NLO and higher order calculations, multi-leg calculations

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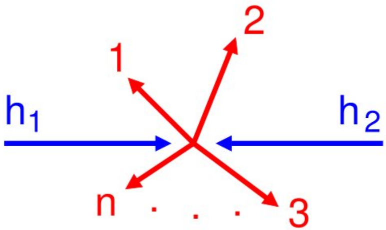
Outline

- numerical recursion for tree-level amplitudes
- color representation
- (BCFW recursion)
- one-loop amplitudes
- (collinear factorization)
- singularities of tree-level matrix elements
- subtraction for real radiation integrals

Collinear factorization

To separate a perturbatively calculable from the universal in hadron scattering.



PDFs are related to the structure of the hadrons, universal to the scattering process

$$\sigma_{h_1,h_2\to n}(p_1,p_2) = \sum_{a,b}\int dx_1\,dx_2\,\boxed{f_a(x_1,\mu)\,f_b(x_2,\mu)}\,\hat{\sigma}_{a,b\to n}(x_1p_1,x_2p_2;\mu)$$

$$\hat{\sigma}_{a,b\to n}(p_a,p_b;\mu) = \boxed{\int d\Phi(p_a,p_b\to\{p\}_n)}\,\boxed{|\mathcal{M}_{a,b\to n}(p_a,p_b\to\{p\}_n;\mu)|^2}\,\boxed{\mathcal{O}(p_a,p_b,\{p\}_n)}$$

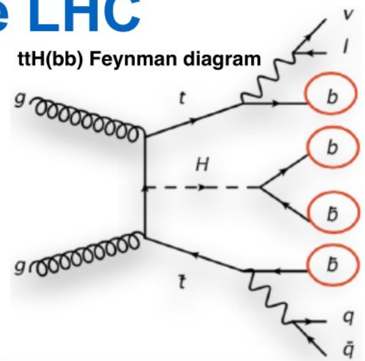
Phase space (includes spin/color summation) governs the kinematics

Matrix element (squared) contains model parameters, governs the dynamics

Observable, imposes phase space cuts

ttH Production at the LHC

Events triggered by single lepton triggers



Single Lepton channel

- One leptonic W decay
- One electron or one muon
- At least 4 jets
- At least 2 b-tagged jets

Di-lepton channel

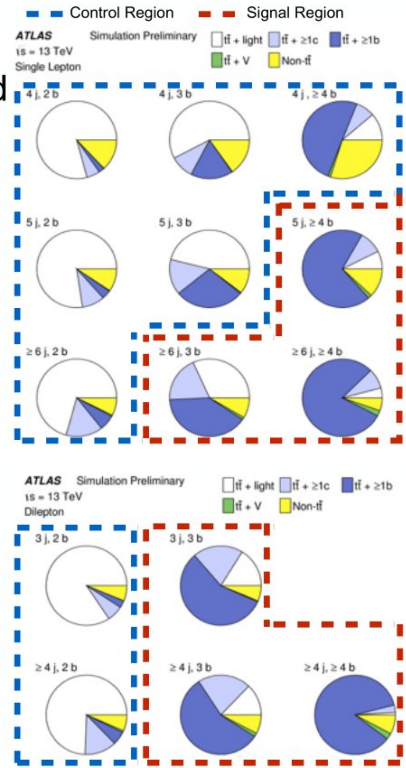
- Two leptonic W decays
- Two opposite charge light leptons (e,μ)
- At least 3 jets
- At least 2 b-tagged jets

- Very challenging analysis :
  - 4 b-jets in final state
  - Large background from $t\bar{t}$+jets
  - Strategy : **Divide** into different regions

- This dataset used 3.2 fb^{-1} from 2015 & 10.0 fb^{-1} from 2016.

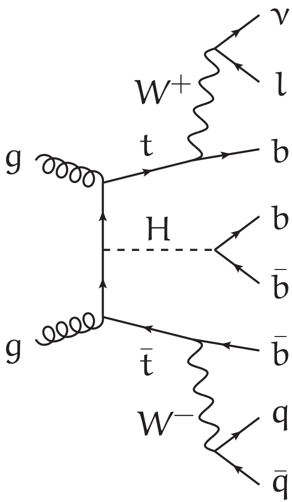
Signal & Background Composition

- The ttH signal process is modelled using MadGraph5_aMC@NLO+Pythia8
- Dominating background ($t\bar{t}$+jets)
  - $t\bar{t}$+ ≥ 1 b jets
  - $t\bar{t}$+ ≥ 1 c jets   Powheg+Pythia6
  - $t\bar{t}$+ light jets
- Other backgrounds
  - $t\bar{t}$+V
  - Non-$t\bar{t}$
    Single Top, W/Z+jets, Diboson
    Multi-jet (Fakes and non-prompt)



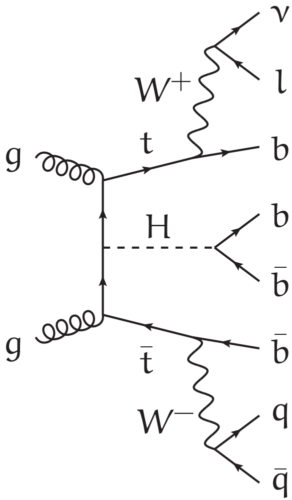
High multiplicity

- signal: $\mathcal{O}(1 \times 10^3)$ graphs
- background: $\mathcal{O}(3 \times 10^3)$ graphs



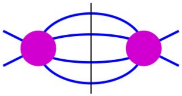
High multiplicity

- signal: $\mathcal{O}(1 \times 10^3)$ graphs
- background: $\mathcal{O}(3 \times 10^3)$ graphs
- 1-loop signal: $\mathcal{O}(2 \times 10^5)$ graphs
- 1-loop background: $\mathcal{O}(3 \times 10^5)$ graphs
- real(extra gluon) signal: $\mathcal{O}(1 \times 10^4)$ graphs
- real(extra gluon) background: $\mathcal{O}(5 \times 10^4)$ graphs



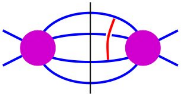
Leading Order:

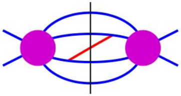
$$\hat{\sigma}^{\mathrm{LO}}_{a,b\to n} = \int d\Phi_n \left|\mathcal{M}^{(0)}_{a,b\to n}\right|^2 \mathcal{O}^{\mathrm{LO}}_n$$



Next-to-Leading Order:

$$\hat{\sigma}^{\mathrm{NLO}}_{a,b\to n} = \int d\Phi_n \, 2\Re\left(\mathcal{M}^{(0)}_{a,b\to n}\mathcal{M}^{(1)}_{a,b\to n}\right) \mathcal{O}^{\mathrm{LO}}_n + \int d\Phi_{n+1} \left|\mathcal{M}^{(0)}_{a,b\to n+1}\right|^2 \mathcal{O}^{\mathrm{NLO}}_{n+1}$$





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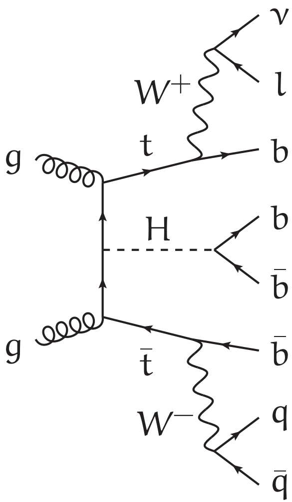


- need milions of evaluations in practical Monte Carlo calculations
- need many partonic processes
- want to study many other processes besides $t\bar{t}H$ production
- need automation to deal with these multiplicities

Numerical evaluation of amplitudes

Phase space integration

$$\langle \mathcal{O} \rangle = \int d\Phi_n(\{p\}_n)\, |\mathcal{M}_n(\{p\}_n)|^2\, \mathcal{O}_n(\{p\}_n)$$

has to be done by Monte Carlo.

Numerical evaluation of amplitudes

Phase space integration

has to be done by Monte Carlo.

Helicity and Color summation

does not involve complicated restrictions like "cuts"

$$\langle \mathcal{O} \rangle = \int d\Phi_n(\{p\}_n) \sum_{\{\lambda\}_n} \sum_{\{a\}_n} |\mathcal{M}_n(\{p\}_n, \{\lambda\}_n, \{a\}_n)|^2 \, \mathcal{O}_n(\{p\}_n)$$

and could be conceived to be performed algebraïcly. For many-particle final states, however, this leads to *huge expressions* for

$$\sum_{\{\lambda\}_n} \sum_{\{a\}_n} |\mathcal{M}_n(\{p\}_n, \{\lambda\}_n, \{a\}_n)|^2$$

The alternative is to treat helicity and color summation on the same footing as phase space integration, and just evaluate

$$\mathcal{M}_n(\{p\}_n, \{\lambda\}_n, \{a\}_n)$$

numerically as function of momentum- helicity- and color-configurations.

Numerical evaluation of amplitudes

- Expressions in terms of invariants for scattering amplitudes involving several particles tend to become huge.

- Eventual goal is (just) their repeated numerical evaluation in a MC.

Avoid expressions completely!

- Essentially the only expressions involved should be the vertices and the propagators of the field theory.

- Use an algorithm to evaluate scattering amplitudes, given the numerical values of momenta, helicity and color degrees of freedom of the external particles as initial input.

Zero-dimensional field theory

Consider ϕ^3-theory on a single space-time point

$$Z[J] = \int_{-\infty}^{\infty} d\phi \, \exp\left\{\frac{i}{\hbar}\left[J\phi + S(\phi)\right]\right\} \quad , \quad S(\phi) = -\frac{m^2}{2}\phi^2 - \frac{g}{6}\phi^3 \ , \ \mathrm{Im}(m^2 < 0)$$

We trivially have the linear Dyson-Schwinger equation

$$0 = \int_{-\infty}^{\infty} d\phi \, \frac{\hbar}{i}\frac{d}{d\phi} \exp\left\{\frac{i}{\hbar}\left[J\phi + S(\phi)\right]\right\} = \left(J - \frac{\hbar}{i} m^2 \frac{d}{dJ} + \frac{\hbar^2 g}{2}\frac{d^2}{dJ^2}\right) Z[J]$$

$Z[J]$ generates zero-dimensional "Green functions", connected "Green functions" generated by

$$W[J] = \ln Z[J]$$

Non-linear Dyson-Schwinger equation

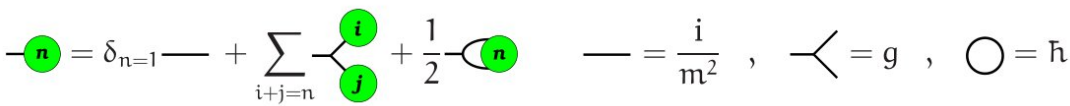
$$0 = J + i m^2 \frac{dW[J]}{dJ} + \frac{g}{2}\left[\hbar \frac{d^2 W[J]}{dJ^2} + \left(\frac{dW[J]}{dJ}\right)^2\right]$$

Zero-dimensional field theory

Dyson-Schwinger equation for Green functions from $\frac{dW[J]}{dJ} = \sum_{n=0}^{\infty} \frac{C_{n+1} J^n}{n!}$

$$\frac{C_{n+1}}{n!} = \frac{i}{m^2}\left(\delta_{n=1} + g \sum_{i+j=n} \frac{C_{i+1}}{i!}\frac{C_{j+1}}{j!} + \frac{\hbar g}{2}\frac{C_{n+2}}{n!}\right)$$

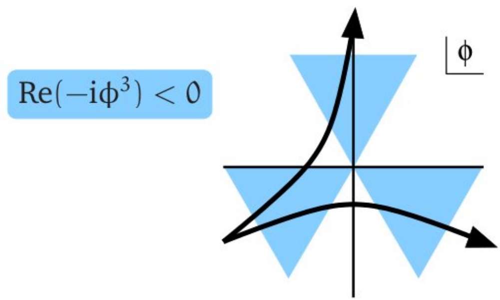
We may cast the equation into a graphical form



Solutions for tadpole

$$C_1^{\mathrm{pert}} \propto \frac{\hbar g^2}{m^4} \qquad C_1^{\mathrm{non-pert}} \propto \frac{m^2}{g}$$

Non-perturbative solution corresponds to other integration contour in the complex ϕ-plane in the definition of $Z[J]$.



Zero-dimensional field theory

Introduce more zero-dimensional points

$$S(\phi) = -\sum_{k,l} \frac{1}{2} A_{k,l}\phi_k\phi_l - \sum \frac{g}{6}\phi_l^3 \quad , \quad \mathrm{Im}(A_{k,k} < 0)$$

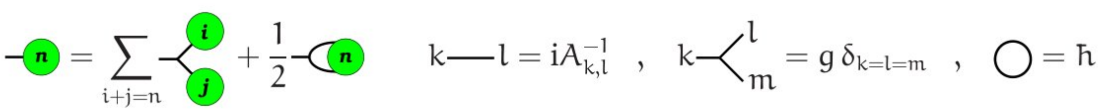
Dyson-Schwinger equation

$$0 = J_k + i\sum_l A_{k,l}\frac{\partial W[J]}{\partial J_l} + \frac{g}{2}\left[\hbar \frac{\partial^2 W[J]}{\partial J_k^2} + \left(\frac{\partial W[J]}{\partial J_k}\right)^2\right]$$

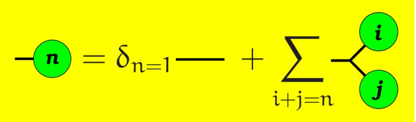
Expand generating function in terms of Green functions

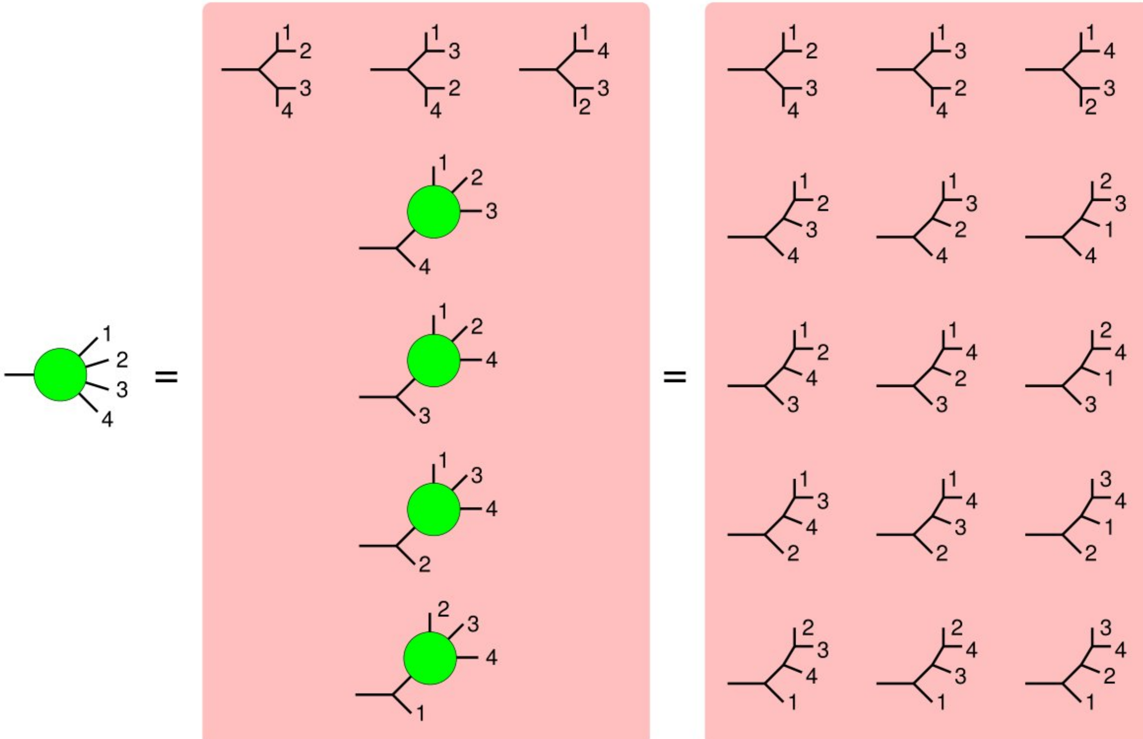
$$\frac{\partial W[J]}{\partial J_l} = \sum_{i_1+i_2+\cdots+i_k=n} C_{l;i_1 i_2\cdots i_k}\frac{J_1^{i_1}}{i_1!}\frac{J_2^{i_2}}{i_2!}\cdots\frac{J_k^{i_k}}{i_k!}$$

Graphical interpretation

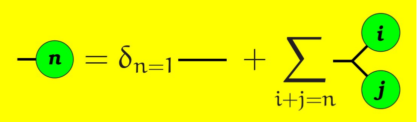


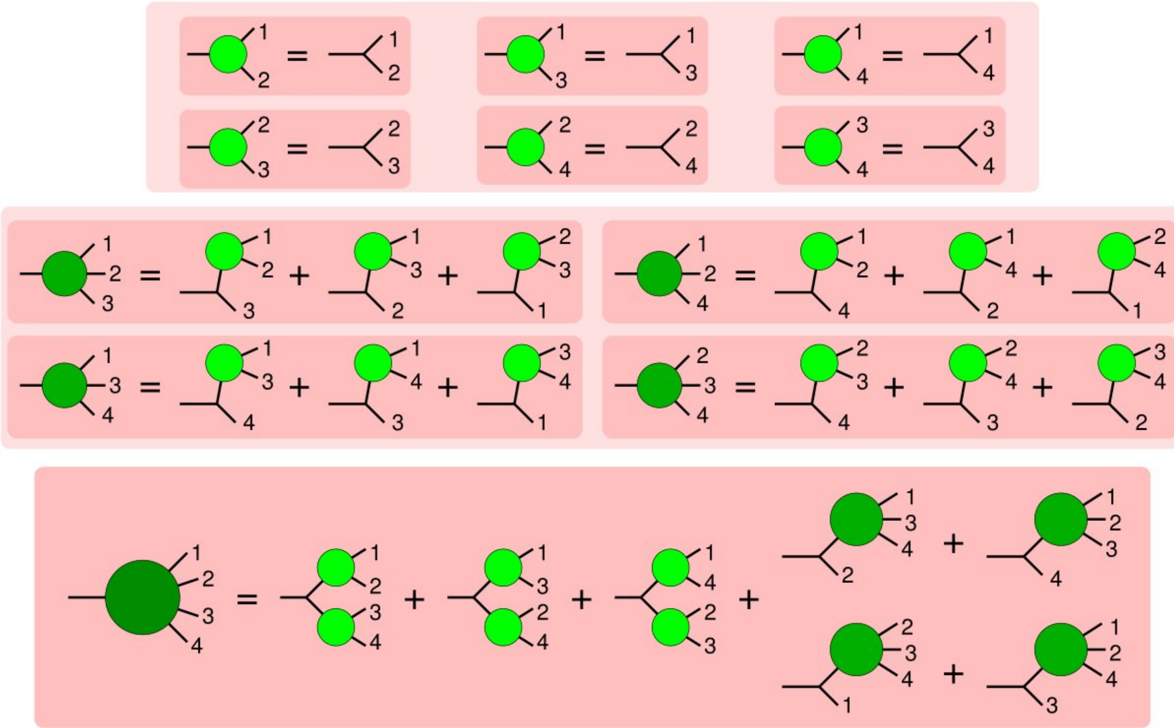
Tree-level recursion





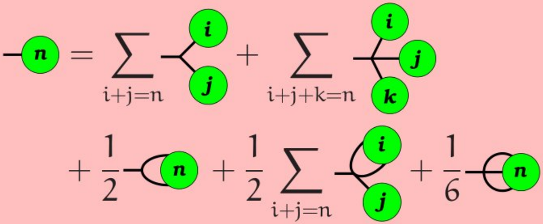
Tree-level recursion



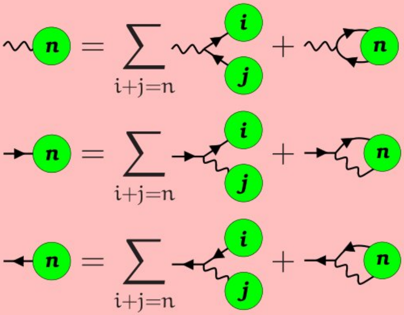


Perturbative Dyson-Schwinger recursion

Theories with four-point vertices:



Theories with more types of currents:



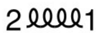
Currents may have several components.

- distinguishable external lines correspond to on-shell particles
  $\Longrightarrow$ polarization vectors, spinors, 1
- sum of momenta of on-shell lines is equal to momentum of off-shell line
- vertices directly from Feynman rules in momentum space
- off-shell line carries propagator from Feynman rules, in any gauge
- on-shell $(n+1)$-leg amplitude
  - from current with n on-shell legs
  - by omitting the final propagator
  - and contracting with pol.vec. or spinor instead

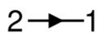
History

- Berends, Giele 1987: planar multi-gluon amplitudes
- Caravaglios, Moretti 1995: formulation for arbitrary lagrangians
- Draggiotis, Kleiss, Papadopoulos 1998: multi-gluon amplitudes
- Caravaglios, Mangano, Moretti, Pittau 1998: multi-jet processes
- Kanaki, Papadopoulos 1999: HELAC (standard model)
- Moretti, Ohl, Reuter 2001: O'Mega
- Mangano, Moretti, Piccinini, Pittau, Polosa 2003: ALPGEN
- Gleisberg, Hoeche 2008: Comix
- Kleiss, van den Oord 2011: Camorra
- Actis, Denner, Hofer, Scharf, Uccirati 2012: Recola (one-loop)

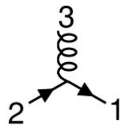
QCD Feynman rules



$$= \frac{-i}{p^2} \eta^{\mu_1 \mu_2} \delta^{a_1 a_2}$$

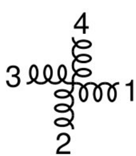


$$= \frac{i}{\not{p} - m} \delta_{i_1 i_2}$$



$$= -ig\, T^{a_3}_{i_1 i_2}\, \gamma^{\mu_3}$$

$$= g\, f^{a_1 a_2 a_3} \left[(p_1 - p_2)^{\mu_3} \eta^{\mu_1 \mu_2} + (p_2 - p_3)^{\mu_1} \eta^{\mu_2 \mu_3} + (p_3 - p_1)^{\mu_2} \eta^{\mu_3 \mu_1} \right]$$



$$\begin{aligned} = ig^2 \big[& \left(f^{a_1 a_3 b} f^{a_2 a_4 b} - f^{a_1 a_4 b} f^{a_3 a_2 b} \right) \eta^{\mu_1 \mu_2} \eta^{\mu_3 \mu_4} \\ & + \left(f^{a_1 a_2 b} f^{a_3 a_4 b} - f^{a_1 a_4 b} f^{a_2 a_3 b} \right) \eta^{\mu_1 \mu_3} \eta^{\mu_2 \mu_4} \\ & + \left(f^{a_1 a_3 b} f^{a_4 a_2 b} - f^{a_1 a_2 b} f^{a_3 a_4 b} \right) \eta^{\mu_1 \mu_4} \eta^{\mu_3 \mu_2} \big] \end{aligned}$$

Treating *eg.* gluon off-shell currents as 8×4-dimensional vectors $A^{a,\mu}$ and performing all contractions in each vertex is out of the question.

Sum over spins and colors

Calculation of a cross section requires phase space integration and summation over spins and colors.

$$\sigma = \int d\Phi \sum_{\text{spin}} \sum_{\text{color}} |\mathcal{M}(\Phi, \text{spin}, \text{color})|^2 \; \mathcal{O}(\Phi)$$

- Phase space must we dealt with within a Monte Carlo approach (that's why we need to be able to evaluate scattering amplitudes numerically efficiently)
- Spin may be dealt with within a Monte Carlo approach:

$$\sum_{+,-} \Rightarrow \int_0^1 d\rho \quad , \quad \varepsilon^{\mu}(\rho) = u_+(\rho)\varepsilon_+^{\mu} + u_-(\rho)\varepsilon_-^{\mu} \quad , \quad \int_0^1 u_i(\rho)u_j(\rho)^* = \delta_{i,j}$$

  - random helicities: $u_{\pm}(\rho) = \sqrt{2}\,\theta(\pm(\frac{1}{2} - \rho))$
  - random polarizations: $u_{\pm}(\rho) = e^{\pm i\pi\rho}$
- Color may be dealt with also within a Monte Carlo approach

What color representation to use?

Color-flow representation

$$\sum_a |\mathcal{A}^a|^2 = \sum_{a,b} \delta^{ab} \mathcal{A}^a \mathcal{A}^{b*} = \sum_{a,b} 2\mathrm{Tr}\{T^a T^b\} \mathcal{A}^a \mathcal{A}^{b*} = \sum_{i,j} |\mathcal{A}^i_j|^2 \ , \quad \mathcal{A}^i_j = \sqrt{2}(T^a)^i_j \mathcal{A}^a$$

Contract all external gluons with $\sqrt{2}(T^a)^i_j$
and replace in all gluon propagators $\delta^{ab} = 2\mathrm{Tr}\{T^a T^b\}$
Color structure of the vertices become

$$\text{3-gluon:} \quad 2^{3/2} f^{abc} (T^a)^{i_1}_{j_1} (T^b)^{i_2}_{j_2} (T^c)^{i_3}_{j_3} = \frac{-i}{\sqrt{2}} \left(\delta^{i_1}_{j_2} \delta^{i_2}_{j_3} \delta^{i_3}_{j_1} - \delta^{i_1}_{j_3} \delta^{i_2}_{j_1} \delta^{i_3}_{j_2} \right)$$

$$\begin{aligned} \text{4-gluon:} \quad & 4(f^{abe} f^{cde} - f^{ade} f^{bce}) (T^a)^{i_1}_{j_1} (T^b)^{i_2}_{j_2} (T^c)^{i_3}_{j_3} (T^d)^{i_4}_{j_4} \\ & = \frac{-1}{2} \Big(2\delta^{i_1}_{j_2} \delta^{i_2}_{j_3} \delta^{i_3}_{j_4} \delta^{i_4}_{j_1} + 2\delta^{i_1}_{j_4} \delta^{i_2}_{j_1} \delta^{i_3}_{j_2} \delta^{i_4}_{j_3} \\ & \quad - \delta^{i_1}_{j_2} \delta^{i_2}_{j_4} \delta^{i_3}_{j_1} \delta^{i_4}_{j_3} - \delta^{i_1}_{j_3} \delta^{i_2}_{j_1} \delta^{i_3}_{j_4} \delta^{i_4}_{j_2} - \delta^{i_1}_{j_3} \delta^{i_2}_{j_4} \delta^{i_3}_{j_2} \delta^{i_4}_{j_1} - \delta^{i_1}_{j_4} \delta^{i_2}_{j_3} \delta^{i_3}_{j_1} \delta^{i_4}_{j_2} \Big) \end{aligned}$$

$$\text{quark-gluon:} \quad \sqrt{2}\, (T^a)^{i_1}_{j_1} (T^b)^{i_2}_{j_2} = \frac{1}{\sqrt{2}} \left(\delta^{i_1}_{j_2} \delta^{i_2}_{j_1} - \frac{1}{N_c} \delta^{i_1}_{j_1} \delta^{i_2}_{j_2} \right)$$

$1/N_c$ contribution in quark-gluon vertex, but trivial gluon propagator: $\delta^{i_1}_{j_2} \delta^{i_2}_{j_1}$

Color-dressed amplitudes

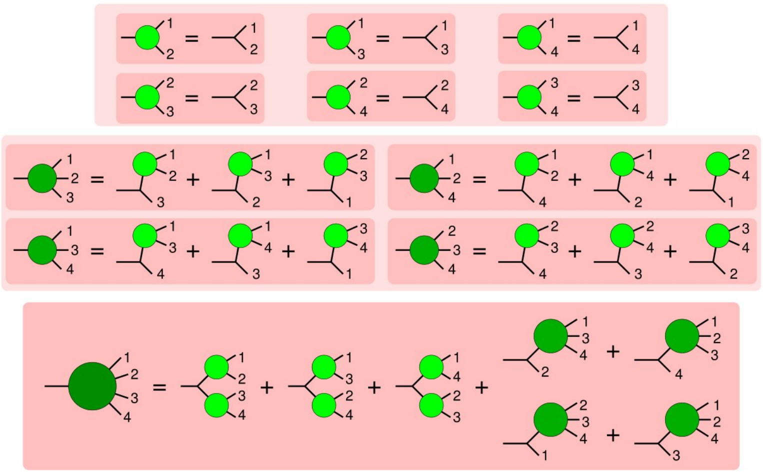
Worek, Papadopoulos 2003
Duhr, Hoeche, Maltoni 2006

Given the external colors $(i_1, j_1), (i_2, j_2), \ldots, (i_n, j_n)$ the internal colors are almost fixed.

$$(i_1, j_1) = (1, 2) \;\; (i_2, j_2) = (2, 3) \quad \longrightarrow \quad \delta^{i_1}_{j_2}\delta^{i_2}_{j_3}\delta^{i_3}_{j_1} - \delta^{i_1}_{j_3}\delta^{i_2}_{j_1}\delta^{i_3}_{j_2} \neq 0 \;\Leftrightarrow\; (i_3, j_3) = (1, 3)$$

$$(i_1, j_1) = (3, 2) \;\; (i_2, j_2) = (2, 3) \quad \longrightarrow \quad (i_3, j_3) = (2, 2) \;\; (i_3, j_3) = (3, 3)$$

Full colored amplitude can be evaluated quickly (no summations in vertices) provided the skeleton can be quickly constructed event by event.



Color-connected amplitudes

Scattering amplitude with n color pairs can be expressed as

$$\mathcal{M}^{i_1 i_2 \cdots i_n}_{j_1 j_2 \cdots j_n} = \sum_{\text{all perm.}} \delta^{i_1}_{j_{\sigma(1)}} \delta^{i_2}_{j_{\sigma(2)}} \ldots \delta^{i_n}_{j_{\sigma(n)}} \mathcal{A}_\sigma(1, 2, \ldots, n)$$

where $\mathcal{A}_\sigma(1, 2, \ldots, n)$ does not depend on the external color, but may depend on N_c. For small n, the explicit color sum is more efficient than color sampling

$$\sum_{\text{color}} |\mathcal{M}|^2 = \sum_{\sigma,\sigma'} N_c^{y(\sigma,\sigma')} \mathcal{A}_\sigma \mathcal{A}^*_{\sigma'}$$

where $y(\sigma, \sigma')$ is the number of common cycles in σ and σ'.

The DS skeleton for $\mathcal{A}_\sigma$ can be found from $\mathcal{M}$, by imagining that $N_c = n$, and assigning the external color configuration

$$(1, \sigma(1)) \;\; (2, \sigma(2)) \;\; \cdots \;\; (n, \sigma(n))$$

and multiplying quark-gluon vertices by $-i\sqrt{N_c}$ if they involve an internal gluon with $i = j$.

Color summation: planar decomposition

There exist other decompositions of amplitudes with gluons and/or (anti-)quarks in which the color content C is factorized from the spin/kinematics content A

$$\mathcal{M} = \sum_{\sigma \in S_n} C(\sigma)\, A(\sigma) \qquad , \qquad \sum_{\text{color}} |\mathcal{M}|^2 = \sum_{\sigma \in S_n} \sum_{\tau \in S_n} \left(\sum_{\text{color}} C(\sigma) C(\tau)^* \right) A(\sigma) A(\tau)^*$$

The dual amplitudes A can be calculated by planar DS recursion.
The number n is the number of color pairs $n_{\text{gluon}} + \frac{1}{2}(n_{\text{quark}} + n_{\text{antiquark}})$, except

- when there are only gluons: $n = n_{\text{gluon}} - 2 = \boxed{n_{\text{parton}} - 2}$ del Duca, Dixon, Maltoni 1999

$$C(\sigma) = f^{a_{n+2} a_{\sigma(1)} b_1} f^{b_1 a_{\sigma(2)} b_2} \ldots f^{b_{n-2} a_{\sigma(n-1)} b_{n-1}} f^{b_{n-1} a_{\sigma(n)} a_{n+1}}$$

- when there are only gluons plus a quark-pair: $n = n_{\text{gluon}} = \boxed{n_{\text{parton}} - 2}$

$$C(\sigma) = (T^{a_{\sigma(1)}} T^{a_{\sigma(2)}} \ldots T^{a_{\sigma(n-1)}} T^{a_{\sigma(n)}})_{ji}$$

For only gluons plus 2 quark-pairs, also $n = n_{\text{gluon}} + \frac{1}{2}(n_{\text{quark}} + n_{\text{antiquark}}) = \boxed{n_{\text{parton}} - 2}$

Weyl spinors for light-like momenta

$$|p] = \begin{pmatrix} L(p) \\ \mathbf{0} \end{pmatrix} \qquad L(p) = \frac{1}{\sqrt{|p_0+p_3|}} \begin{pmatrix} -p_1 + ip_2 \\ p_0 + p_3 \end{pmatrix}$$

$$|p\rangle = \begin{pmatrix} \mathbf{0} \\ R(p) \end{pmatrix} \qquad R(p) = \frac{\sqrt{|p_0+p_3|}}{p_0+p_3} \begin{pmatrix} p_0 + p_3 \\ p_1 + ip_2 \end{pmatrix}$$

Dual spinors are defined without complex conjugation

$$[p| = \big((\mathcal{E}L(p))^T, \mathbf{0} \big)$$
$$\langle p| = \big(\mathbf{0}, (\mathcal{E}^T R(p))^T \big)$$
$$\mathcal{E} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\langle p||q] = [p||q\rangle = 0$$
$$\langle p||p\rangle = [p||p] = 0$$
$$|p\rangle[p| + |p]\langle p| = \not{p} = \gamma_\mu p^\mu$$
$$\not{p}|p\rangle = \not{p}|p] = 0 \quad , \quad \langle p|\not{p} = [p|\not{p} = 0$$
$$p^\mu = \tfrac{1}{2}\langle p|\gamma^\mu|p]$$

$$\langle pq\rangle \equiv \langle p||q\rangle \quad , \quad [pq] \equiv [p||q]$$
$$\langle qp\rangle = -\langle pq\rangle \quad , \quad [qp] = -[pq]$$
$$\langle pq\rangle[qp] = 2p\cdot q$$
$$\langle p|\not{k}|q] = [q|\not{k}|p\rangle$$
$$\langle p|\not{r}|q] = \langle pr\rangle[rq]$$

Schouten identity

$$\frac{|q\rangle\langle p|}{\langle pq\rangle} + \frac{|p\rangle\langle q|}{\langle qp\rangle} + \frac{|q][p|}{[pq]} + \frac{|p][q|}{[qp]} = 1$$

BCFW recursion

Britto, Cachazo, Feng, Witten 2005

Multi-gluon amplitudes have much simpler expressions than one would expect from the Feynman graphs, in particular the MHV amplitudes:

$$\mathcal{A}(i^-, j^-, (\text{the rest})^+) = \frac{\langle p_i p_j \rangle^4}{\langle p_1 p_2 \rangle \langle p_2 p_3 \rangle \cdots \langle p_{n-2} p_{n-1} \rangle \langle p_{n-1} p_n \rangle \langle p_n p_1 \rangle}$$

BCFW recursion allows for easy construction of such simple expressions

- it is a recursion of *on-shell amplitudes*, rather than off-shell Green functions
- it is most efficiently applied as a recursion of *expressions*
- it is easily proven using Cauchy's theorem

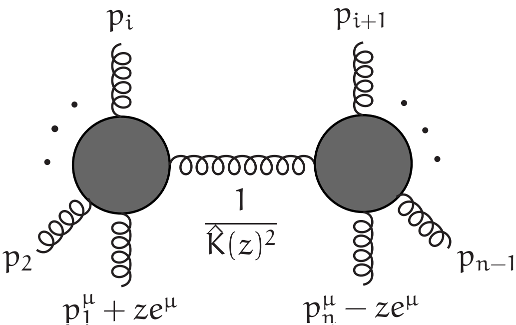
For a rational function f of a complex variable z which vanishes at infinity, we have

$$\oint_R \frac{dz}{2\pi i} \frac{f(z)}{z} \overset{R\to\infty}{=} 0 \qquad \Rightarrow \qquad f(0) = \sum_i \frac{\text{Residue}(f \;@\; z = z_i)}{-z_i}$$

This is applied to amplitudes by turning them into functions of a complex variable by analytical continuation of the momenta to complex values.

BCFW recursion

Britto, Cachazo, Feng, Witten 2005



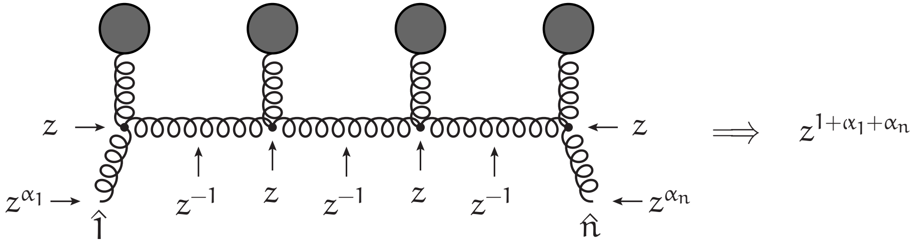
$$\hat{K}^\mu(z) = p_1^\mu + \cdots + p_i^\mu + ze^\mu$$
$$= -p_{i+1}^\mu - \cdots - p_n^\mu + ze^\mu$$

$$\hat{K}(z)^2 = 0 \quad \Leftrightarrow \quad z = -\frac{(p_1 + \cdots + p_i)^2}{2(p_2 + \cdots + p_i)\cdot e}$$

$$\mathcal{A}(1^+, 2, \ldots, n{-}1, n^-) = \sum_{i=2}^{n-1} \sum_{h=+,-} \mathcal{A}(\hat{1}^+, 2, \ldots, i, -\hat{K}_{1,i}^h) \frac{1}{K_{1,i}^2} \mathcal{A}(\hat{K}_{1,i}^{-h}, i{+}1, \ldots, n{-}1, \hat{n}^-)$$

BCFW recursion

Britto, Cachazo, Feng, Witten 2005



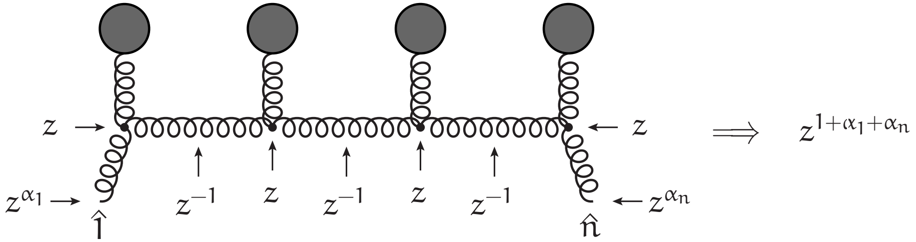
- Does not work for all combinations of shift vector and helicities of shifted gluons, amplitude does not vanish as function of z at infinity for all of them,
- but working choices do exist for all helicity amplitudes.
- Starting point of the recursion are 3-point amplitudes, which do not necessarily vanish when momenta are shifted

$$\mathcal{A}(1^+,2^-,3^-) = \frac{\langle 23\rangle^3}{\langle 31\rangle\langle 12\rangle} \quad , \quad \mathcal{A}(1^-,2^+,3^+) = \frac{[32]^3}{[21][13]}$$

$$\mathcal{A}(1^+,2,\ldots,n-1,n^-) = \sum_{i=2}^{n-1}\sum_{h=+,-}\mathcal{A}(\hat{1}^+,2,\ldots,i,-\hat{K}_{1,i}^{h})\,\frac{1}{K_{1,i}^2}\,\mathcal{A}(\hat{K}_{1,i}^{-h},i+1,\ldots,n-1,\hat{n}^-)$$

BCFW recursion

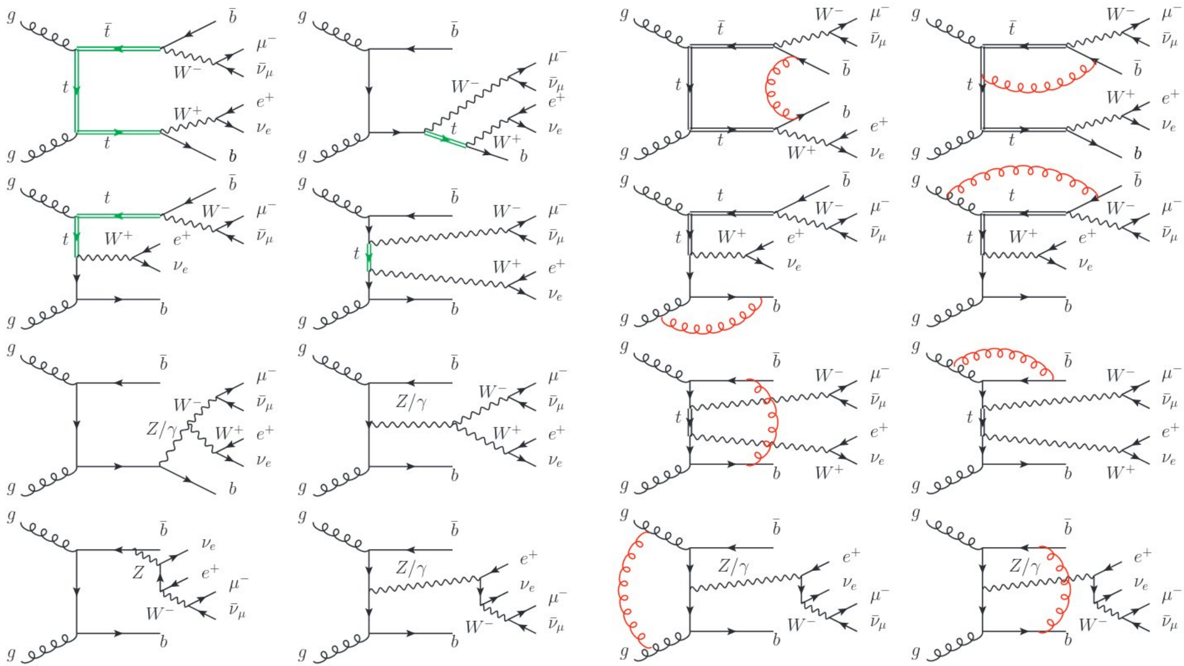
Britto, Cachazo, Feng, Witten 2005



- Does not work for all combinations of shift vector and helicities of shifted gluons, amplitude does not vanish as function of z at infinity for all of them,
- but working choices do exist for all helicity amplitudes.
- Starting point of the recursion are 3-point amplitudes, which do not necessarily vanish when momenta are shifted
- numerical recursion competitive up to 9 external gluons
- efficiency can be improved by hard-wiring few-gluon expressions
- generalized to include quarks Luo, Wen 2005

One-loop amplitudes

Example: a selection of tree-level and one-loop graphs for $gg \to b\bar{b}\,\mu^-\bar{\nu}_\mu\,e^+\nu_e$

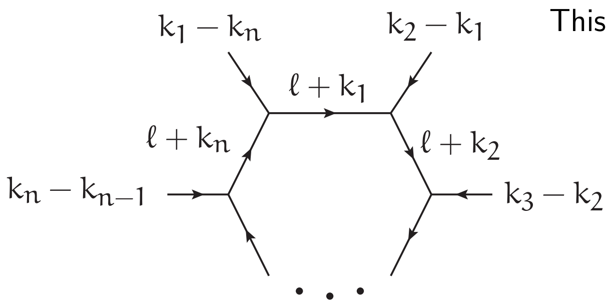


One-loop amplitudes

Loop integral can be expressed in terms of universal up-to-4-point scalar loop integrals.

$$\int d^\omega \ell \, \frac{N(\ell)}{D_1(\ell)D_2(\ell)\cdots D_n(\ell)} = \sum_{i_1<i_2<i_3<i_4} c_4(i_1,i_2,i_3,i_4) \int \frac{d^\omega \ell}{D_{i_1}(\ell)D_{i_2}(\ell)D_{i_3}(\ell)D_{i_4}(\ell)}$$
$$+ \sum_{i_1<i_2<i_3} c_3(i_1,i_2,i_3) \int \frac{d^\omega \ell}{D_{i_1}(\ell)D_{i_2}(\ell)D_{i_3}(\ell)}$$
$$+ \sum_{i_1<i_2} c_2(i_1,i_2) \int \frac{d^\omega \ell}{D_{i_1}(\ell)D_{i_2}(\ell)} + \sum_{i_1} c_1(i_1) \int \frac{d^\omega \ell}{D_{i_1}(\ell)}$$
$$+ \mathcal{R} + \mathcal{O}(\omega - 4)$$

$$\boxed{D_i(\ell) = (\ell + k_i)^2 - m_i^2}$$



This graph just depicts the momentum flow:

- off-shell currents may be attached to external lines
- vertices may actually be 4-point vertices

One-loop amplitudes

Loop integral can be expressed in terms of universal up-to-4-point scalar loop integrals.

$$\int d^\omega \ell \, \frac{N(\ell)}{D_1(\ell) D_2(\ell) \cdots D_n(\ell)} = \sum_{i_1<i_2<i_3<i_4} c_4(i_1,i_2,i_3,i_4) \int \frac{d^\omega \ell}{D_{i_1}(\ell) D_{i_2}(\ell) D_{i_3}(\ell) D_{i_4}(\ell)}$$
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$$+ \ \mathcal{R} \ + \ \mathcal{O}(\omega - 4)$$

$$\boxed{D_i(\ell) = (\ell + k_i)^2 - m_i^2}$$

- integral over rational function of ℓ leads to (poly-)logarithms of rational functions of squared momenta and masses. $\mathcal{R}$ are remnant rational (non-logarithmic) terms.

- integrals require dimensional regularization because of UV and IR divergencies

$$\int d^\omega \ell \, \frac{N(\ell)}{D_1(\ell) D_2(\ell) \cdots D_n(\ell)} = \frac{I_{-2}}{(\omega-4)^2} + \frac{I_{-1}}{\omega - 4} + I_0 + \mathcal{O}(\omega - 4)$$

$\mathcal{O}(\omega - 4)$ can be neglected at NLO

One-loop amplitudes

Loop integral can be expressed in terms of universal up-to-4-point scalar loop integrals.

$$\int d^\omega\ell \, \frac{N(\ell)}{D_1(\ell)D_2(\ell)\cdots D_n(\ell)} = \sum_{i_1<i_2<i_3<i_4} c_4(i_1,i_2,i_3,i_4) \int \frac{d^\omega\ell}{D_{i_1}(\ell)D_{i_2}(\ell)D_{i_3}(\ell)D_{i_4}(\ell)}$$

$$+ \sum_{i_1<i_2<i_3} c_3(i_1,i_2,i_3) \int \frac{d^\omega\ell}{D_{i_1}(\ell)D_{i_2}(\ell)D_{i_3}(\ell)}$$

$$\boxed{D_i(\ell) = (\ell + k_i)^2 - m_i^2}$$

$$+ \sum_{i_1<i_2} c_2(i_1,i_2) \int \frac{d^\omega\ell}{D_{i_1}(\ell)D_{i_2}(\ell)} + \sum_{i_1} c_1(i_1) \int \frac{d^\omega\ell}{D_{i_1}(\ell)}$$

$$+ \ \mathcal{R} \ + \ \mathcal{O}(\omega-4)$$

- the master integrals are universal functions depending on numerical values of momenta and masses, and several programs exist to evaluate them
  LoopTools, QCDLoop, OneLOop, Golem, Collier

- the determination of the coefficients had long been approached only algebraicly/analytically and was a major bottleneck for a long time

One-loop amplitudes with OPP

Ossola, Papadopoulos, Pittau 2006
Ellis, Giele, Kunszt 2007

Loop integral can be expressed in terms of universal up-to-4-point scalar loop integrals.

$$\int d^\omega \ell \, \frac{N(\ell)}{D_1(\ell)D_2(\ell)\cdots D_n(\ell)} = \sum_{i_1<i_2<i_3<i_4} c_4(i_1,i_2,i_3,i_4) \int \frac{d^\omega \ell}{D_{i_1}(\ell)D_{i_2}(\ell)D_{i_3}(\ell)D_{i_4}(\ell)}$$
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$$+ \mathcal{R} + \mathcal{O}(\omega - 4)$$

$$\boxed{D_i(\ell) = (\ell + k_i)^2 - m_i^2}$$

Almost the same relation for the integrand, before integration.

$$\frac{N(\ell)}{D_1(\ell)D_2(\ell)\cdots D_n(\ell)} = \sum_{i_1<i_2<i_3<i_4} \frac{c_4(i_1,i_2,i_3,i_4) + \tilde{c}_4(\ell;i_1,i_2,i_3,i_4)}{D_{i_1}(\ell)D_{i_2}(\ell)D_{i_3}(\ell)D_{i_4}(\ell)}$$
$$+ \sum_{i_1<i_2<i_3} \frac{c_3(i_1,i_2,i_3) + \tilde{c}_3(\ell;i_1,i_2,i_3)}{D_{i_1}(\ell)D_{i_2}(\ell)D_{i_3}(\ell)}$$
$$+ \sum_{i_1<i_2} \frac{c_2(i_1,i_2) + \tilde{c}_2(\ell;i_1,i_2)}{D_{i_1}(\ell)D_{i_2}(\ell)} + \sum_{i_1} \frac{c_1(i_1) + \tilde{c}_1(\ell;i_1)}{D_{i_1}(\ell)}$$

One-loop amplitudes with OPP

Ossola, Papadopoulos, Pittau 2006
Ellis, Giele, Kunszt 2007

- the coefficients $c_i(.)$ are exactly the ones we need.
- the polynomials $\tilde{c}_i(\ell,.)$ have only few coefficients, and integrate to zero.
- any chosen value of ℓ leads to an equation $\longrightarrow$ all coefficients can be determined.
- *smart* choices of ℓ put denominators to zero, and give rise to so-called *multiple cuts*. Using these, the matrix equation can be triangulated.
- requires efficient evaluation of $N(\ell)$.

$$\frac{N(\ell)}{D_1(\ell)D_2(\ell)\cdots D_n(\ell)} = \sum_{i_1<i_2<i_3<i_4} \frac{c_4(i_1,i_2,i_3,i_4)+\tilde{c}_4(\ell;i_1,i_2,i_3,i_4)}{D_{i_1}(\ell)D_{i_2}(\ell)D_{i_3}(\ell)D_{i_4}(\ell)}$$
$$+\sum_{i_1<i_2<i_3} \frac{c_3(i_1,i_2,i_3)+\tilde{c}_3(\ell;i_1,i_2,i_3)}{D_{i_1}(\ell)D_{i_2}(\ell)D_{i_3}(\ell)}$$
$$+\sum_{i_1<i_2} \frac{c_2(i_1,i_2)+\tilde{c}_2(\ell;i_1,i_2)}{D_{i_1}(\ell)D_{i_2}(\ell)} + \sum_{i_1} \frac{c_1(i_1)+\tilde{c}_1(\ell;i_1)}{D_{i_1}(\ell)}$$

One-loop amplitudes with OPP

Ossola, Papadopoulos, Pittau 2006
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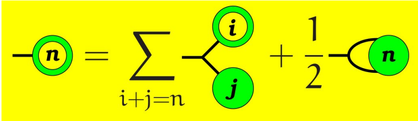
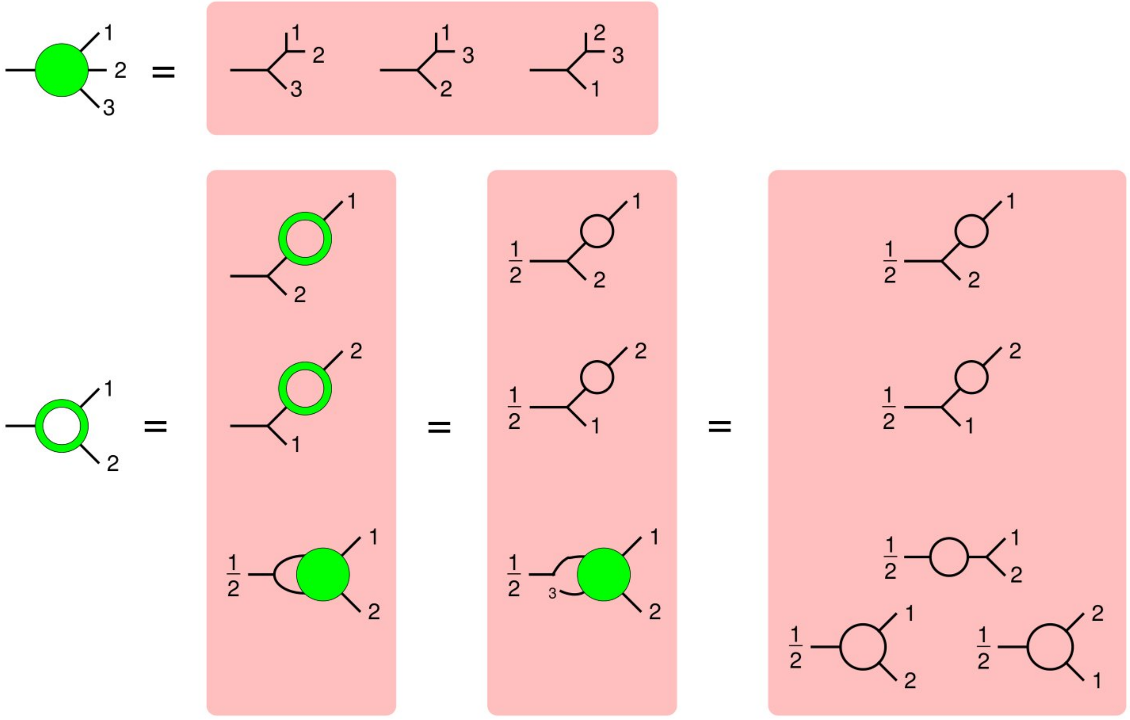
- *smart* choices of ℓ put denominators to zero, and give rise to so-called *multiple cuts*. Using these, the matrix equation can be triangulated.
- requires efficient evaluation of $N(\ell)$.

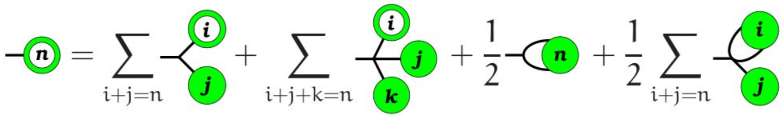
- part of the rational terms $\mathcal{R}$, related to the mismatch between 4-dimenional and ω-dimensional denominators, are provided within the method.
- the other part, related to the mismatch between 4-dimenional and ω-dimensional numerator has to be calculated separately and follows the structure of renormalization counter terms. Garzelli, Malamos, Pittau 2010
- these "complications" can be avoided by including master integrals with 5 denominators in dimensions higher than 4. Giele, Kunszt, Melnikov 2008

One-loop amplitudes recursively

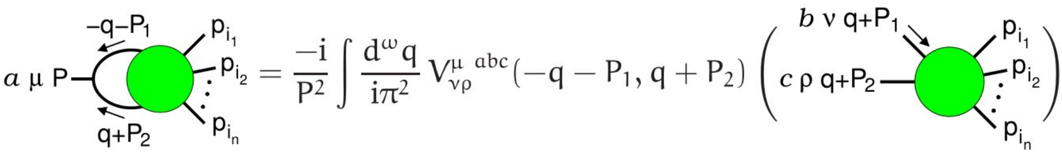




One-loop recursion:



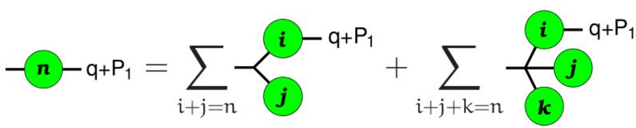
Actual loops generated by last two terms. Third term in more detail:



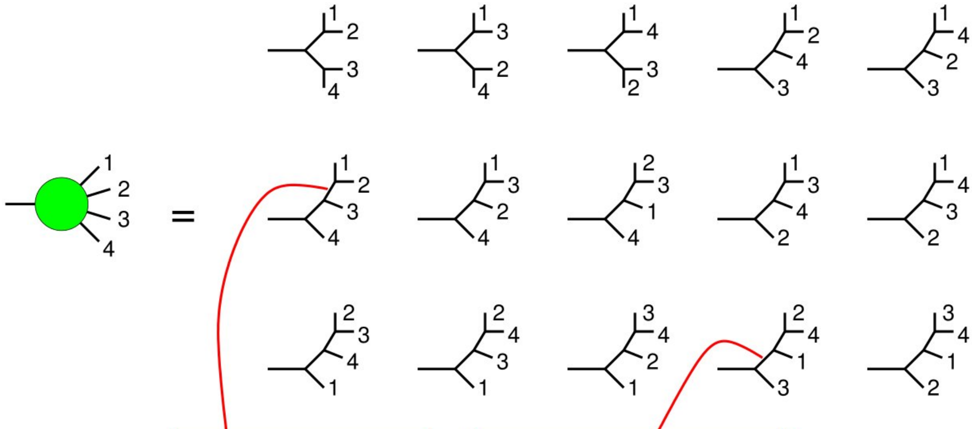
Momentum conservation:

$$P_2 - P_1 = P = p_{i_1} + p_{i_2} + \cdots + p_{i_n}$$

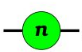
Integrand satisfies tree-level recursion:



One-loop amplitudes recursively



$$\frac{1}{(p_1+p_2)^2-m^2} \qquad \frac{1}{(p_1+p_2+p_4)^2-m^2}$$



$$\text{q+P}_1 = \sum_{|\mathcal{D}|\le n+1} \sum_{r=0}^{|\mathcal{D}|-1} \mathcal{G}^{bc\,\nu\rho}_{\nu_1\nu_2\cdots\nu_r}(\mathcal{D}) \frac{q^{\nu_1} q^{\nu_2}\cdots q^{\nu_r}}{\prod_{j\in\mathcal{D}}[(q+p_j)^2-m_j^2]}$$

Tensor integrals

We can increase the basis of universal loop integrals to *tensor* integrals

$$\mathcal{T}_{\nu_1\nu_2\cdots\nu_r}(\mathcal{D}) = \int d^\omega q \, \frac{q_{\nu_1} q_{\nu_2} \cdots q_{\nu_r}}{\prod_{j\in\mathcal{D}} D_j}$$

Many efficient techniques exist to evaluate them recursively.

- Passarino, Veltman 1979
- del Aguila, Pittau 2004
- AvH, Vollinga, Weinzierl 2005
- Binoth, Guillet, Heinrich, Pilon, Schubert 2005
- Denner, Dittmaier 2005
- Diakonidis, Fleischer, Riemann, Tausk 2009
- `Collier`: Denner, Dittmaier, Hofer 2016

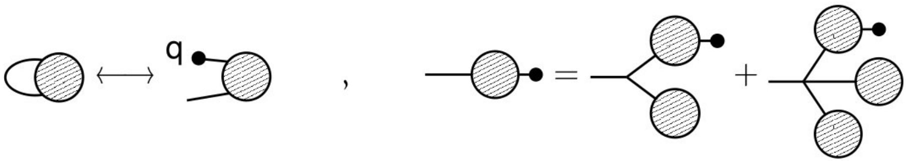
Originally the coefficients $\mathcal{G}$ in an expansion of the one-loop amplitude

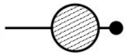
$$\mathcal{M}^{(1)} = \sum_r \sum_{\mathcal{D}} \mathcal{G}^{\nu_1\nu_2\cdots\nu_r}(\mathcal{D})\, \mathcal{T}_{\nu_1\nu_2\cdots\nu_r}(\mathcal{D})$$

were determined analytically as expressions.

Planar one-loop multi-gluon amplitudes

One-loop benchmark Giele, Zanderighi 2008, Lazopoulos 2008



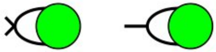
μ  $\lambda = \displaystyle\sum_{\mathcal{D}\subset\{i-1,i,\ldots,j\}} \sum_{r=0}^{|\mathcal{D}|-1} \mathcal{G}^{\lambda\mu}_{\nu_1\nu_2\cdots\nu_r}(\mathcal{D}) \frac{q^{\nu_1}q^{\nu_2}\cdots q^{\nu_r}}{\prod_{j\in\mathcal{D}}(q+p_{1,j})^2}$

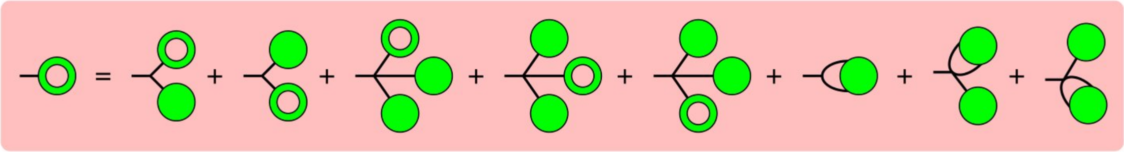
Separate the q-dependence from the 3-point vertex

$$V^{\mu}_{\nu\rho}(q+p_1,p_2) = V^{\mu}_{\nu\rho}(p_1,p_2) + X^{\mu}_{\sigma\nu\rho}q^{\sigma}$$

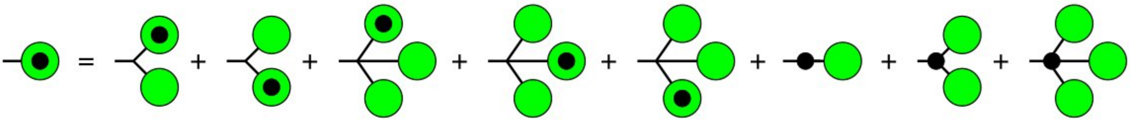
$$\mathcal{G}^{\lambda\mu}_{\nu_1\nu_2\cdots\nu_r}(\mathcal{D},k,j) = \mathcal{G}^{\lambda\nu}_{\nu_1\nu_2\cdots\nu_r}(\mathcal{D},k)\left[V^{\mu}_{\nu\rho}A^{\rho}_{k+1,j} + \sum_{l=k+1}^{j-1} W^{\mu}_{\nu\rho\sigma}A^{\rho}_{k+1,l}A^{\sigma}_{l+1,j}\right]$$
$$+\,\mathcal{G}^{\lambda\nu}_{\nu_1\nu_2\cdots\nu_{r-1}}(\mathcal{D},k)\,X^{\mu}_{\nu_r\nu\rho}A^{\rho}_{k+1,j}\ ,$$

Planar one-loop multi-gluon amplitudes

1. calculate all necessary tensor integrals $\mathcal{T}_{\nu_1\nu_2\cdots\nu_r}(\mathcal{D})$
2. calculate all tree-level off-shell currents $A^{\mu}_{i,j}$
3. calculate the coefficients $\mathcal{G}^{\nu_1\nu_2\cdots\nu_r}(\mathcal{D})$
4. calculate all currents 
5. calculate one-loop currents via

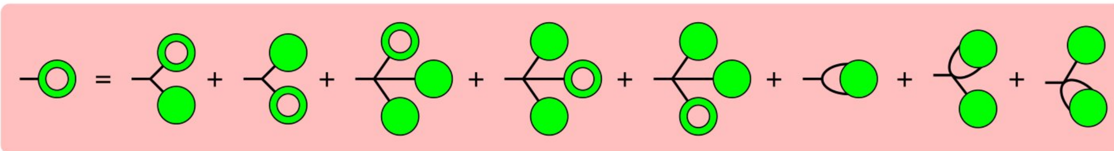


6. calculate missing $\mathcal{R}$ terms via



Planar one-loop multi-gluon amplitudes

1. calculate all necessary tensor integrals $\mathcal{T}_{\nu_1\nu_2\cdots\nu_r}(\mathcal{D})$
2. calculate all tree-level off-shell currents $A^{\mu}_{i,j}$
3. calculate the coefficients $\mathcal{G}^{\nu_1\nu_2\cdots\nu_r}(\mathcal{D})$
4. calculate all currents 
5. calculate one-loop currents via



Generalization to general one-loop amplitudes

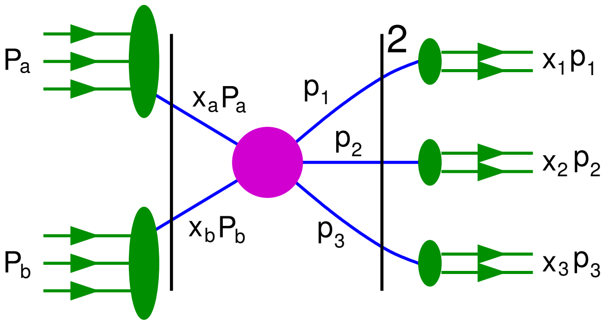
- Recola Actis, Denner, Hofer, Scharf, Uccirati 2012
- (Graph-by-graph) OpenLoops Cascioli, Maierhofer, Pozzorini 2012

Collinear factorization

Naïve parton model for hadron scattering

$$d\sigma(P_a, P_b \to \{P_i\}) = \int \prod_{j=a,b,1,\ldots} dx_j f_j(x_j) \;\; d\hat{\sigma}(p_a, p_b \to \{p_i\})$$

- the parton densities f_a and fragmentation function f_i describe physics of long time scales
  - not calculable within perturbation theory
  - universal to the hard scattering process
  - to be extracted from experiments

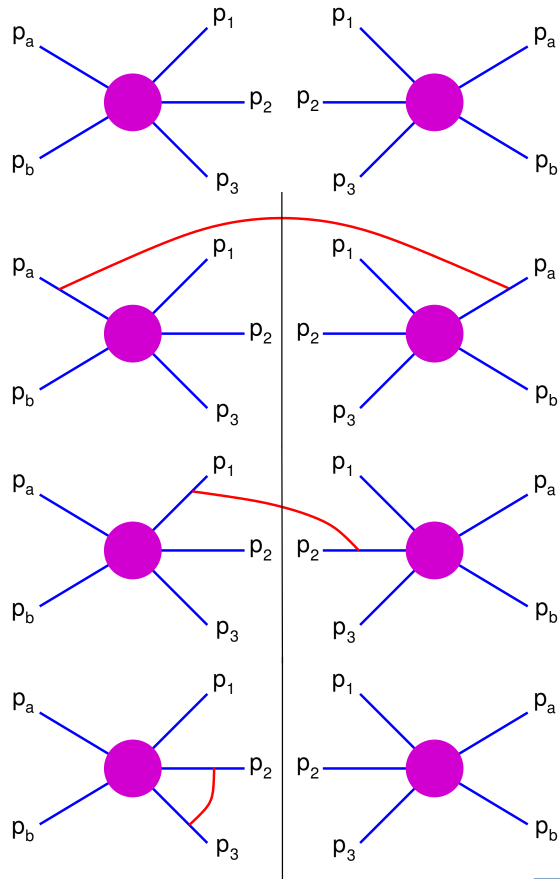


- the partonic scattering cross section $\hat{\sigma}$ describes physics of short time scales, and should be calculable within perturbative QCD
  - Asymptotic freedom: small coupling for high energy

Perturbative QCD

For the squared scattering amplitude $|\mathcal{M}|^2$

- blue lines represent identified partons
- mirrored graphs represent $\mathcal{M}$ and $\mathcal{M}^*$
- first square represents leading order
- higher orders by adding one coupling, that is two 3-point vertices connected by gluon integrated over its phase space
- gluons crossing the cut are real
  - are on-shell
  - participate in momentum conservation
- gluons not crossing the cut are virtual, are off-shell



Trouble with divergences

Integrating over the phase space of the extra, unobserved gluons, one encounters mass or IR divergences stemming form non-integratible singularities:

soft singularities: $\int_0 \frac{d|k|}{|k|}$

appear because all components of integration momentum k may become arbitrarily small

collinear singularities: $\int_0 \frac{d\theta}{1-\cos\theta}$

appear because integration momentum k may become arbitrarily collinear with massless parton momentum p_i

- all soft singularities cancel with each other, as predicted by the Kinoshita-Lee-Nauenberg theorem
- collinear singularities do not all cancel

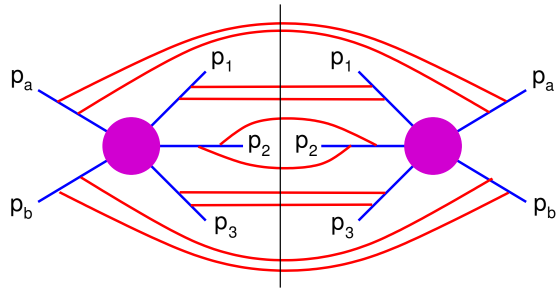
R.K. Ellis, H. Georgi, M. Machacek, H.D. Politzer, G.G. Ross, 1979:
Non-cancelling collinear divergences can be indentified with the external partons, and can be factorized, to all orders in QCD.

Factorization by EGMPR

Partonic cross section can be written as

$$d\sigma(p_a, p_b \to \{p_i\}) = \int \prod_{j=a,b,1,\ldots} dy_j\, \Gamma_j(y_j;\mu)\;\; d\tilde{\sigma}(q_a, q_b \to \{q_i\};\mu)$$

$$q_{a/b} = y_{a/b} p_{a/b} \quad , \quad p_i = y_i q_i$$



- corrected partonic cross section $d\tilde{\sigma}$ is free of IR singularities
- singularities are factored into the Γ_j
- requires the introduction of an arbitrary factorization scale μ
- formula has the same form as the original factorization formula
- absorb the Γ_j formally into the f_j:

$$\tilde{f}_j(x_j;\mu) = \int dy\, dz\, f_j(y)\Gamma(z;\mu)\delta(x - yz)$$

Factorization

"Renormalized" formula for hard scattering cross section

$$d\sigma(P_a, P_b \to \{P_i\}) = \int \prod_{j=a,b,1,\ldots} dy_j \tilde{f}_j(y_j;\mu) \;\; d\tilde{\sigma}(p_a, p_b \to \{p_i\};\mu)$$

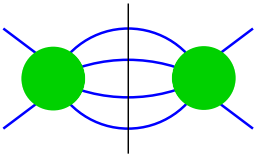
$$p_{a/b} = x_{a/b} P_{a/b} \quad , \quad P_i = x_i p_i$$

- $d\sigma$ does not depend on μ, while the $\tilde{f}_j$ and $\tilde{\sigma}$ do
- μ may be put equal to renormalization scale
- because their dependence on μ is known, QCD evolution can be applied to the $\tilde{f}_j$
- $d\tilde{\sigma}$ may be sensitive logarithms of ratios of various scales in the process, which are remnants from the cancellations
- these may need to be resummed to all orders in the coupling in certain kinematical regions (*eg* via parton shower)
- higher fixed order terms may be needed to reduce scale dependence
- jet-algorithm, or even just phase space cuts, play role of fragmentation functions

Ingredients for NLO calculations

LO calculation

$$\langle \mathcal{O} \rangle^{\mathrm{LO}} = \int \mathrm{d}\Phi_n \, |\mathcal{M}_n^{(0)}|^2 \, \mathcal{O}_n^{\mathrm{LO}} \quad =$$

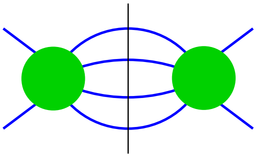


- The observable $\mathcal{O}_n^{\mathrm{LO}}$ represents some interesting distribution, and includes phase space cuts avoiding any pair of partons to become collinear, and any parton to become soft.
- $\mathcal{M}_n^{(0)}$ is the Born (tree-level) matrix element.

Ingredients for NLO calculations

LO calculation

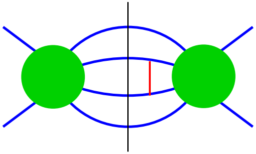
$$\langle \mathcal{O} \rangle^{\mathrm{LO}} = \int d\Phi_n \, |\mathcal{M}_n^{(0)}|^2 \, \mathcal{O}_n^{\mathrm{LO}} \quad =$$



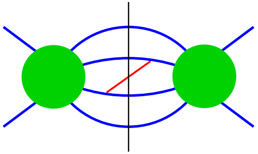
NLO calculation: add

$$\langle \mathcal{O} \rangle^{\mathrm{NLO}} = \int d\Phi_n \, 2\Re\big(\mathcal{M}_n^{(0)} \mathcal{M}_n^{(1)}\big) \, \mathcal{O}_n^{\mathrm{LO}} + \int d\Phi_{n+1} \, |\mathcal{M}_{n+1}^{(0)}|^2 \, \mathcal{O}_{n+1}^{\mathrm{NLO}}$$

$=$



$+$

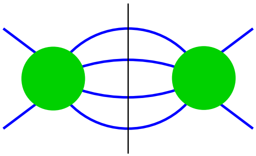


- $\mathcal{M}_n^{(1)}$ is the one-loop amplitude
- $\mathcal{M}_{n+1}^{(0)}$ is the real-radiation (tree-level) matrix element with one more parton;
- $\mathcal{O}_{n+1}^{\mathrm{NLO}}$ includes a jet algorithm that allows *one pair of partons* to become collinear, and *one parton* to become soft;

Ingredients for NLO calculations

LO calculation

$$\langle \mathcal{O} \rangle^{\mathrm{LO}} = \int \mathrm{d}\Phi_n \, |\mathcal{M}_n^{(0)}|^2 \, \mathcal{O}_n^{\mathrm{LO}} \quad =$$



NLO calculation: add

$$\begin{aligned}
\langle \mathcal{O} \rangle^{\mathrm{NLO}} &= \int \mathrm{d}\Phi_n \, 2\mathfrak{R}\big(\mathcal{M}_n^{(0)} \mathcal{M}_n^{(1)}\big) \, \mathcal{O}_n^{\mathrm{LO}} + \int \mathrm{d}\Phi_{n+1} \, |\mathcal{M}_{n+1}^{(0)}|^2 \, \mathcal{O}_{n+1}^{\mathrm{NLO}} \\
&= \int \mathrm{d}\Phi_n \left[2\mathfrak{R}\big(\mathcal{M}_n^{(0)} \mathcal{M}_n^{(1)}\big) + \int \mathrm{d}\Phi_1 \, \mathcal{S}_{n+1} + \mathcal{C}_n \right] \mathcal{O}_n^{\mathrm{LO}} \\
&+ \int \mathrm{d}\Phi_{n+1} \left[\, |\mathcal{M}_{n+1}^{(0)}|^2 \, \mathcal{O}_{n+1}^{\mathrm{NLO}} - \mathcal{S}_{n+1} \mathcal{O}_n^{\mathrm{LO}} \, \right]
\end{aligned}$$

- get finite phase space integrals with the help of subtraction
- demands factorization both of phase space and singularities
- remnant collinear divergences related to initial-state partons require separate subtraction term

Literature for NLO subtraction methods

S. Catani, M. Seymour,
A General algorithm for calculating jet cross-sections in NLO QCD,
Nucl.Phys. B485 (1997) 291-419, Erratum: Nucl.Phys. B510 (1998) 503-504,
`hep-ph/9607318`

G. Somogyi, Z. Trocsanyi,
A New subtraction scheme for computing QCD jet cross sections at next-to-leading order accuracy,
`hep-ph/0609041`

R. Frederix, S. Frixione, F. Maltoni, T. Stelzer, *Automation of next-to-leading order computations in QCD: The FKS subtraction*,
JHEP 0910 (2009) 003
arXiv:1405.0301

Singularities of multi-gluon MEs

$$|\mathcal{M}_4|^2 = 2g^4N^2(N^2-1)\left(\sum_{i<j}^{4} s_{ij}^4\right)\left(\frac{1}{s_{12}s_{23}s_{34}s_{41}} + \frac{1}{s_{13}s_{32}s_{24}s_{41}} + \frac{1}{s_{12}s_{24}s_{43}s_{31}}\right)$$

$$|\mathcal{M}_5|^2 = 2g^6N^3(N^2-1)\left(\sum_{i<j}^{5} s_{ij}^4\right)\sum_{\text{permutations}} \frac{1}{s_{12}s_{23}s_{34}s_{45}s_{51}} \qquad \boxed{s_{ij} = 2p_i\cdot p_j}$$

4- and 5- gluon tree-level matrix elements summed over color and helicity have reasonable simple expressions, but are very instructive regarding the singularity structure of multi-parton matrix elements in general.

- obviously singular when any $s_{ij} \to 0$
- actual singularities are, however, in $E_j = p_j^0$ and θ_{ij} instead of $s_{ij} = E_iE_j(1-\cos\theta_{ij})$.

$$E_j \to 0 \quad \Rightarrow \quad |\mathcal{M}|^2 \propto \frac{1}{E_j^2} \quad \Rightarrow \quad d^4p_j\delta(p_j^2)|\mathcal{M}|^2 = \frac{d^3p_j}{2E_j}|\mathcal{M}|^2 \propto \frac{dE_j}{E_j}$$

$$\theta_{ij} \to 0 \quad \Rightarrow \quad |\mathcal{M}|^2 \propto \frac{1}{1-\cos\theta_{ij}} \quad \Rightarrow \quad d^4p_j\delta(p_j^2)|\mathcal{M}|^2 \propto \frac{d\cos\theta_{ij}}{1-\cos\theta_{ij}}$$

Singularities of multi-gluon MEs

$$|\mathcal{M}_4|^2 = 2g^4N^2(N^2-1)\left(\sum_{i<j}^{4} s_{ij}^4\right)\left(\frac{1}{s_{12}s_{23}s_{34}s_{41}} + \frac{1}{s_{13}s_{32}s_{24}s_{41}} + \frac{1}{s_{12}s_{24}s_{43}s_{31}}\right)$$

$$|\mathcal{M}_5|^2 = 2g^6N^3(N^2-1)\left(\sum_{i<j}^{5} s_{ij}^4\right)\sum_{\text{permutations}} \frac{1}{s_{12}s_{23}s_{34}s_{45}s_{51}}$$

$$\boxed{s_{ij} = 2p_i \cdot p_j}$$

4- and 5- gluon t ... ave reasonable simple expression ... cture of multi-parton matrix ele...

Eventually, we are worrying about the integral

$$\int d\Phi_{n+1}\left[\,|\mathcal{M}^{(0)}_{n+1}|^2\,\mathcal{O}^{\mathrm{NLO}}_{n+1} - \mathcal{S}_{n+1}\mathcal{O}^{\mathrm{LO}}_{n}\,\right]$$

Both $\mathcal{O}^{\mathrm{NLO}}_{n+1}$ and $\mathcal{O}^{\mathrm{LO}}_{n}$ avoid any regions of phase space with more than one $E \to 0$ or more than one $\theta \to 0$.

- obviously singu...
- actual singular... $1-\cos\theta_{ij})$.

$$E_j \to 0 \quad \Rightarrow \quad |\mathcal{M}|^2 \propto \frac{1}{E_j^2} \quad \Rightarrow \quad d^4p_j\,\delta(p_j^2)\,|\mathcal{M}|^2 = \frac{d^3p_j}{2E_j}|\mathcal{M}|^2 \propto \frac{dE_j}{E_j}$$

$$\theta_{ij} \to 0 \quad \Rightarrow \quad |\mathcal{M}|^2 \propto \frac{1}{1-\cos\theta_{ij}} \quad \Rightarrow \quad d^4p_j\,\delta(p_j^2)\,|\mathcal{M}|^2 \propto \frac{d\cos\theta_{ij}}{1-\cos\theta_{ij}}$$

Soft behavior of multi-gluon MEs

$$|\mathcal{M}_4|^2 = 2g^4 N^2 (N^2-1) \left(\sum_{i<j}^{4} s_{ij}^4\right) \left(\frac{1}{s_{12}s_{23}s_{34}s_{41}} + \frac{1}{s_{13}s_{32}s_{24}s_{41}} + \frac{1}{s_{12}s_{24}s_{43}s_{31}}\right)$$

$$|\mathcal{M}_5|^2 = 2g^6 N^3 (N^2-1) \left(\sum_{i<j}^{5} s_{ij}^4\right) \sum_{\text{permutations}} \frac{1}{s_{12}s_{23}s_{34}s_{45}s_{51}}$$

$$\boxed{s_{ij} = 2p_i \cdot p_j}$$

$$\frac{1}{s_{12}s_{23}s_{34}s_{45}s_{51}} = \frac{s_{41}}{s_{45}s_{51}} \times \frac{1}{s_{12}s_{23}s_{34}s_{41}}$$

$$|\mathcal{M}_5|^2 \xrightarrow{p_5 \to 0} 2g^6 N^3 (N^2-1) \left(\sum_{i<j}^{4} s_{ij}^4\right) \left[\frac{s_{41}}{s_{45}s_{51}} \left(\frac{1}{s_{12}s_{23}s_{34}s_{41}} + \frac{1}{s_{13}s_{32}s_{24}s_{41}}\right) + \frac{s_{32}}{s_{35}s_{52}} \left(\frac{1}{s_{24}s_{41}s_{13}s_{32}} + \frac{1}{s_{21}s_{14}s_{43}s_{32}}\right) + \cdots\right]$$

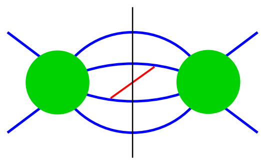
$$|\mathcal{M}_5|^2 \xrightarrow{p_5 \to 0} g^2 \sum_{i>j}^{4} \frac{s_{ij}}{s_{i5}s_{5j}} N |\mathcal{M}_4(i,j)|^2$$

Soft behavior of multi-gluon MEs

$$|\mathcal{M}_4|^2 = 2g^4 N^2 (N^2-1) \left(\sum_{i<j}^{4} s_{ij}^4\right) \left(\frac{1}{s_{12}s_{23}s_{34}s_{41}} + \frac{1}{s_{13}s_{32}s_{24}s_{41}} + \frac{1}{s_{12}s_{24}s_{43}s_{31}}\right)$$

$$|\mathcal{M}_5|^2 = \ldots \qquad \boxed{s_{ij} = 2p_i \cdot p_j}$$

For arbitrary multi-parton processes:



$$|\mathcal{M}_{n+1}|^2 \xrightarrow{p_{n+1}\to 0} -\mu^{2\varepsilon} g^2 \sum_{i\neq j} \frac{s_{ij}}{s_{i,n+1}s_{n+1,j}} \langle \mathcal{M}_n | \mathbf{T}_i \cdot \mathbf{T}_j | \mathcal{M}_n \rangle$$

$$|\mathcal{M}_5|^2 \xrightarrow{p_5\to \ldots} \ldots \left.\frac{\ldots}{s_{32}}\right) + \cdots \Big]$$

$$|\mathcal{M}_5|^2 \xrightarrow{p_5\to 0} g^2 \sum_{i>j}^{4} \frac{s_{ij}}{s_{i5}s_{5j}} N |\mathcal{M}_4(i,j)|^2$$

Collinear behavior of multi-gluon MEs

$$|\mathcal{M}_4|^2 = 2g^4N^2(N^2-1)\left(\sum_{i<j}^{4} s_{ij}^4\right)\left(\frac{1}{s_{12}s_{23}s_{34}s_{41}}+\frac{1}{s_{13}s_{32}s_{24}s_{41}}+\frac{1}{s_{12}s_{24}s_{43}s_{31}}\right)$$

$$|\mathcal{M}_5|^2 = 2g^6N^3(N^2-1)\left(\sum_{i<j}^{5} s_{ij}^4\right)\sum_{\text{permutations}}\frac{1}{s_{12}s_{23}s_{34}s_{45}s_{51}}$$

$$\boxed{s_{ij} = 2p_i\cdot p_j}$$

Collinear limit: $p_5 \to zp_4'\,,\quad p_4 \to (1-z)p_4'$

$$\sum_{i<j}^{5} s_{ij}^4 \longrightarrow s_{12}^4+s_{13}^4+s_{23}^4+(1-z)^4\left(s_{14}^4+s_{24}^4+s_{34}^4\right)+z^4\left(s_{14}^4+s_{24}^4+s_{34}^4\right)$$

Momentum conservation: $p_1+p_2+p_3+p_4'=0 \;\Rightarrow\; s_{34}=s_{12}\,,\; s_{24}=s_{13}\,,\; s_{14}=s_{23}$

$$\sum_{i<j}^{5} s_{ij}^4 \longrightarrow \left(1+(1-z)^4+z^4\right)\left(s_{12}^4+s_{13}^4+s_{23}^4\right)$$

$$|\mathcal{M}_5|^2 \longrightarrow g^2\,N\,\frac{1+(1-z)^4+z^4}{z(1-z)}\,\frac{1}{s_{45}}\,|\mathcal{M}_4|^2$$

Collinear behavior of multi-gluon MEs

$$\left|\mathcal{M}_4\right|^2 = 2g^4 N^2(N^2-1) \left(\sum_{i<j}^{4} s_{ij}^4\right) \left(\frac{1}{s_{12}s_{23}s_{34}s_{41}} + \frac{1}{s_{13}s_{32}s_{24}s_{41}} + \frac{1}{s_{12}s_{24}s_{43}s_{31}}\right)$$

$$\left|\mathcal{M}_5\right|^2 = 2g^6 N^3(N^2-1) \left(\sum_{i<j}^{5} s_{ij}^4\right) \sum_{\text{permutations}} \frac{1}{s_{12}s_{23}s_{34}s_{45}s_{51}} \qquad \boxed{s_{ij} = 2p_i \cdot p_j}$$

Collinear limit: $p_5 \to z p_4'\,, \quad p_4 \to (1-z) p_4'$

$$\sum_{i<j}^{5} s_{ij}^4 \longrightarrow s_{12}^4 + s_{13}^4 + s_{23}^4 + (1-z)^4\left(s_{14}^4 + s_{24}^4 + s_{34}^4\right) + z^4\left(s_{14}^4 + s_{24}^4 + s_{34}^4\right)$$

Momentum conservation: $p_1 + p_2 + p_3 + p_4' = 0 \;\Rightarrow\; s_{34} = s_{12}\,,\; s_{24} = s_{13}\,,\; s_{14} = s_{23}$

$$\sum_{i<j}^{5} s_{ij}^4 \longrightarrow \left(1 + (1-z)^4 + z^4\right)\left(s_{12}^4 + s_{13}^4 + s_{23}^4\right)$$

$$\left|\mathcal{M}_5\right|^2 \longrightarrow g^2\, 2N \left(\frac{z}{1-z} + \frac{1-z}{z} + z(1-z)\right) \frac{1}{s_{45}} \left|\mathcal{M}_4\right|^2$$

Collinear behavior of multi-parton MEs

$$p_i^\mu = z p_{j'}^\mu + k_T^\mu - \frac{k_T^2}{2z(p_{j'} \cdot n)} n^\mu \quad , \quad p_j^\mu = (1-z) p_{j'}^\mu - k_T^\mu - \frac{k_T^2}{2(1-z)(p_{j'} \cdot n)} n^\mu$$

$$\left| \mathcal{M}_{n+1} \right|^2 \xrightarrow{k_T \to 0} \mu^{2\varepsilon} g^2 \frac{2}{s_{ij}} \langle \mathcal{M}_n | P_{ij}(z, k_T, \varepsilon) | \mathcal{M}_n \rangle$$

for example $< P_{gg}(z, \varepsilon) >= 2N \left(\frac{z}{1-z} + \frac{1-z}{z} + z(1-z) \right)$

Subtraction and phase space

$$\langle \mathcal{O} \rangle^{\mathrm{NLO}} = \int d\Phi_n \left[2\mathfrak{R}\left(\mathcal{M}_n^{(0)}\mathcal{M}_n^{(1)}\right) + \int d\Phi_1\, \mathcal{S}_{n+1} + \mathcal{C}_n \right] \mathcal{O}_n^{\mathrm{LO}}$$
$$+ \int d\Phi_{n+1} \left[\, |\mathcal{M}_{n+1}^{(0)}|^2\, \mathcal{O}_{n+1}^{\mathrm{NLO}} - \mathcal{S}_{n+1}\mathcal{O}_n^{\mathrm{LO}} \, \right]$$

$$d\Phi_n = \left(\prod_{i=1}^{n} d^4 p_i\, \delta_+(p_i^2 - m_i^2) \right) \delta^4 \left(\sum_{i=1}^{n} p_i - P \right)$$

There are objects living in n-particle phase space underneath the $(n+1)$-particle phase space integral. We need phase space mappings $T^{(\omega)}_{n\leftarrow n+1} : \Phi_{n+1} \longrightarrow \Phi_n$ for the various singularities, labelled ω

$$\int d\Phi_{n+1} \left[\, |\mathcal{M}_{n+1}^{(0)}|^2\, \mathcal{O}_{n+1}^{\mathrm{NLO}} - \sum_{\omega} \Theta_{n+1}^{(\omega)} \mathcal{K}_{n+1}^{(\omega)} \mathcal{F}_n^{(\omega)} \circ T^{(\omega)}_{n\leftarrow n+1}\, \mathcal{O}_n^{\mathrm{LO}} \circ T^{(\omega)}_{n\leftarrow n+1} \, \right]$$

Subtraction and phase space

$$\langle \mathcal{O} \rangle^{\mathrm{NLO}} = \int d\Phi_n \left[2\Re\left(\mathcal{M}_n^{(0)} \mathcal{M}_n^{(1)}\right) + \int d\Phi_1 \, \mathcal{S}_{n+1} + \mathcal{C}_n \right] \mathcal{O}_n^{\mathrm{LO}}$$
$$+ \int d\Phi_{n+1} \left[|\mathcal{M}_{n+1}^{(0)}|^2 \, \mathcal{O}_{n+1}^{\mathrm{NLO}} - \mathcal{S}_{n+1} \mathcal{O}_n^{\mathrm{LO}} \right]$$

$$d\Phi_n = \left(\prod_{i=1}^{n} d^4 p_i \, \delta_+(p_i^2 - m_i^2) \right) \delta^4 \left(\sum_{i=1}^{n} p_i - P \right)$$

There are objects living in n-particle phase space underneath the $(n+1)$-particle phase space integral. We need phase space mappings $T^{(\omega)}_{n \leftarrow n+1} : \Phi_{n+1} \longrightarrow \Phi_n$ for the various singularities, labelled ω

$$\int d\Phi_{n+1} \left[|\mathcal{M}_{n+1}^{(0)}|^2 \, \mathcal{O}_{n+1}^{\mathrm{NLO}} - \sum_{\omega} \Theta_{n+1}^{(\omega)} \mathcal{K}_{n+1}^{(\omega)} \mathcal{F}_n^{(\omega)} \circ T^{(\omega)}_{n \leftarrow n+1} \, \mathcal{O}_n^{\mathrm{LO}} \circ T^{(\omega)}_{n \leftarrow n+1} \right]$$

$\mathcal{O}_n^{\mathrm{LO}}$ lives in Φ_n and needs $T^{(\omega)}_{n \leftarrow n+1}$

Subtraction and phase space

$$\langle \mathcal{O} \rangle^{\rm NLO} = \int d\Phi_n \left[2\Re\left(\mathcal{M}_n^{(0)}\mathcal{M}_n^{(1)}\right) + \int d\Phi_1\, \mathcal{S}_{n+1} + \mathcal{C}_n \right] \mathcal{O}_n^{\rm LO}$$
$$+ \int d\Phi_{n+1} \left[|\mathcal{M}_{n+1}^{(0)}|^2\, \mathcal{O}_{n+1}^{\rm NLO} - \mathcal{S}_{n+1} \mathcal{O}_n^{\rm LO} \right]$$

$$d\Phi_n = \left(\prod_{i=1}^{n} d^4 p_i\, \delta_+(p_i^2 - m_i^2) \right) \delta^4 \left(\sum_{i=1}^{n} p_i - P \right)$$

There are objects living in n-particle phase space underneath the $(n+1)$-particle phase space integral. We need phase space mappings $T^{(\omega)}_{n \leftarrow n+1} : \Phi_{n+1} \longrightarrow \Phi_n$ for the various singularities, labelled ω

$$\int d\Phi_{n+1} \left[|\mathcal{M}_{n+1}^{(0)}|^2\, \mathcal{O}_{n+1}^{\rm NLO} - \sum_{\omega} \Theta_{n+1}^{(\omega)} \mathcal{K}_{n+1}^{(\omega)} \mathcal{F}_n^{(\omega)} \circ T^{(\omega)}_{n \leftarrow n+1}\, \mathcal{O}_n^{\rm LO} \circ T^{(\omega)}_{n \leftarrow n+1} \right]$$

eg. something like $\dfrac{s_{i(\omega),k(\omega)}}{s_{i(\omega),j(\omega)}\, s_{j(\omega),k(\omega)}}$ lives in Φ_{n+1}

eg. something like $\langle \mathcal{M}_n | \mathbf{T}_{i(\omega)} \cdot \mathbf{T}_{k(\omega)} | \mathcal{M}_n \rangle$ lives in Φ_n and needs $T^{(\omega)}_{n \leftarrow n+1}$

Subtraction and phase space

$$\langle \mathcal{O} \rangle^{\mathrm{NLO}} = \int d\Phi_n \left[2\Re\left(\mathcal{M}_n^{(0)}\mathcal{M}_n^{(1)}\right) + \int d\Phi_1\, \mathcal{S}_{n+1} + \mathcal{C}_n \right] \mathcal{O}_n^{\mathrm{LO}}$$
$$+ \int d\Phi_{n+1} \left[|\mathcal{M}_{n+1}^{(0)}|^2\, \mathcal{O}_{n+1}^{\mathrm{NLO}} - \mathcal{S}_{n+1}\mathcal{O}_n^{\mathrm{LO}} \right]$$

$$d\Phi_n = \left(\prod_{i=1}^{n} d^4 p_i\, \delta_+(p_i^2 - m_i^2) \right) \delta^4 \left(\sum_{i=1}^{n} p_i - P \right)$$

There are objects living in n-particle phase space underneath the $(n+1)$-particle phase space integral. We need phase space mappings $T^{(\omega)}_{n\leftarrow n+1} : \Phi_{n+1} \longrightarrow \Phi_n$ for the various singularities, labelled ω

$$\int d\Phi_{n+1} \left[|\mathcal{M}_{n+1}^{(0)}|^2\, \mathcal{O}_{n+1}^{\mathrm{NLO}} - \sum_{\omega} \Theta^{(\omega)}_{n+1} \mathcal{K}^{(\omega)}_{n+1} \mathcal{F}^{(\omega)}_n \circ T^{(\omega)}_{n\leftarrow n+1}\, \mathcal{O}_n^{\mathrm{LO}} \circ T^{(\omega)}_{n\leftarrow n+1} \right]$$

possibility to restrict phase space to singularities ω

Subtraction and phase space

$$\langle \mathcal{O} \rangle^{\mathrm{NLO}} = \int d\Phi_n \left[2\mathfrak{R}\left(\mathcal{M}_n^{(0)} \mathcal{M}_n^{(1)}\right) + \int d\Phi_1\, \mathcal{S}_{n+1} + \mathcal{C}_n \right] \mathcal{O}_n^{\mathrm{LO}} + \int d\Phi_{n+1} \left[|\mathcal{M}_{n+1}^{(0)}|^2\, \mathcal{O}_{n+1}^{\mathrm{NLO}} - \mathcal{S}_{n+1} \mathcal{O}_n^{\mathrm{LO}} \right]$$

$$d\Phi_n = \left(\prod_{i=1}^{n} d^4 p_i\, \delta_+(p_i^2 - m_i^2) \right) \delta^4 \left(\sum_{i=1}^{n} p_i - P \right)$$

There are objects living in n-particle phase space underneath the $(n+1)$-particle phase space integral. We need phase space mappings $T^{(\omega)}_{n \leftarrow n+1} : \Phi_{n+1} \longrightarrow \Phi_n$ for the various singularities, labelled ω

$$\int d\Phi_{n+1} \left[|\mathcal{M}_{n+1}^{(0)}|^2\, \mathcal{O}_{n+1}^{\mathrm{NLO}} - \sum_{\omega} \Theta_{n+1}^{(\omega)} \mathcal{K}_{n+1}^{(\omega)} \mathcal{F}_n^{(\omega)} \circ T^{(\omega)}_{n \leftarrow n+1}\, \mathcal{O}_n^{\mathrm{LO}} \circ T^{(\omega)}_{n \leftarrow n+1} \right]$$

We also need the inverse $T^{(\omega)}_{n \rightarrow n+1} : \Phi_n \times \Phi_1 \longrightarrow \Phi_{n+1}$ in order to exactly match the integrated subtraction terms.

In practice we also need them to *efficiently* generate phase space for the real-subtracted integral.

FKS subtraction

Suppose the phase space restrictions satisfy

$$\Theta^{(i,j)}_{n+1} \longrightarrow 0 \ \text{ if } \begin{cases} \text{any other energy than } E_j \text{ goes to zero} \\ \text{any other angle than } \theta_{ij} \text{ goes to zero} \end{cases} \qquad \text{and} \qquad \sum_{i\neq j} \Theta^{(i,j)}_{n+1} = 1$$

$$\int d\Phi_{n+1} \left[|\mathcal{M}^{(0)}_{n+1}|^2 \, \mathcal{O}^{\text{NLO}}_{n+1} - \sum_{i\neq j} \Theta^{(i,j)}_{n+1} \mathcal{K}^{(i,j)}_{n+1} \mathcal{F}^{(i,j)}_{n} \circ T^{(i,j)}_{n\leftarrow n+1} \, \mathcal{O}^{\text{LO}}_{n} \circ T^{(i,j)}_{n\leftarrow n+1} \right]$$
$$= \sum_{i\neq j} \int d\Phi_{n+1} \, \Theta^{(i,j)}_{n+1} \left[|\mathcal{M}^{(0)}_{n+1}|^2 \, \mathcal{O}^{\text{NLO}}_{n+1} - \mathcal{K}^{(i,j)}_{n+1} \mathcal{F}^{(i,j)}_{n} \circ T^{(i,j)}_{n\leftarrow n+1} \, \mathcal{O}^{\text{LO}}_{n} \circ T^{(i,j)}_{n\leftarrow n+1} \right]$$

Each integral sees at most one singularity $E_j \longrightarrow 0$ and at most one singularity $\theta_{ij} \longrightarrow 0$. Now we only need the inverse phase space mapping $T^{(i,j)}_{n\rightarrow n+1} : \Phi_n \times \Phi_1 \longrightarrow \Phi_{n+1}$

$$\int d\Phi_{n+1} \, \Theta^{(i,j)}_{n+1} \left[|\mathcal{M}^{(0)}_{n+1}|^2 \, \mathcal{O}^{\text{NLO}}_{n+1} - \mathcal{K}^{(i,j)}_{n+1} \mathcal{F}^{(i,j)}_{n} \circ T^{(i,j)}_{n\leftarrow n+1} \, \mathcal{O}^{\text{LO}}_{n} \circ T^{(i,j)}_{n\leftarrow n+1} \right]$$
$$= \int d\Phi_n \times d\Phi_1 \, \Theta^{(i,j)}_{n+1} \circ T^{(i,j)}_{n\rightarrow n+1} \left[\left(|\mathcal{M}^{(0)}_{n+1}|^2 \, \mathcal{O}^{\text{NLO}}_{n+1} \right) \circ T^{(i,j)}_{n\rightarrow n+1} - \mathcal{K}^{(i,j)}_{n+1} \circ T^{(i,j)}_{n\rightarrow n+1} \mathcal{F}^{(i,j)}_{n} \, \mathcal{O}^{\text{LO}}_{n} \right]$$

FKS subtraction

Suppose the phase space restrictions satisfy

$$\Theta_{n+1}^{(i,j)} \to 0 \ \text{ if } \begin{cases} \text{any other energy than } E_j \text{ goes to zero} \\ \text{any other angle than } \theta_{ij} \text{ goes to zero} \end{cases} \qquad \text{and} \qquad \sum_{i\neq j} \Theta_{n+1}^{(i,j)} = 1$$

Example for $n+1=3$:

$$\Theta_3^{(1,2)} = \frac{\frac{1}{E_2\theta_{12}}}{\frac{1}{E_1\theta_{12}} + \frac{1}{E_1\theta_{13}} + \frac{1}{E_2\theta_{12}} + \frac{1}{E_2\theta_{23}} + \frac{1}{E_3\theta_{13}} + \frac{1}{E_3\theta_{23}}}$$

satisfies the requirements, and also

$$\Theta_{n+1}^{(i,j)} + \Theta_{n+1}^{(j,i)} \xrightarrow{\theta_{ij}\to 0} 1 \quad \text{and} \quad \sum_i \Theta_{n+1}^{(i,j)} \xrightarrow{E_j \to 0} 1$$

In practice, more sophisticated choices than $\dfrac{1}{E_j\theta_{ij}}$ are used.

NLO programs

Complete stand-alone programs for multi-leg NLO calculations:

- MadGraph5_aMC@NLO
- HelacNLO
- WHIZARD

Programs for one-loop amplitudes, to be combined with *eg.* Sherpa

- Njet
- Gosam
- BlackHat
- OpenLoops