

Scalar one-loop Feynman integrals in arbitrary space-time dimension

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Why one-loop Feynman integrals? And why in $D = 4 + 2n - 2\epsilon$ dimensions?

Based on [1, 2], I began in 1980 to calculate Feynman integrals, see [Mann, Riemann, 1983](#) [3]: “Effective Flavor Changing Weak Neutral Current In The Standard Theory And Z Boson Decay”

Basics

The seminal papers on 1-loop Feynman integrals:

’t Hooft, Veltman, 1978 [1]: “Scalar oneloop integrals”

Passarino, Veltman, 1978 [2]: “One Loop Corrections for e^+e^- Annihilation into $\mu^+\mu^-$ in the Weinberg Model”

Interest in “modern” developments for the calculation of 1-loop integrals from basically two sides

1. For many-particle calculations, there **appear inverse Gram determinants from tensor reductions** in the answers.

These $1/G(p_i)$ may diverge, because Gram dets can exactly vanish: $G(p_i) \equiv 0$.

One may perform tensor reductions so that no inverse Grams appear, but one has to buy 1-loop integrals in higher dimensions, $D = 4 + 2n - 2\epsilon$. See [4, 5]

Interest in “modern” developments for the calculation of 1-loop integrals from basically two sides

2. Higher-order loop calculations need h.o. contributions from ϵ -expansions of 1-loops:

$$1/(d-4) = -1/(2\epsilon) \text{ and } \Gamma(\epsilon) = a/\epsilon + c + \epsilon + \dots$$

A Seminal paper was on ϵ -terms of 1-loop functions:

Nierste, Müller, Böhm, 1992 [6]: “Two loop relevant parts of D-dimensional massive scalar one loop integrals”

1-loop integrals in D dimensions

A general solution in D dimensions was derived in another 2 seminal papers:

Tarasov, 2000 [7] and **Fleischer, Jegerlehner, Tarasov, 2003 [8]**: “A New hypergeometric representation of one loop scalar integrals in d dimensions”

I was wondering if the results of **Fleischer/Jegerlehner/Tarasov (2003)** are sufficiently general for practical, black-box applications, and saw a need of creating a software solution in terms of contemporary mathematics.

So we decided to study the issue from scratch in 2 steps:

1st step: Re-derive analytical expressions for scalar one-loop integrals as meromorphic functions of **arbitrary** space-time dimension D

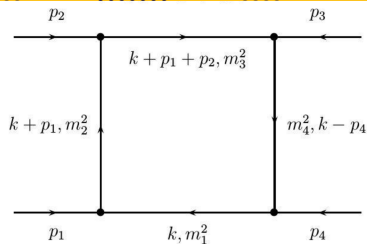
- 2-point functions: Gauss hypergeometric functions ${}_2F_1$ [9]
- 3-point functions: additional Kamp'e de F'eriet functions F_1 ; there are the Appell functions F_1, \dots, F_4 [10]
- 4-point functions: additional Lauricella-Saran functions F_S [11]

2nd step:

Derive the Laurent expansions around the singular points of these functions.

This talk:

- Analytical expressions for self-energies, vertices, boxes
- Numerical checks



$$J_N \equiv J_N(d; \{p_i p_j\}, \{m_i^2\}) = \int \frac{d^d k}{i\pi^{d/2}} \frac{1}{D_1^{\nu_1} D_2^{\nu_2} \dots D_N^{\nu_N}} \quad (1)$$

with

$$D_i = \frac{1}{(k + q_i)^2 - m_i^2 + i\epsilon}. \quad (2)$$

$$\nu_i = 1, \quad \sum_{i=1}^n p_i = 0 \quad (3)$$

$$J_n = (-1)^n \Gamma(n - d/2) \int_0^1 \prod_{j=1}^n dx_j \delta\left(1 - \sum_{i=1}^n x_i\right) \frac{1}{F_n(x)^{n-d/2}} \quad (4)$$

Here, the F -function is the **second Symanzik polynomial**.

It is derived from the propagators (2),

$$M^2 = x_1 D_1 + \dots + x_N D_N = k^2 - 2Qk + J. \quad (5)$$

Using $\delta(1 - \sum x_i)$ under the integral in order to transform linear terms in x into quadratic ones, we may obtain:

$$F_n(x) = -\left(\sum_i x_i\right) J + Q^2 = \frac{1}{2} \sum_{i,j} x_i Y_{ij} x_j - i\epsilon, \quad (6)$$

The Y_{ij} are elements of the **Cayley matrix**, introduced for a systematic study of one-loop n -point Feynman integrals e.g. in [12]

$$Y_{ij} = Y_{ji} = m_i^2 + m_j^2 - (q_i - q_j)^2. \quad (7)$$

There are $N_n = \frac{1}{2}n(n+1)$ **different** Y_{ji} **for** n -**point functions:** $N_3 = 6, N_4 = 10, N_5 = 15$.

The operator $\mathbf{k}^- \dots$

\dots will reduce an n -point Feynman integral J_n to an $(n-1)$ -point integral J_{n-1} by shrinking the propagator $1/D_k$

$$\mathbf{k}^- J_n = \mathbf{k}^- \int \frac{d^d k}{i\pi^{d/2}} \frac{1}{\prod_{j=1}^n D_j} = \int \frac{d^d k}{i\pi^{d/2}} \frac{1}{\prod_{j \neq k, j=1}^n D_j}. \quad (8)$$

Mellin-Barnes representation

$$\frac{1}{(1+z)^\lambda} = \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} ds \frac{\Gamma(-s) \Gamma(\lambda+s)}{\Gamma(\lambda)} z^s = {}_2F_1 \left[\begin{matrix} \lambda, b; \\ b; \end{matrix} -z \right]. \quad (9)$$

It is valid if $|\text{Arg}(z)| < \pi$ and the integration contour has to be chosen such that the poles of $\Gamma(-s)$ and $\Gamma(\lambda+s)$ are well-separated. The right hand side of (9) is identified as Gauss' hypergeometric function. For more details see [13]).

F -function and Gram and Cayley determinants

Gram and Cayley det's are introduced by Melrose [12] (1965). The Cayley determinant $\lambda_{12\dots N}$ is composed of the

$Y_{ij} = m_i^2 + m_j^2 - (q_i - q_j)^2$ introduced in (7), and its determinant is:

$$\lambda_n \equiv \lambda_{12\dots n} = \begin{vmatrix} Y_{11} & Y_{12} & \dots & Y_{1n} \\ Y_{12} & Y_{22} & \dots & Y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ Y_{1n} & Y_{2n} & \dots & Y_{nn} \end{vmatrix}. \quad (10)$$

We also define the $(n-1) \times (n-1)$ dimensional Gram determinant $g_n \equiv g_{12\dots n}$,

$$G_n \equiv G_{12\dots n} = - \begin{vmatrix} (q_1 - q_n)^2 & (q_1 - q_n)(q_2 - q_n) & \dots & (q_1 - q_n)(q_{n-1} - q_n) \\ (q_1 - q_n)(q_2 - q_n) & (q_2 - q_n)^2 & \dots & (q_2 - q_n)(q_{n-1} - q_n) \\ \vdots & \vdots & \ddots & \vdots \\ (q_1 - q_n)(q_{n-1} - q_n) & (q_2 - q_n)(q_{n-1} - q_n) & \dots & (q_{n-1} - q_n)^2 \end{vmatrix}. \quad (11)$$

Both determinants are independent of a common shifting of the momenta q_i . Further, the Gram det G_n is independent of the propagator masses.

Co-factors of the Cayley matrix

One further notation will be introduced, namely that of **co-factors of the Cayley matrix**. Also called **signed minors** in e.g. [12, 14]):

$$\left(\begin{array}{ccc} j_1 & j_2 & \cdots j_m \\ k_1 & k_2 & \cdots k_m \end{array} \right)_n. \quad (12)$$

The signed minors are determinants, labeled by those **rows j_1, j_2, \dots, j_m and columns k_1, k_2, \dots, k_m which have been discarded from the definition of the Cayley determinant $(\)_n$** , with a sign convention.

$$\text{sign} \left(\begin{array}{ccc} j_1 & j_2 & \cdots j_m \\ k_1 & k_2 & \cdots k_m \end{array} \right)_n = (-1)^{j_1+j_2+\cdots+j_m+k_1+k_2+\cdots+k_m} \times \text{Signature}[j_1, j_2, \dots, j_m] \times \text{Signature}[k_1, k_2, \dots, k_m]$$

Here, `Signature` (defined like the Mathematica command) gives the sign of permutations needed to place the indices in increasing order.

Cayley matrix, by definition:

$$\lambda_n = \left(\begin{array}{c} 0 \\ 0 \end{array} \right)_n. \quad (14)$$

Further, it is (see [15]):

$$-\frac{1}{2} \partial_i \lambda_n \equiv -\frac{1}{2} \frac{\partial \lambda_n}{\partial m_i^2} = \left(\begin{array}{c} 0 \\ i \end{array} \right)_n. \quad (15)$$

Rewriting the F -function further, exploring the $x_n = 1 - \sum x_i \dots$

The elimination of one of the x_i creates linear terms in $F(x)$.

$$F_n(x) = x^T G_n x + 2H_n^T x + K_n. \quad (16)$$

The $F_n(x)$ may be cast by shifts $x \rightarrow (x - y)$ into the form

$$F_n(x) = (x - y)^T G_n (x - y) + r_n - i\varepsilon = \Lambda_n(x) + r_n - i\varepsilon = \Lambda_n(x) + R_n, \quad (17)$$

$$\Lambda_n(x) = (x - y)^T G_n (x - y), \quad (18)$$

and

$$r_n = K_n - H_n^T G_n^{-1} H_n = -\frac{\lambda_n}{g_n} =! -\frac{\begin{pmatrix} 0 \\ 0 \end{pmatrix}_n}{\binom{}{n}}. \quad (19)$$

The inhomogeneity $R_n = r_n - i\varepsilon$ carries the $i\varepsilon$ -prescription.

The linear shifts y_i

The $(n - 1)$ components y_i of the vector y appearing here in $F_n(x)$ are:

$$y_i = - \left(G_n^{-1} K_n \right)_i, \quad i \neq n \quad (20)$$

The following relations are also valid:

$$y_i = \frac{\partial r_n}{\partial m_i^2} = -\frac{1}{g_n} \frac{\partial \lambda_n}{\partial m_i^2} = -\frac{\partial_i \lambda_n}{g_n} = \frac{2}{g_n} \begin{pmatrix} 0 \\ i \end{pmatrix}_n, \quad i = 1 \cdots n. \quad (21)$$

The auxiliary condition $\sum_i^n y_i = 1$ is fulfilled.

We see that the notations for the F -function are finally independent of the choice of the variable which was eliminated by use of the δ -function in the integrand of (4).

The inhomogeneity R_n is the only variable carrying the causal $i\epsilon$ -prescription, while e.g. $\Lambda(x)$ and the y_i are by definition real quantities.

The recursion relation for J_n

One may use the Mellin-Barnes relation (9) in order to decompose the integrand of J_n given in (4) as follows:

$$\begin{aligned} \frac{1}{[F(x)]^{n-\frac{d}{2}}} &\equiv \frac{1}{[\Lambda_n(x) + R_n]^{n-\frac{d}{2}}} \equiv \frac{R_n^{-(n-\frac{d}{2})}}{\left[1 + \frac{\Lambda_n(x)}{R_n}\right]^{n-\frac{d}{2}}} \\ &= \frac{R_n^{-(n-\frac{d}{2})}}{2\pi i} \int_{-i\infty}^{+i\infty} ds \frac{\Gamma(-s) \Gamma(n - \frac{d}{2} + s)}{\Gamma(n - \frac{d}{2})} \left[\frac{\Lambda_n(x)}{R_n}\right]^s, \end{aligned} \quad (22)$$

for $|\text{Arg}(\Lambda_n/R_n)| < \pi$. The condition always applies. Further, the integration path in the complex s -plane separates the poles of $\Gamma(-s)$ and $\Gamma(n - \frac{d}{2} + s)$. As a result of (22), the Feynman parameter integral of J_n becomes homogeneous:

$$K_n = \prod_{j=1}^{n-1} \int_0^{1-\sum_{i=j+1}^{n-1} x_i} dx_j \left[\frac{\Lambda_n(x)}{R_n}\right]^s \equiv \int dS_{n-1} \left[\frac{\Lambda_n(x)}{R_n}\right]^s. \quad (23)$$

The recursion relation for J_n

In order to solve the integral in (23), we consider the differential operator \hat{P}_n [16, 17],

$$\hat{P}_n \left[\frac{\Lambda_n(x)}{R_n} \right]^s \equiv \sum_{i=1}^{n-1} \frac{1}{2} (x_i - y_i) \frac{\partial}{\partial x_i} \left[\frac{\Lambda_n(x)}{R_n} \right]^s = s \left[\frac{\Lambda_n(x)}{R_n} \right]^s. \quad (24)$$

This eigenvalue relation allows to introduce the operator \hat{P}_n into the integrand of (23):

$$K_n = \int dS_{n-1} \frac{\hat{P}_n}{s} \left[\frac{\Lambda_n(x)}{R_n} \right]^s = \frac{1}{2s} \sum_{i=1}^{n-1} \prod_{k=1}^{n-1} \int_0^{u_k} dx'_k (x_i - y_i) \frac{\partial}{\partial x_i} \left[\frac{\Lambda_n(x)}{R_n} \right]^s. \quad (25)$$

After a series of manipulations in order to perform one of the x -integrations – by partial integration, eating the corresponding differential – one arrives at:

$$J_n = \frac{(-1)^n}{2\pi i} \int_{-i\infty}^{+i\infty} ds \frac{\Gamma(-s) \Gamma(n - \frac{d}{2} + s) \Gamma(s+1)}{2 \Gamma(s+2)} \left(\frac{1}{R_n} \right)^{n-\frac{d}{2}} \\ \times \sum_{i=1}^n \left\{ \left(\frac{\partial r_n}{\partial m_i^2} \right) \int dS_{n-2}^{(i)} \left[\frac{F_{n-1}^{(i)}}{R_n} - 1 \right]^s \right\} \quad (26)$$

We stress again that only the R_n carries an $i\epsilon$. Now it is important to eliminate the term (-1) from the combination $(F_{n-1}^{(i)}/R_n - 1)^s$ under the Mellin-Barnes integral over s , because then we arrive at a sum over the n different $(n-1)$ -point functions arising from skipping a propagator from the original integral. In fact, this may be arranged using the following relation for $(-z) = F/R - 1$ [18]:

$$\int_{-i\infty}^{+i\infty} ds \frac{\Gamma(-s) \Gamma(a+s) \Gamma(b+s)}{\Gamma(c+s)} (-z)^s \quad (27)$$

$$= \int_{-i\infty}^{+i\infty} ds \frac{\Gamma(-s) \Gamma(a+b-c-s) \Gamma(c-a+s) \Gamma(c-b+s)}{\Gamma(c-a) \Gamma(c-b)} (1-z)^{c-a-b+s},$$

provided that $|\text{Arg}(-z)| < 2\pi$.

We arrive at the following recursion relation:

The recursion relation for 1-loop n -point functions

$$J_n(d, \{q_i, m_i^2\}) = \frac{-1}{2\pi i} \int_{-i\infty}^{+i\infty} ds \frac{\Gamma(-s) \Gamma(\frac{d-n+1}{2} + s) \Gamma(s+1)}{2\Gamma(\frac{d-n+1}{2})} R_n^{-s} \times \sum_{k=1}^n \left(\frac{1}{r_n} \frac{\partial r_n}{\partial m_k^2} \right) \mathbf{k}^- J_n(d+2s; \{q_i, m_i^2\}). \quad (28)$$

The cases $G_n = 0$ and $\lambda_n = r_n = 0$ prevent the use of the Mellin-Barnes transformation. They are simpler than what we have to do here. Details are given elsewhere.

1-point function, or tadpole

$$J_1(d; m^2) = \int \frac{d^d k}{i\pi^{d/2}} \frac{1}{k^2 - m^2 + i\epsilon} = -\frac{\Gamma(1-d/2)}{(m^2 - i\epsilon)^{1-d/2}}. \quad (29)$$

Comments

1. In Tarasov 2003 [8], a recursion was derived where our Mellin-Barnes integral is replaced by an infinite sum to be solved. Formulae for 2,3,4-point functions are given.
2. A 4-point function is a 3-fold integral. With AMBRE, we get up to 15-fold integrals instead.
3. See Johann Usovitch's talk: Integrand is equally integrable for Euklidean and Minkoswkian cases.
No Gram=0 problem.

The 2-point function

From our recursion relation (28), taken at $n = 2$ and using the expression (29) with $d \rightarrow d + 2s$ for the one-point functions under the integral, one gets the following representation:

$$\begin{aligned}
 J_2(D; q_1, m_1^2, q_2, m_2^2) &= \frac{e^{\epsilon\gamma_E}}{2\pi i} \int_{-i\infty}^{+i\infty} ds \frac{\Gamma(-s) \Gamma\left(\frac{D-1}{2} + s\right) \Gamma(s+1)}{2 \Gamma\left(\frac{D-1}{2}\right)} R_2^s \\
 &\times \left[\frac{1}{r_2} \frac{\partial r_2}{\partial m_2^2} \frac{\Gamma\left(1 - \frac{D+2s}{2}\right)}{(m_1^2)^{1 - \frac{D+2s}{2}}} + (m_1^2 \leftrightarrow m_2^2) \right]. \quad (30)
 \end{aligned}$$

One may close the integration contour of the MB-integral in (30) to the right, apply the Cauchy theorem and collect the residua originating from two series of zeros of arguments of Γ -functions at $s = m$ and $s = m - d/2 - 1$ for $m \in \mathbb{N}$.

The first series stems from the MB-integration kernel, the other one from the dimensionally shifted 1-point functions.

And then summing up in terms of Gauss' hypergeometric functions.

The 2-point function (slightly rewritten), $R_2 \equiv R_{12}$

$$\begin{aligned}
 J_2(d; Q^2, m_1^2, q_2, m_2^2) &= -\frac{\Gamma\left(2 - \frac{d}{2}\right) \Gamma\left(\frac{d}{2} - 1\right)}{(d-2) \Gamma\left(\frac{d}{2}\right)} \frac{\partial_2 R_2}{R_2} \\
 &\quad \left[(m_1^2)^{\frac{d}{2}-1} {}_2F_1\left[1, \frac{d}{2} - \frac{1}{2}; \frac{m_1^2}{R_2}\right] + \frac{R_2^{\frac{d}{2}-1}}{\sqrt{1 - \frac{m_1^2}{R_2}}} \sqrt{\pi} \frac{\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{d}{2} - \frac{1}{2}\right)} \right] \\
 &\quad + (m_1^2 \leftrightarrow m_2^2)
 \end{aligned} \tag{31}$$

The representation (31) is valid for $\left|\frac{m_1^2}{r_{12}}\right| < 1$, $\left|\frac{m_2^2}{r_{12}}\right| < 1$ and $\text{Re}\left(\frac{d-2}{2}\right) > 0$.

The result is in agreement with Eqn. (53) of Tarasov et al. (2003) [8].

The 3-point function

According to the master formula (28), we can write the massive 3-point function as a sum of three terms:

$$J_3 = J_{123} + J_{231} + J_{312}, \quad (32)$$

using the representation for e.g. J_{123}

$$\begin{aligned}
 J_{123}(d, \{q_i, m_i^2\}) &= -\frac{e^{\epsilon\gamma_E}}{2\pi i} \int_{-i\infty}^{+i\infty} ds \frac{\Gamma(-s) \Gamma(\frac{d-2+2s}{2}) \Gamma(s+1)}{2 \Gamma(\frac{d-2}{2})} R_3^{-s} \\
 &\times \frac{1}{r_3} \frac{\partial r_3}{\partial m_3^2} J_2(d+2s; q_1, m_1^2, q_2, m_2^2). \quad (33)
 \end{aligned}$$

Here, $J_2(d+2s; q_1, m_1^2, q_2, m_2^2)$ is given by (31), taken at $d+2s$ dimensions. By performing the Mellin-Barnes integrals, one gets three terms, each consisting of eight series, from taking the residues by closing the integration contours to the right; one of the three terms is:

$$\begin{aligned}
 J_{123} &= \Gamma\left(2 - \frac{d}{2}\right) R_{123}^{\frac{d}{2}-2} \times b_{123} \\
 &- \frac{\sqrt{\pi} \Gamma\left(2 - \frac{d}{2}\right) \Gamma\left(\frac{d}{2} - 1\right)}{\Gamma\left(\frac{d-1}{2}\right)} \frac{\partial_3 \lambda_{123}}{\lambda_{123}} \frac{R_{12}^{\frac{d}{2}-1}}{4\lambda_{12}} \left[\frac{\partial_2 \lambda_{12}}{\sqrt{1 - \frac{m_1^2}{R_{12}}}} + \frac{\partial_1 \lambda_{12}}{\sqrt{1 - \frac{m_2^2}{R_{12}}}} \right] \times {}_2F_1\left[\begin{matrix} \frac{d-2}{2}, 1; \\ \frac{d-1}{2}; \end{matrix} \frac{R_{12}}{R_{123}}\right] \\
 &+ \frac{2}{d-2} \Gamma\left(2 - \frac{d}{2}\right) \frac{\partial_3 \lambda_{123}}{\lambda_{123}} \times \left[\frac{\partial_2 \lambda_{12}}{\sqrt{1 - \frac{m_1^2}{R_{12}}}} \frac{(m_1^2)^{\frac{d}{2}-1}}{4\lambda_{12}} F_1\left(\frac{d-2}{2}; 1, \frac{1}{2}; \frac{d}{2}; \frac{m_1^2}{R_{123}}, \frac{m_1^2}{R_{12}}\right) \right. \\
 &\left. + (m_1^2 \leftrightarrow m_2^2) \right],
 \end{aligned} \tag{34}$$

and

$$\begin{aligned}
 b_{123} &= -\frac{1}{2g_{12}} \frac{\partial_3 \lambda_{123}}{\lambda_{123}} \left(\frac{\partial_2 \lambda_{12}}{\sqrt{1 - \frac{m_1^2}{R_{12}}}} + \frac{\partial_1 \lambda_{12}}{\sqrt{1 - \frac{m_2^2}{R_{12}}}} \right) {}_2F_1\left[\begin{matrix} 1, 1; \\ \frac{3}{2}; \end{matrix} \frac{R_{12}}{R_{123}}\right] \\
 &- \frac{\partial_3 \lambda_{123}}{\lambda_{123}} \left\{ \frac{\partial_2 \lambda_{12}}{\sqrt{1 - \frac{m_1^2}{R_{12}}}} \frac{m_1^2}{4\lambda_{12}} F_1\left(1; 1, \frac{1}{2}; 2; \frac{m_1^2}{R_{123}}, \frac{m_1^2}{R_{12}}\right) + (m_1^2 \leftrightarrow m_2^2) \right\},
 \end{aligned} \tag{35}$$

where $\partial_i \lambda_{j\dots}$ is defined in (21). The representation (32) is valid for $\text{Re}(d - 2/2) > 0$. The conditions $|m_i^2/R_{ij}| < 1$, $|R_{ij}/R_{ijk}| < 1$ had to be met during the derivation. The result may be analytically continued in a straightforward way, however, in the complete complex domain.

The functions ${}_2F_1$ and F_1 of the b_{ijk} -terms are met by setting $d = 4$ in the corresponding functions J_{ijk} of the general J_3 .

An alternative writing of $J_3 = J_{123} + J_{231} + J_{312}$ is, with $R_3 = R_{123}$, $R_2 = R_{12}$ etc. here:

The massive vertex

$$\begin{aligned}
 J_{123} = & \Gamma\left(2 - \frac{d}{2}\right) \frac{\partial_3 r_3}{r_3} \frac{\partial_2 r_2}{r_2} \frac{r_2}{2\sqrt{1 - m_1^2/r_2}} \\
 & \left[-R_2^{d/2-2} \frac{\sqrt{\pi}}{2} \frac{\Gamma\left(\frac{d}{2} - 1\right)}{\Gamma\left(\frac{d}{2} - \frac{1}{2}\right)} {}_2F_1\left[\frac{d-2}{2}, 1; \frac{R_2}{R_3}\right] + R_3^{d/2-2} {}_2F_1\left[1, 1; \frac{R_2}{R_3}\right] \right] \\
 & + \Gamma\left(2 - \frac{d}{2}\right) \frac{\partial_3 r_3}{r_3} \frac{\partial_2 r_2}{r_2} \frac{m_1^2}{4\sqrt{1 - m_1^2/r_2}} \\
 & \left[+ \frac{2(m_1^2)^{d/2-2}}{d-2} F_1\left(\frac{d-2}{2}; 1, \frac{1}{2}; \frac{d}{2}; \frac{m_1^2}{R_3}, \frac{m_1^2}{R_2}\right) - R_3^{d/2-2} F_1\left(1; 1, \frac{1}{2}; 2; \frac{m_1^2}{R_3}, \frac{m_1^2}{R_2}\right) \right] \\
 & + (m_1^2 \leftrightarrow m_2^2)
 \end{aligned}$$

For $d \rightarrow 4$, both the [...] approach zero.

So the J_3 is finite in this limit, as it should be for massive 3-point function.

For the 3-point function, we look at the expression $J_{123} + J_{231} + J_{312}$.

We should agree with Eqn. (74) to (76) of Tarasov (2003).

Our terms with d -dimensional F_1 and ${}_2F_1$ do agree exactly, but $b_{123} + b_{231} + b_{312}$ looks quite different.

Tarasov (2003) [8], Eqns. (73) and (75)

There are kinematic conditions on internal momenta $q_{ij}^2 = (q_i - q_j)^2$ to be respected; the b_3 -term of Tarasov becomes:

$$\begin{aligned}
 J_3(b_3) &= \theta(-G_3) \times \theta(q_{ij}^2) \times \theta\left(\frac{m_i^2}{r_3} - 1\right) \\
 &\quad \times \frac{\Gamma(2 - d/2)}{\lambda_3} \left(2^{3/2} \pi \sqrt{-G_3} R_3^{d/2-1}\right)
 \end{aligned} \tag{36}$$

Otherwise:

$$J_3(b_3) = b_3 = 0. \tag{37}$$

Numerics for 3-point functions, table 1

$[p_i^2], [m_i^2]$	[+100, +200, +300], [10, 20, 30]	
G_{123}	-160000	
λ_{123}	-8860000	
m_i^2/r_{123}	-0.180587, -0.361174, -0.541761	
m_i^2/r_{12}	-0.97561, -1.95122, -2.92683	
m_i^2/r_{23}	-0.39801, -0.79602, -1.19403	
m_i^2/r_{31}	-0.180723, -0.361446, -0.542169	
$\sum J$ -terms	(0.019223879 - 0.007987267 I)	
$\sum b_3$ -terms	0	
J_3 (TR)	(0.019223879 - 0.007987267 I)	
b_3 -term	(-0.089171509 + 0.069788641 I)	+ (0.022214414)/eps
$b_3 + \sum J$ -terms	(-0.012307377 - 0.009301346 I)	
J_3 (OT)	$\sum J$ -terms, b_3 -term $\rightarrow 0$, OK	
MB suite		
(-1)*fiesta3	-(0.012307 + 0.009301 I)	+ (8*10-6 + 0.00001 I) pm4)
LoopTools/FF, ϵ^0	0.0192238790286244077-0.00798726725497102795 i	

Table 1: Numerics for a vertex in space-time dimension $d = 4 - 2\epsilon$. Causal $\epsilon = 10^{-20}$. Red input quantities (external momenta shown here!) suggest that, according to Eqn. (73) in Tarasov (2003) [8], one has to set $b_3 = 0$.

Although b_3 of [8] deviates from our vanishing value, it has to be set to zero, $b_3 \rightarrow 0$.

The results of both calculations for J_3 agree for this case.

Numerics for 3-point functions, table 2

$[p_i^2], [m_i^2]$	$[-100, +200, -300], [10, 20, 30]$	
G_{123}	480000	
λ_3	-19300000	
m_i^2/r_3	0.248705, 0.497409, 0.746114	
m_i^2/r_{12}	0.248447, 0.496894, 0.745342	
m_i^2/r_{23}	-0.39801, -0.79602, -1.19403	
m_i^2/r_{31}	0.104895, 0.20979, 0.314685	
$\sum J$ -terms	$(-0.012307377 - 0.056679689 \text{ I})$	$+ (+ 0.012825498 \text{ I})/\text{eps}$
$\sum b_3$ -terms	$(+ 0.047378343 \text{ I})$	$- (+ 0.012825498 \text{ I})/\text{eps}$
$J_3(\text{TR})$	$(-0.012307377 - 0.009301346 \text{ I})$	
b_3 -term	$(+ 0.047378343 \text{ I})$	$- (+ 0.012825498 \text{ I})/\text{eps}$
$b_3 + \sum J$ -terms	$(-0.012307377 - 0.009301346 \text{ I})$	
$J_3(\text{OT})$	$\sum J$ -terms, b_3 -term $\rightarrow 0$, gets wrong	
MB suite		
$(-1)^*\text{fiesta3}$	$(-0.012307 + 0.009301 \text{ I})$	$+ (8*10^{-6} + 0.00001 \text{ I}) \text{ pm4}$
LoopTools/FF, ϵ^0	$-0.0123073773677820630 - 0.0093013461700863289 \text{ i}$	

Table 2: Numerics for a vertex in space-time dimension $d = 4 - 2\epsilon$. Causal $\epsilon = 10^{-20}$. Red input quantities suggest that, according to eq. (73) in Tarasov2003 [8], one has to set $b_3 = 0$. Further, we have set in the numerics for eq. (75) of Tarasov2003 [8] that $\text{Sqrt}[-g_{123} + \text{I}*\text{epsil}]$, what looks counter-intuitive for a "momentum"-like function.

Both results agree if we do not set Tarasov's $b_3 \rightarrow 0$.

Numerics for 3-point functions, table 3

p_i^2	-100,-200,-300	
m_i^2	10,20,30	
G_{123}	-160000	
λ_{123}	15260000	
m_i^2/r_{123}	0.104849, 0.209699, 0.314548	
m_i^2/r_{12}	0.248447, 0.496894, 0.745342	
m_i^2/r_{23}	0.133111, 0.266223, 0.399334	
m_i^2/r_{31}	0.104895, 0.20979, 0.314685	
$\sum J$ -terms	(0.0933877 - 0 I)	- (0.0222144 - 0 I)/eps
$\sum b$ -terms	-0.101249	+ 0.0222144/eps
$J_3(\text{TR})$	(-0.00786155 - 0 I)	
b_3	(-0.101249 + 0 I)	+ (0.0222144 + 0 I)/eps
b_3+J -terms	(-0.007861546 + 0 I)	
$J_3(\text{OT})$	b_3+J -terms \rightarrow OK	
MB suite	-0.007862014, 5.002549159*10-6, 0	
(-1)*fiesta3	-(0.007862)	+ (6*10-6 + 6*10-6 I pm10)
LoopTools/FF, ϵ^0	-0.00786154613229082290	

Table 3: Numerics for a vertex in space-time dimension $d = 4 - 2\epsilon$. Causal $\epsilon = 10^{-20}$.

Agreement with Tarasov (2003).

Numerics for 3-point functions, table 4

p_i^2	+100, -200, +300	
m_i^2	10, 20, 30	
G_{123}	480000	
λ_{123}	4900000	
m_i^2/r_{123}	-0.979592, -1.95918, -2.93878	
m_i^2/r_{12}	-0.97561, -1.95122, -2.92683	
m_i^2/r_{23}	0.133111, 0.266223, 0.399334	
m_i^2/r_{31}	-0.180723, -0.361446, -0.542169	
$\sum J$ -terms	(0.006243624 - 0.018272524 l)	
$\sum b_3$ -terms	0	
$J_3(\text{TR})$	(0.006243624 - 0.018272524 l)	
b_3 -term	(0.040292491 + 0.029796253 l)	+ (- 0.012825498 l)/eps
$b_3 + \sum J$ -terms	(-0.012307377 - 0.009301346 l)	+ (4*-18 - 6*-18 l)/eps
$J_3(\text{OT})$	$\sum J$ -terms, b_3 -term $\rightarrow 0$, OK	
MB suite		
(-1)*fiesta3	-(-0.006322 + 0.014701 l)	+ (0.000012 + 0.000014 l) pm
LoopTools/FF, ϵ^0	0.00624362477277410 - 0.01827252404872805 i	

Table 4: Numerics for a vertex in space-time dimension $d = 4 - 2\epsilon$. Causal $\epsilon = 10^{-20}$. Red input quantities suggest that, according to eq. (73) in Tarasov2003 [8], one has to set $b_3 = 0$.

Agreement with Tarasov (2003) due to setting $b_3 = 0$ there.

The 4-point function

According to the master formula (28), we can write the massive 4-point function as a sum of four terms:

$$J_4 = J_{1234} + J_{2341} + J_{3412} + J_{4123}, \quad (38)$$

Each of the four terms has the structure

$$J_{1234} = \frac{\Gamma\left(2 - \frac{d}{2}\right) \Gamma\left(\frac{d}{2} - 1\right)}{\Gamma\left(\frac{d-3}{2}\right)} \times (r_{1234})^{\frac{d}{2}-2} \times \hat{b}_{1234} + \Gamma(2 - d/2) \times \hat{J}_{1234}^d \quad (39)$$

The pre-factor is singular: $\Gamma(2 - d/2) = 1/\epsilon + \dots$ for $d \geq 4 - 2\epsilon$.

We agree for \hat{J}_{1234}^d etc. with Tarasov (2003) [8].
For the b_4 -term, the situation is a bit unclear.

The boundary term \hat{b}_{1234} is independent of d :

$$\begin{aligned}
 \hat{b}_{1234} &= \frac{1}{2} \left(\frac{b_{123}}{r_{1234}} \frac{\partial r_{1234}}{\partial m_4^2} \right) \frac{\sqrt{\pi}}{\sqrt{1 - r_{123}/r_{1234}}} \\
 &+ \sqrt{\pi} \left(\frac{1}{r_{1234}} \frac{\partial r_{1234}}{\partial m_4^2} \right) \left(\frac{1}{r_{123}} \frac{\partial r_{123}}{\partial m_3^2} \right) \left(\frac{1}{4g_{12}} \right) \times \\
 &\times \left[\frac{\partial_2 \lambda_{12}}{\sqrt{1 - m_1^2/r_{12}}} + \frac{\partial_1 \lambda_{12}}{\sqrt{1 - m_2^2/r_{12}}} \right] \left(\frac{1}{\sqrt{1 - r_{12}/r_{123}}} \right) F_1 \left(\frac{1}{2}; 1, \frac{1}{2}; \frac{3}{2}; \frac{r_{12}}{r_{1234}}, \frac{r_{12}}{r_{123}} \right) \quad (40) \\
 &+ \sqrt{\pi} \left(\frac{1}{r_{1234}} \frac{\partial r_{1234}}{\partial m_4^2} \right) \left(\frac{1}{r_{123}} \frac{\partial r_{123}}{\partial m_3^2} \right) \times \\
 &\times \left[\left(\frac{\partial_2 \lambda_{12}}{1 - \frac{m_1^2}{r_{12}}} \right) \left(\frac{m_1^2}{8\lambda_{12}} \right) \left(\frac{r_{123}}{r_{123} - m_1^2} \right) \right. \\
 &\times F_S \left(\frac{1}{2}, 1, 1; 1, 1, \frac{1}{2}; 2, 2, 2; \frac{m_1^2}{r_{1234}}, \frac{m_1^2}{m_1^2 - r_{123}}, \frac{m_1^2}{m_1^2 - r_{12}} \right) + (1 \leftrightarrow 2) \left. \right] \\
 &+ (2, 3, 1) + (3, 1, 2).
 \end{aligned}$$

The boundary term b_4 has not been exactly defined in [8], concerning the kinematic conditions. We did not perform massive numerical tests.

and

$$\begin{aligned}
\hat{J}_{1234} &= (r_{1234})^{\frac{d}{2}-2} \times b_{1234} \\
&\quad - \frac{1}{2} \left(\frac{1}{r_{1234}} \frac{\partial r_{1234}}{\partial m_4^2} \right) {}_2F_1 \left[\frac{d-3}{2}, 1; \frac{r_{123}}{r_{1234}} \right] \times (r_{123})^{\frac{d}{2}-2} \times b_{123} \\
&\quad + \sqrt{\pi} \frac{\Gamma(\frac{d}{2}-1)}{\Gamma(\frac{d-1}{2})} \left(\frac{1}{r_{1234}} \frac{\partial r_{1234}}{\partial m_4^2} \right) \left(\frac{1}{r_{123}} \frac{\partial r_{123}}{\partial m_3^2} \right) \left[\frac{\partial_2 \lambda_{12}}{\sqrt{1-\frac{m_1^2}{r_{12}}}} + \frac{\partial_1 \lambda_{12}}{\sqrt{1-\frac{m_2^2}{r_{12}}}} \right] \times \\
&\quad \quad \quad \times \left(\frac{r_{12}^{\frac{d}{2}-1}}{8\lambda_{12}} \right) \left(\frac{1}{\sqrt{1-\frac{r_{12}}{r_{123}}}} \right) F_1 \left(\frac{d-3}{2}; 1, \frac{1}{2}; \frac{d-1}{2}; \frac{r_{12}}{r_{1234}}, \frac{r_{12}}{r_{123}} \right) \\
&\quad - \frac{\Gamma(\frac{d}{2}-1)}{\Gamma(\frac{d}{2})} \left(\frac{1}{r_{1234}} \frac{\partial r_{1234}}{\partial m_4^2} \right) \left(\frac{1}{r_{123}} \frac{\partial r_{123}}{\partial m_3^2} \right) \times \\
&\quad \quad \times \left[\left(\frac{r_{123}}{r_{123}-m_1^2} \right) \left(\frac{r_{12}}{r_{12}-m_1^2} \right) \left(\frac{\partial_2 \lambda_{12}}{8\lambda_{12}} \right) (m_1^2)^{\frac{d}{2}-1} \times \right. \\
&\quad \quad \quad \times F_5 \left(\frac{d-3}{2}, 1, 1, 1, 1, \frac{1}{2}; \frac{d}{2}, \frac{d}{2}, \frac{d}{2}; \frac{m_1^2}{r_{1234}}, \frac{m_1^2}{m_1^2-r_{123}}, \frac{m_1^2}{m_1^2-r_{12}} \right) + (1 \leftrightarrow 2) \left. \right] \\
&\quad + (2, 3, 1) + (3, 1, 2).
\end{aligned} \tag{41}$$

An alternative writing of $J_4 = J_{1234} + J_{2341} + J_{3412} + J_{4123}$ is, with $R_4 = R_{1234}$, $R_3 = R_{123}$, $R_2 = R_{12}$ etc. here:

The massive box function

$$\begin{aligned}
 J_{1234} = & \Gamma\left(2 - \frac{d}{2}\right) \frac{\partial_4 r_4}{r_4} \left\{ \right. \\
 & \left[\frac{b_{123}}{2} \left(-R_3^{d/2-2} {}_2F_1\left[\frac{d-3}{2}, 1; \frac{R_2}{R_3}\right] + R_4^{d/2-2} \sqrt{\pi} \frac{\Gamma\left(\frac{d}{2}-1\right)}{\Gamma\left(\frac{d}{2}-\frac{3}{2}\right)} {}_2F_1(d \rightarrow 4) \right) \right] \\
 & + \frac{\Gamma\left(\frac{d}{2}-1\right)}{\Gamma\left(\frac{d}{2}-\frac{3}{2}\right)} \frac{\sqrt{\pi}}{4} \frac{\partial_3 r_3}{r_3} \frac{\partial_2 r_2}{\sqrt{1-m_1^2/R_2}} {}_2F_1\left[\frac{1}{2}, 1; \frac{R_2}{R_3}\right] \\
 & \left[+ \frac{R_2^{d/2-2}}{d-3} F_1\left(\frac{d-3}{2}; 1, \frac{1}{2}; \frac{d-1}{2}; \frac{R_2}{R_4}, \frac{R_2}{R_3}\right) - R_4^{d/2-2} F_1(d \rightarrow 4) \right] \\
 & \frac{m_1^2}{8} \frac{\Gamma\left(\frac{d}{2}-1\right)}{\Gamma\left(\frac{d}{2}-\frac{3}{2}\right)} \frac{\partial_3 r_3}{r_3} \frac{\partial_2 r_2}{r_2} \frac{r_3}{r_3-m_1^2} \frac{r_2}{r_2-m_1^2} \\
 & \left[- (m_1^2)^{d/2-2} \frac{\Gamma\left(\frac{d}{2}-3/2\right)}{\Gamma\left(\frac{d}{2}\right)} F_S(d/2-3/2, 1, 1, 1, 1, d/2, d/2, d/2, d/2, \frac{m_1^2}{R_4}, \frac{m_1^2}{m_1^2-R_3}, \frac{m_1^2}{m_1^2-R_2}) \right. \\
 & \left. + R_4^{d/2-2} \sqrt{\pi} F_S(d \rightarrow 4) \right] + (m_1^2 \leftrightarrow m_2^2) \left. \right\} \quad (42)
 \end{aligned}$$

For $d \rightarrow 4$, all three [...] approach zero.

So that the massive J_4 gets finite then: OK.

Summary

- **We derived a new recursion relation for one-loop scalar Feynman integrals:** self-energies, vertices, boxes etc.
The condition $\nu_i = 1$ seems to be essential for that.
A generalization to multiloops seems to be not straightforward or impossible.
- **Solving the recursions in terms of special functions reproduces essential parts of the results of Tarasov et al. from 2003.**
- **Concerning their b_3 -terms, we see a need of improvement compared to their paper, if their result is not just wrong in some kinematical situations.** Our conclusions concerning that depend somewhat on an interpretation of their text.
- **We derived a new series of Mellin-Barnes representations: 1-dim. for self-energies, 2-dim. for vertices, and 3-dim. for box diagrams** for the most general massive kinematics. Compared to dim=3, 5, 9 respectively, in the “conventional” Mellin-Barnes-approach.
This is worked out by Johann Usovitsch, see talk.
Again, we see no direct generalization to multi-loops.
- The special case of **vanishing Gram determinant** $G_n = 0$ is not covered. But small Gram determinants are, and one has to take measures to get reasonable numerics. → **Small Gram dets are very interesting, but nothing is done here.**
But see numerics of J. Usovitsch, see talk.

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