Effective potential at 3-loops

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outline

- effective potential and its evaluation
- scalar sector and spontaneous symmetry breaking
- differential equations and cyclotomic polylogarithms
- analytical results in the scalar sector
- numerical evaluation of cyclotomic polylogarithms with differential equations
- conclusions

motivation

- the conceptions of effective potential is important to undestand the spontaneous symmetry breaking
- in the SM we can study the behaviour of the effective potential at high energies with the renormalization group tools
- in the stability scenario we can derive restrictions on Higgs/top masses
- some literature about such analysis
  Degrassi et al. 12', 13'
  Bezrukov, Kalmykov, Kniehl, Shaposhnikov 12'
  Bednyakov,Pikelner,Kniehl,OV 15'
- required intput:
  - $V_{\text{eff}}(\phi)$
  - renomalization group functions
  - "low" energy matching relations

previous calcuations of $V_{\mathrm{eff}}(\phi)$ in the SM

- 1-loop — Sher 1989
- 2-loop — Ford, Jack, Jones 1992
- 2-loop for general theory — Martin 2002
- 3-loop at leading order Yukawa — Martin 2013
- 4-loop at leading order QCD — Martin 2015
- 3-loop (numerically using M3VIL library) — Martin 2017

effective action

- partitioin function

$$Z[j] = \int \mathcal{D}\phi \, \exp\left(iS[\phi] + i \int j\phi \, dx \right)$$

- effective action

$$\Gamma[\phi] = \ln Z[j] - \int j\phi \, dx$$

where j is a solution to

$$\frac{\delta \ln Z[j]}{\delta j} = \phi$$

effective action

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where j is a solution to

$$\frac{\delta \ln Z[j]}{\delta j} = \phi$$

- $\Gamma[\phi]$ can be represented as a formal series

$$\Gamma[\phi] = \sum_{n=0}^{\infty} \int \Gamma^{(n)}(x_1, \ldots, x_n)\phi(x_1) \ldots \phi(x_n) \, dx_1 \ldots dx_n$$

where $\Gamma^{(n)}(x_1, \ldots, x_n)$ are 1PI irreducible Green's functions

effective potential

- alternatively we can do an expansion around $\phi = \mathrm{const}$

$$\Gamma[\phi] = \int dx \left[U_0(\phi) + \frac{1}{2!}(\partial\phi)^2 U_2(\phi) + \frac{1}{4!}(\partial\phi)^4 U_4(\phi) + \dots \right]$$

effective potential

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- then

$$V_{\mathrm{eff}}(\phi) = -U_0(\phi)$$

potential in the SM

- scalar sector lagrangian in the SM

$$\mathcal{L}_{\text{scalar}} = (\partial\Phi^\dagger)(\partial\Phi) - m^2\Phi^\dagger\Phi - \lambda(\Phi^\dagger\Phi)^2$$

we can choose one of the vacua

$$\Phi(x) = \begin{pmatrix} \frac{\phi + H(x) + iG^0(x)}{\sqrt{2}} \\ G^+(x) \end{pmatrix}$$

- this brings to loop expansion

$$V_{\text{eff}}(\phi) = -\frac{m^2}{2}\phi^2 + \frac{\lambda}{4}\phi^4 + V_1(\phi) + V_2(\phi) + \ldots$$

standard evaluation

- calculation in background field ϕ_b
- in the constant background field diagrams can be resummed to give propagators with "effective masses" (depending on the gauge!)
- in all calculations the Landau gauge is used to avoid mixing scalar-vector; in this cases propagators have "masses"

$$
\begin{aligned}
M_H^2 &= -m^2 + 3\lambda\phi^2 \\
M_G^2 &= -m^2 + \lambda\phi^2 \\
M_t^2 &= \frac{1}{2} y_t^2 \phi^2 \\
M_W^2 &= \frac{1}{4} g^2 \phi^2 \quad \text{and} \quad 0 \\
M_Z^2 &= \frac{1}{4} (g'^2 + g^2) \phi^2 \quad \text{and} \quad 0
\end{aligned}
$$

minimum of the potential

- minimum (solving order by order)

$$\frac{\partial V_{\rm eff}}{\partial \phi} = 0$$

- at lowest order

$$\phi_{\rm min} = \sqrt{\frac{m^2}{\lambda}}$$

which corresponds to "mass" of goldstone

$$G(\phi_{\rm min}) = 0$$

- higher orders $\phi_{\rm min} = \phi_0 + \phi_1 + \phi_2 + \dots$

$$\phi_1 = -V_1'/V_0''$$
$$\phi_2 = -(V_2' + \phi_1 V_1'' + \frac{1}{2} V_0''')/V_0''$$
$$\phi_3 = -(V_3' + \phi_1 V_2'' + \frac{1}{2}\phi_2 V_1''' + \phi_1\phi_2 V_0''' + \frac{1}{6} V_0'''')/V_0''$$

- V_j are singular in the limit $G \to 0$ but all ϕ_j should be finite

3-loop matching relation for the Higgs self-coupling

- running Higgs self-coupling

$$\lambda(\mu) = \frac{G_F}{\sqrt{2}} M_H^2 \Big(1 + \delta_H(\mu)\Big)$$

  where $\delta_H(\mu)$ can be evaluated from the Higgs self-energy
- pole mass of the Higgs boson as a solution to equation

$$M_H^2 = m^2 + 3\lambda v^2 + \Pi_{HH}(M_H^2)$$

- 3-loop result for $\Pi_{HH}(M_H^2)$ in the gaugeless limit of the SM
- estimation of the pole mass from the potential

$$M_H^2 = 2\lambda v^2 + \left(-\frac{1}{\phi}\frac{\partial}{\partial\phi} + \frac{\partial^2}{\partial\phi^2}\right) V_{\text{eff}}(\phi)\Bigg|_{\phi=v}$$

more general scalar Lagrangian

$$\mathcal{L}_S = \underbrace{\frac{{m_H}^2}{2}H^2 + \frac{{m_G}^2}{2}G_i^2}_{\text{mass terms}} + \underbrace{\frac{\tau_0}{6}H^3 + \frac{\tau_i}{6}HG_i^2}_{\text{triple interaction}} + \underbrace{\frac{\lambda_0}{24}H^4 + \frac{\lambda_i}{12}H^2G_i^2 + \frac{\lambda_{ij}}{24}G_i^2G_j^2}_{\text{quartic interaction}}$$

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- $O(N)$ symmetric scalar theory with $\langle\varphi_1\rangle = \phi \neq 0$ and $\langle\varphi_i\rangle = 0,\ j > 1$

$$\mathcal{L} = \frac{m^2}{2}\varphi^2 + \frac{\lambda}{24}(\varphi^2)^2$$

$$\tau_0 = \tau_i = \lambda\phi, \quad \lambda_0 = \lambda_i = \lambda_{ij} = \lambda \quad m_H = m^2 + \frac{\lambda}{2}\phi^2, m_G = m^2 + \frac{\lambda}{6}\phi^2$$

- Standard Model in the broken phase

$$\mathcal{L} = m^2\Phi^\dagger\Phi + \frac{\lambda}{6}(\Phi^\dagger\Phi)^2, \quad \Phi = \frac{1}{\sqrt{2}}(\phi + H + iG_0, G_r + iG_i)^T$$

$$\tau_0 = \tau_i = \lambda\phi, \quad \lambda_0 = \lambda_i = \lambda_{ij} = \lambda \quad m_H = m^2 + \frac{\lambda}{2}\phi^2, m_G = m^2 + \frac{\lambda}{6}\phi^2$$

known analytical results for the effective potential

- 2-loop
- SM [Ford,Jack,Jones'93]
- general theory [Martin'01]

- 3-loop
- Massless broken $\mathcal{O}(N)$ symmetric φ^4 theory (only $\mathcal{O}(\frac{1}{\varepsilon})$ part) $m_G^2 = 3m_H^2$ [Chung,Chung'97;Kotikov'98]
- Single component masive φ^4 theory only m_H [Chung,Chung'99]

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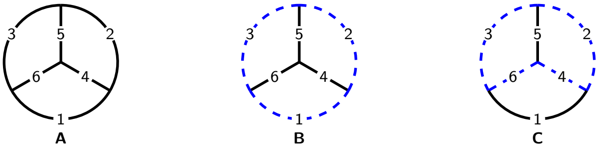
Our goal:

General case $m \neq 0$ with two scales $m_G \neq m_H$

Three-loop topologies for vacuum integrals



- Topology **A** is single scale, reduction using MATAD [Steinhauser'00] master integrals known up to the weight 6 [Kniehl,Pikelner,OV'17]
- Topologies **B** and **C** have 11 master integrals each depending on a single variable $x = \left(\frac{m_G}{m_H}\right)^2$, reduction using LiteRed [Lee'14]
- Differentiating in x and reducing back to the set of the master integrals we obtain closed system of 11 differential equations:

$$\partial_x J_a(\varepsilon, x) = M_{ab}(\varepsilon, x) J_b(\varepsilon, x)$$

- We are looking for solution as expansion in $\varepsilon = 2 - d/2$

Differential equations and canonical basis

- set of the master integrals is not unique
- uniform basis: the coefficients of ε-expansion have uniform transcendental weight
- after apropriate change of basis $\vec{J} = T\vec{g}$, the DE system has the form [Henn'13]:

$$\partial_x g_a(x) = \varepsilon M_{ab}(x) g_b(x)$$

- solution in each order in ε can be written explicitly upto a constant vector

$$g_a\{\varepsilon^n\}(y) = \int dy\, M_{ab}(y)\, g_b\{\varepsilon^{n-1}\}(y) + \mathcal{C}_{a,n}$$

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- for system solvable in terms of GPL, algorithmic ways of canonical basis construction exists [Lee'14] and [Meyer'16] with public implementations `Fuchsia` [Gituliar,Magerya'17], `epsilon` [Prausa'17] and `CANONICA` [Meyer'17]

Differential equations and canonical basis

change of variable

rational transformation can be constructed only after apropriate variable change

$$x = \frac{y^2}{(1+y^2)^2}$$

Iterated integrals

$$\int_0^x dz_1 f_1(z_1) \int_0^{z_1} dz_2 f_2(z_2) \int_0^{z_2} dz_3 f_3(z_3) \cdots \int_0^{z_{n-1}} dz_n f_n(z_n)$$

- Harmonic polylogarithms(HPL), include Li_n and $S_{n,p}$

$$f_{-1}(z) = \frac{1}{z-1}, \quad f_0(z) = \frac{1}{z}, \quad f_1(z) = \frac{1}{z+1}$$

- Generalized polylogarithms(GPL), include HPL

$$f_a(z) = \frac{1}{z-a}$$

- Cyclotomic polylogarithms, after factorization over $\mathbb{C}$ and partial fractioning can be reexpressed through GPL

$$f_a^b(z) = \frac{z^b}{\Phi_a(z)}, \quad f_0^0(z) = \frac{1}{z}, \quad \Phi_n(z) = \prod_{\gcd(k,n)=1} \left(z - e^{2\pi i \frac{k}{n}}\right)$$

Cyclotomic polylogarithms integration

- system in canonical basis can be easily decomposed into the form, where $B_{a,b}$ and $C_{a,b}$ are pure numeric matrices and all y dependence is inside functions f_a^b known how to integrate using definition of CPL:

$$
\begin{aligned}
B(y) = \big(&f_0^0 B_{0,0} + f_1^0 B_{1,0} + f_2^0 B_{2,0} + f_3^0 B_{3,0} + f_3^1 B_{3,1} \\
&+ f_4^1 B_{4,1} + f_6^0 B_{6,0} + f_6^1 B_{6,1} + f_{12}^1 B_{12,1} + f_{12}^3 B_{12,3}\big) \\
C(y) = \big(&f_0^0 C_{0,0} + f_1^0 C_{1,0} + f_2^0 C_{2,0} + f_3^0 C_{3,0} + f_3^1 C_{3,1} \\
&+ f_4^1 C_{4,1} + f_6^0 C_{6,0} + f_6^1 C_{6,1} + f_8^3 C_{8,3}\big)
\end{aligned}
$$

- integration constants are fixed from the finite number of terms of small m_G mass expansion ($y \to 0$)
- the $O(\varepsilon^0)$ parts of the three-loop integrals are expressible through the cyclotomic polylogarithms up to the weight four

cyclotomic polylogarithms

$$\mathcal{H}\left[\begin{smallmatrix} a \\ b \end{smallmatrix}; \begin{smallmatrix} a_1 \\ b_1 \end{smallmatrix}; \ldots; \begin{smallmatrix} a_k \\ b_k \end{smallmatrix}\right](x) = \int_0^y dt\ f_a^b(t)\,\mathcal{H}\left[\begin{smallmatrix} a_1 \\ b_1 \end{smallmatrix}; \ldots; \begin{smallmatrix} a_k \\ b_k \end{smallmatrix}\right](t)$$

with f_a^b being inverse cyclotomic polynomials

$$f_0^0(t) = \frac{1}{t}, \quad f_a^b(t) = \frac{y^b}{\Phi_a(t)}$$

e.g.

$$\begin{aligned}
\Phi_1 &= t - 1 \\
\Phi_2 &= t + 1 \\
\Phi_3 &= t^2 + t + 1 \\
\Phi_4 &= t^2 + 1 \\
\Phi_6 &= t^2 - y + 1 \\
\Phi_8 &= t^4 + 1 \\
\Phi_{12} &= t^4 - t^2 + 1
\end{aligned}$$

cyclotomic polylogarithms

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$$\Phi_{12} = t^4 - t^2 + 1$$

example: mercedes C

$$
\begin{aligned}
J^{c}_{111111} = {} & -192\mathcal{H}\begin{bmatrix}0;&0;&0;&0\\0,&0,&0,&3\end{bmatrix} - 384\mathcal{H}\begin{bmatrix}0;&0;&0;&1\\0,&0,&0,&3\end{bmatrix} + 384\mathcal{H}\begin{bmatrix}0;&0;&0;&1\\0,&0,&0,&4\end{bmatrix} + 192\mathcal{H}\begin{bmatrix}0;&0;&0;&0\\0,&0,&0,&6\end{bmatrix} \\
& - 384\mathcal{H}\begin{bmatrix}0;&0;&0;&1\\0,&0,&0,&6\end{bmatrix} + 384\mathcal{H}\begin{bmatrix}0;&0;&0;&3\\0,&0,&0,&8\end{bmatrix} - 256\mathcal{H}\begin{bmatrix}0;&0;&0;&1\\0,&0,&3,&4\end{bmatrix} - 512\mathcal{H}\begin{bmatrix}0;&0;&1;&1\\0,&0,&3,&4\end{bmatrix} + 64\mathcal{H}\begin{bmatrix}0;&0;&1;&0\\0,&0,&4,&3\end{bmatrix} \\
& + 128\mathcal{H}\begin{bmatrix}0;&0;&1;&1\\0,&0,&4,&3\end{bmatrix} - 64\mathcal{H}\begin{bmatrix}0;&0;&1;&0\\0,&0,&4,&6\end{bmatrix} + 128\mathcal{H}\begin{bmatrix}0;&0;&1;&1\\0,&0,&4,&6\end{bmatrix} + 256\mathcal{H}\begin{bmatrix}0;&0;&0;&1\\0,&0,&6,&4\end{bmatrix} - 512\mathcal{H}\begin{bmatrix}0;&0;&1;&1\\0,&0,&6,&4\end{bmatrix} \\
& + 64\mathcal{H}\begin{bmatrix}0;&0;&3;&0\\0,&0,&8,&3\end{bmatrix} + 128\mathcal{H}\begin{bmatrix}0;&0;&3;&1\\0,&0,&8,&3\end{bmatrix} - 64\mathcal{H}\begin{bmatrix}0;&0;&3;&0\\0,&0,&8,&6\end{bmatrix} + 128\mathcal{H}\begin{bmatrix}0;&0;&3;&1\\0,&0,&8,&6\end{bmatrix} - 256\mathcal{H}\begin{bmatrix}0;&0;&1;&1\\0,&3,&4,&4\end{bmatrix} \\
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\end{aligned}
$$

numerical evaluation of cyclotomic polylogarithms

- CPL were concidered in series of papers [Ablinger, Blümlein, Scheider '11, '14, '15]
- relations to nested sums/integrals, Mellin transforms, Shuffle algebra etc. are implemented in package Harmonic Sums [Ablinger 12', 14']
- after factorization of cyclotomic over $\mathbb{C}$ polynomials CPL can be expressend in terms of GPL
  $\Longrightarrow$ numerical implementation in Ginac [Volling, Weinzierl 04']
- our own implementation with the goal of multiprecision evaluation CycloGPL

system of differential equations

- for CPL of weight w introduce $w+1$ dimensional vector of functions

$$f_w = \mathcal{H}\left[\begin{smallmatrix} a_w \\ b_w \end{smallmatrix}; \ldots; \begin{smallmatrix} a_1 \\ b_1 \end{smallmatrix}\right], f_{w-1} = \mathcal{H}\left[\begin{smallmatrix} a_{w-1} \\ b_{w-1} \end{smallmatrix}; \ldots; \begin{smallmatrix} a_1 \\ b_1 \end{smallmatrix}\right], \ldots, f_1 = \mathcal{H}\left[\begin{smallmatrix} a1 \\ b_1 \end{smallmatrix}\right], \quad f_0 = \mathcal{H}\,[]$$

- vector $\mathbf{f} = (f_w, \ldots, f_0)^T$ obeys differential equation

$$\frac{d}{dy}\mathbf{f} = M(y)\,\mathbf{f}$$

- where the $(w+1) \times (w+1)$ matrix $M(y)$ has the form

$$M(y) = \begin{pmatrix} 0 & f_{a_w}^{b_w} & 0 & \ldots & 0 & 0 \\ 0 & 0 & f_{a_{w-1}}^{b_{w-1}} & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 0 & f_{a_1}^{b_1} \\ 0 & 0 & 0 & \ldots & 0 & 0 \end{pmatrix}$$

system of differential equations (2)

- while $0 \leq x = m_1/m_2 \leq \infty$ the variable

$$x = y^2/(1+y^2)^2$$

  is complex and is within unit circle
- the (regular) singular points y_r of differential equation are

$$0, \qquad \infty, \qquad \text{roots of unity } \omega_n, \quad n = 1,2,3,4,6,8,12$$

- we can solve DE at points y_r by Frobenius method
- boundary conditions at $y = 0$ (trivially!)
- matching of solutions in different points in overlapping points [Lee, Smirnov, Smirnov 16']

Frobenius solution of DE: scalar (1-dimensional) example)

$$f'(x) = A(x)f(x)$$

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- Solution at regular point: $A(x) = a_1 + a_2x + a_3x^2 + \dots$
  substitute the Ansatz $f(x) = c_0 + c_1x + x_2x^2 + \dots$
  We get the recurrence for the coefficients

$$nc_n = \sum_{k=1}^{n} a_k c_{n-k}, \qquad n > 0$$

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- Solution at singular point: $A(x) = \frac{a_0}{x} + a_1 + a_2 x + a_3 x^2 + \dots$

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  We get the recurrence for the coefficients

$$nc_n = \sum_{k=1}^{n} a_k c_{n-k}, \qquad n > 0$$

- Solution at singular point: $A(x) = \frac{a_0}{x} + a_1 + a_2 x + a_3 x^2 + \ldots$
  Now the correct Ansatz is:

$$f(x) = x^{a_0}\left(c_0 + c_1 x + c_2 x^2 + \ldots\right)$$

multidimensional case

- in practice we have a system of n DE's

$$\vec{\mathbf{f}}' = M(x)\vec{\mathbf{f}}, \qquad \text{with matrix } M(x)$$

- according to standard DE theory there are exactly n independent solutions ϕ_k; let put them into matrux W with columns ϕ_k, then

$$W' = M(x)W, \qquad \text{and } \det W \not\equiv 0$$

- W is the fundamental solution of the above matrix equation and it is unique up to a constant matrix
- if we fix in some region boundary conditions vector $\vec{b}$, then the solution is

$$\vec{f}(x) = W(x)\vec{b}$$

- in any other region

$$\tilde{W}(x) = W(x)C, \qquad \text{with constant matrix } C$$

general theory in normal case

- to solve the matrix differential eqution

$$W' = M(x)W, \qquad \text{with } M(x) = \frac{M_0}{x} + xM_1 + x^2 M_2 + \dots$$

- use the Ansatz

$$W(x) = x^{M_0}(U_0 + xU_1 + x^2 U_2 + \dots), \qquad \text{with } U_0 = I$$

- then we find U_n from the reccursive procedure

$$U_n M_0 - M_0 U_n + nU_n = \sum_{k=1}^{n} M_k U_{n-k}$$

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solution in the above form exists if

- M_0 has no eigenvalues that differ by an integer

general theory in normal case

- to solve the matrix differential eqution

$$W' = M(x)W, \qquad \text{with } M(x) = \frac{M_0}{x} + xM_1 + x^2 M_2 + \ldots$$

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$$W(x) = x^{M_0}(U_0 + xU_1 + x^2 U_2 + \ldots), \qquad \text{with } U_0 = I$$

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solution in the above form exists if

- M_0 has no eigenvalues that differ by an integer

in our case all eigenvalues are equal to zero!

- *Mathematica* package: 1000 decimals in few seconds

conclusions

- we have evaluated analytically the 3-loop effective potential in the scalar sector of a general theory
- in order to get the ϵ-form of the differential equation system, the change of variable is necessary $\frac{m_1}{m_2} = \frac{y^2}{(1+y^2)^2}$
- the result is expressible in terms of cyclotomic polylogarithms of weight 4 and with bottom indices 1,2,3,4,6,8,12
- the latter are evaluated with high precision using generalized power series solution of the DE system
- we tested whether the result at high energy limit (where we can set $m_1^2 = 3m_2^2$) is expressible in terms of sixth root of unity basis $\longrightarrow$ no

Thank You