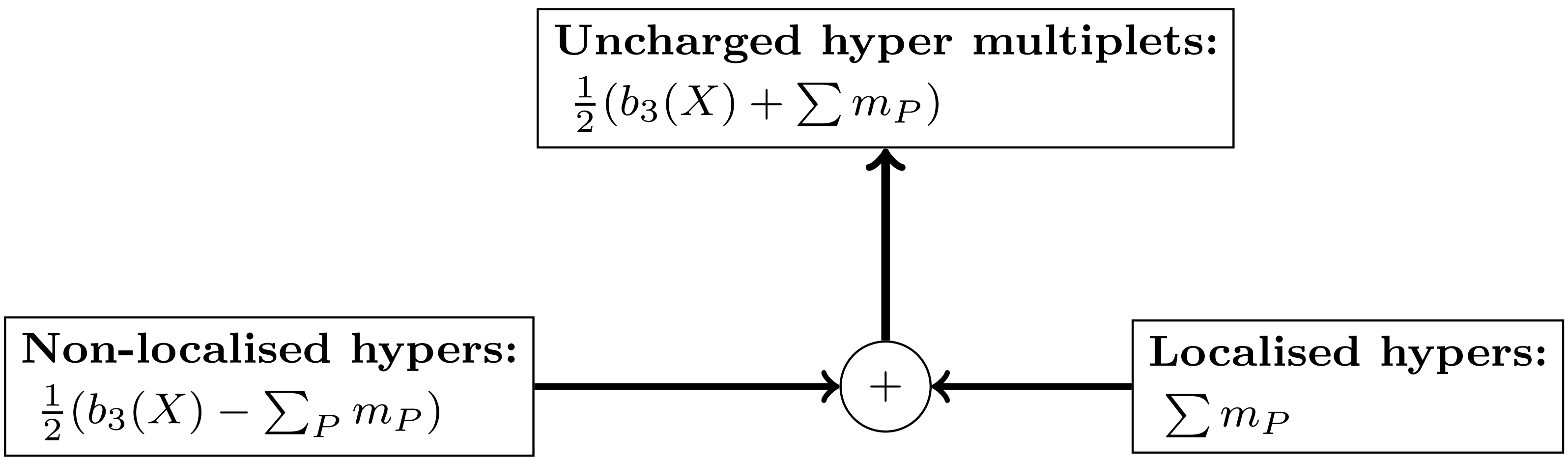
1. For elliptic CY Y_3 the isolated $\mathbb{Q}$-factorial terminal singularity is always of hypersurface type.
2. For these a simple algorithm exists for computation of m_P. [Looijenga '84]
   $m_P = \dim_{\mathbb{C}}(\mathbb{C}[x_i]/\langle \partial f / \partial x_i \rangle)$
3. m_P here counts versal deformations [Saito'71], ...
   $=$ complex structure deformations which change only the singularity

$\mathrm{CxDef}(Y_3)$ and Milnor number

$$\begin{aligned}
\mathrm{CxDef}(Y_3) &= \frac{1}{2} b_3(Y_3) - 1 + \frac{1}{2}\sum_P m_P \\
&= \underbrace{\left[\frac{1}{2} b_3(Y_3) - 1 - \frac{1}{2}\sum_P m_P\right]}_{\text{def. independent of sing.}} + \underbrace{\left[\sum_P m_P\right]}_{\text{def. of sing.}}
\end{aligned}$$

Interpretation: [Arras,Grassi,TW'16]

Isolated $\mathbb{Q}$-factorial terminal sing. hosts m_P uncharged massless hypermultiplets.



Euler characteristic

$b_3(Y_3) \quad \underset{\text{computable via}}{\overset{\text{for singular } Y_3}{\longleftrightarrow}} \quad$ top. Euler characteristic $\chi_{\text{top}}(Y_3)$

For isolated $\mathbb{Q}$-factorial terminal sing.: [Arras,Grassi,TW'16 & to appear w/ Srinivas]

$$\frac{1}{2}\,\chi_{\text{top}}(X) = \text{KaDef}(X) - \text{CxDef}(X) + \frac{1}{2}\sum_P m_P$$

$$\Longrightarrow n_{H_0} = \underbrace{\left(1 + \text{KaDef}(X) - \tfrac{1}{2}\chi_{\text{top}}(X) - \frac{1}{2}\sum_P m_P\right)}_{\text{unlocalised}} + \underbrace{\sum_P m_P}_{\text{localised}}$$

Computation of $\chi_{\text{top}}(Y_3)$:

- Partially resolve Y_3 into $\hat{Y}_3$ by resolving all codim.-one singularities.
- Compute $\chi_{\text{top}}(\hat{Y}_3)$ by summing over all fibers - including the singular fibers in codimension-two.
- Modification of procedure of [Grassi,Morrison '05 & '11]

$\mathrm{CxDef}(Y_3)$ and Milnor number

Gravitational anomaly cancellation in 6d SUGRA:

$$\underbrace{n_H}_{n_{H_0}+n_{H_R}} - n_V + 29 n_T = 273$$

- n_{H_R}: hypers in representation R of gauge algebra
- $n_{H_0} = 1 + h^{1,1}(\hat{Y}_3) - \frac{1}{2}\chi_{\mathrm{top}}(\hat{Y}_3) + \frac{1}{2}\sum_P m_P$: uncharged hypers

Applied to our type II example over base $B_2 = \mathbb{P}^2$: [Arras,Grassi,TW'16]

- $h^{1,1}(\mathbb{P}^2) = 1, \qquad G = \emptyset, \qquad \sum_P m_P = 17 \times 2 = 34$
- $\chi_{\mathrm{top}}(Y_3) = -506$

Anomaly check:

$$n_H = n_{H_0} = 273 \qquad n_T = 0 = n_V = 0 \qquad \checkmark$$

uncharged localised hypers per singularity $= m_P = 2$

More examples

(μ_f, μ_g)	$(1,1)$	$(2,1)$	$(1,5)$	$(1,7)$
fibres	II $\to$ III	II $\to$ IV	III$\to I_0^*$	III$\to I_0^*$
Gauge Group	—	—	$SU(2)$	$SU(2)$
# isolated sing.	17	17	11	11
m_P	2	2	2	4
χ_{top}	-506	-506	-434	-412
$h^{1,1}$	2	2	3	3
Cxdef	272	272	231	231
$n^{\text{loc.}}_{\text{unch.}} = \sum_P m_P$	34	34	22	44
$n^{\text{loc.}}_{\text{unch.}}$ **per locus**	**2**	**2**	**2**	**4**
$n^{\text{unloc.}}_{\text{unch.}} = h^{2,1} + 1 - 1/2 \sum_P m_P$	1 + 238	1 + 238	1 + 209	1 + 187
$n_{\text{ch.}}$	0	0	44	44
$n_{\text{ch.}}$ **per locus**	**0**	**0**	**4**	**4**
irrep	—	—	$2 \times \mathbf{2}$	$2 \times \mathbf{2}$

Conclusions

- **Codim.-two $\mathbb{Q}$-factorial terminal singularities** in ell. 3-folds:
  massless uncharged hypermultiplets
- Quantitative analysis is possible by working with rational homology of singular space and computing $\chi_{\rm top}(Y_3)$

Paete, non dolet!

Open questions:

- Extension to CY 4-folds: Expect same physics in codimension-two
  Natural conjecture for counting (in absence of flux):

  $n_{\rm loc.,uncharged} = h^0(C, \sqrt{K_C})\, m_P$ $\sqrt{K_C}$: spin bundle of curve C

- Effective action of the localised states?

  In some cases, they carry non-trivial 'global' charge, violated by instantons

  What is the situation here?