The ρ parameter at three loops and elliptic integrals

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Motivation

Many problems in perturbative QFT can be (and have been) solved in terms of iterated integrals.

Different kinds of alphabets have arised:

Harmonic polylogarithms:

$$\int_0^x \frac{dy}{y} \int_0^y \frac{dz}{1+z}$$

Goncharov polylogarithms:

$$\int_0^x \frac{dy}{y} \int_0^y \frac{dz}{z-3}$$

Cyclotomic polylogarithms:

$$\int_0^x \frac{dy}{1+y^2} \int_0^y \frac{dz}{1-z+z^2}$$

Root-valued iterated integrals:

$$\int_0^x \frac{dy}{y} \int_0^y \frac{dz}{z\sqrt{1+z}}$$

They all stem from first order factorizable differential equations.

Many interesting problems require the solution of differential equations that are not first order factorizable. The next level of complexity consists therefore of differential equations that factorize to firts order, except for one unfactorizable term of second order that may contain more than three singular points.

The 3-loop 2-mass corrections to the ρ parameter is such a problem. Its study may shed light on the solution of more complicated problems.

The ρ parameter

At tree level:

$$\rho = \frac{M_W^2}{M_Z^2 \cos^2(\theta_W)} = 1$$

Corrections:

$$\rho = 1 + \Delta\rho$$

$$\Delta\rho = \frac{\Sigma_Z(0)}{M_Z^2} - \frac{\Sigma_W(0)}{M_W^2}$$

Here $\Sigma_Z(0)$ and $\Sigma_W(0)$ are the transverse parts of the Z and W boson propagators, respectively, which are defined by

$$\Sigma_{W/Z}(0) = \frac{g_{\mu\nu}}{d} \Pi_{W/Z}$$

where $\Pi_{W/Z}$ are the corresponding polarization functions.

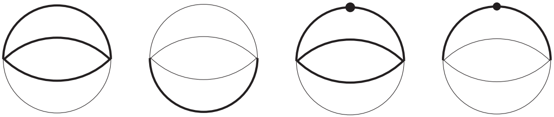
$$\Delta\rho = \frac{3 G_F m_t^2}{8\pi^2\sqrt{2}} \left(\delta^{(0)} + \frac{\alpha_s}{\pi}\delta^{(1)} + \left(\frac{\alpha_s}{\pi}\right)^2 \delta^{(2)} + \mathcal{O}(\alpha_s^3) \right)$$

Most recent calculations of corrections to the ρ parameter:

- L. Avdeev, J. Fleischer, S. Mikhailov and O. Tarasov, "*$\mathcal{O}(\alpha\alpha_s^2)$ correction to the electroweak ρ parameter*," Phys. Lett. B **336** (1994) 560 Erratum: [Phys. Lett. B **349** (1995) 597] [hep-ph/9406363].
- K. G. Chetyrkin, J. H. Kühn and M. Steinhauser, "*Corrections of order $\mathcal{O}(G_F M_t^2 \alpha_s^2)$ to the ρ parameter*," Phys. Lett. B **351** (1995) 331 [hep-ph/9502291].
- R. Boughezal, J. B. Tausk and J. J. van der Bij, "*Three-loop electroweak corrections to the W-boson mass and $\sin^2\theta_{\mathrm{eff}}^{\mathrm{lept}}$ in the large Higgs mass limit*," Nucl. Phys. B **725** (2005) 3 [hep-ph/0504092].
- K. G. Chetyrkin, M. Faisst, J. H. Kühn, P. Maierhofer and C. Sturm, "*Four-Loop QCD Corrections to the ρ Parameter*," Phys. Rev. Lett. **97** (2006) 102003 [hep-ph/0605201].
- R. Boughezal and M. Czakon, "*Single scale tadpoles and $\mathcal{O}(G_F m_t^2 \alpha_s^3)$ corrections to the ρ parameter*," Nucl. Phys. B **755** (2006) 221 [hep-ph/0606232].
- J. Grigo, J. Hoff, P. Marquard and M. Steinhauser, "*Moments of heavy quark correlators with two masses: exact mass dependence to three loops*," Nucl. Phys. B **864** (2012) 580 [arXiv:1206.3418 [hep-ph]].

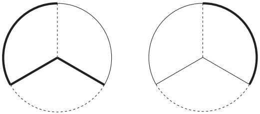
Most master integrals can be obtained entirely in terms of harmonic polylogarithms (the corresponding differential equations factorize to first order).

The are four master integrals, however, for which the differential equation associated to the $\mathcal{O}(\varepsilon^0)$ term is of second order and non-factorizable:



$J_{8a}^{(3)}$ $J_{8b}^{(3)}$ $J_{9a}^{(3)}$ $J_{9b}^{(3)}$

In the case of two other master integrals, the associated first order differential equations involves the integrals shown above in the inhomogeneous part:



$J_{10a}^{(3)}$ $J_{10b}^{(3)}$

$$
\begin{aligned}
J_{8a}^{(3)}(x) &= \mathcal{N}^3 m_1^4 \Bigg[\frac{1}{\varepsilon^3}(1+x^2) + \frac{1}{\varepsilon^2}\left(\frac{15}{4} + 4x^2 - \frac{x^4}{12} - 3x^2 \ln(x) \right) \\
&\quad + \frac{1}{\varepsilon}\left(\frac{65}{8} + 10x^2 - \frac{5}{8}x^4 + \frac{1}{2}(x^4 - 24x^2)\ln(x) + 3x^2 \ln^2(x) \right) \\
&\quad + f_{8a}(x) \Bigg],
\end{aligned}
$$

$$
\begin{aligned}
J_{8b}^{(3)}(x) &= \mathcal{N}^3 m_1^4 \Bigg[\frac{1}{\varepsilon^3}(x^2+x^4) + \frac{1}{\varepsilon^2}\Bigg(-\frac{1}{12} + 4x^2 + \frac{15}{4}x^4 - 3x^2 \ln(x) \\
&\quad - 6x^4 \ln(x) \Bigg) + \frac{1}{\varepsilon}\Bigg(\frac{65}{8}x^4 + 10x^2 - \frac{5}{8} - \frac{1}{2}(24x^2 + 45x^4)\ln(x) \\
&\quad + (3x^2 + 18x^4)\ln^2(x) \Bigg) + f_{8b}(x) \Bigg]
\end{aligned}
$$

and similar for $J_{8a}^{(3)}$ and $J_{8b}^{(3)}$. Here x is the ratio of the masses $(0 < x < 1)$ and $\mathcal{N} = \left(\frac{\mu^2}{m_1^2}\right)^{\varepsilon} i\pi^{D/2}\Gamma(3 - D/2)$ with $D = 4 - 2\varepsilon$.

The $\mathcal{O}(\varepsilon^0)$ terms satisfy the follwing systems of differential equations:

$$\frac{d}{dx}\begin{pmatrix} f_{8a}(x) \\ f_{9a}(x) \end{pmatrix} = \begin{pmatrix} \frac{4}{x} & \frac{6}{x} \\ \frac{4(x^2-3)}{x(x^2-9)(x^2-1)} & \frac{2(x^4-9)}{x(x^2-9)(x^2-1)} \end{pmatrix} \otimes \begin{pmatrix} f_{8a}(x) \\ f_{9a}(x) \end{pmatrix} + \begin{pmatrix} R_{8a}(x) \\ R_{9a}(x) \end{pmatrix}$$

and

$$\frac{d}{dx}\begin{pmatrix} f_{8b}(x) \\ f_{9b}(x) \end{pmatrix} = \begin{pmatrix} \frac{4}{x} & \frac{2}{x} \\ \frac{4(3x^2-1)}{x(9x^2-1)(x^2-1)} & \frac{2(9x^4-1)}{x(9x^2-1)(x^2-1)} \end{pmatrix} \otimes \begin{pmatrix} f_{8b}(x) \\ f_{9b}(x) \end{pmatrix} + \begin{pmatrix} R_{8b}(x) \\ R_{9b}(x) \end{pmatrix},$$

The inhomogeneous parts, $R_{8a}(x)$, $R_{9a}(x)$, $R_{8b}(x)$ and $R_{9b}(x)$, are given in terms of rational functions and powers of $\ln(x)$. For example,

$$\begin{aligned} R_{8a}(x) &= \frac{15(-13-16x^2+x^4)}{4x} - 3x(-24+x^2)\ln(x) - 18x\ln^2(x) \\ R_{9a}(x) &= \frac{1755+1863x^2-1255x^4+157x^6}{12x(x^2-9)(x^2-1)} - \frac{x(324-145x^2+15x^4)}{(x^2-9)(x^2-1)}\ln(x) \\ &\quad + \frac{2x(45-17x^2+2x^4)}{(x^2-9)(x^2-1)}\ln^2(x) - \frac{16x^3}{3(x^2-9)(x^2-1)}\ln^3(x). \end{aligned}$$

and similar for $R_{8b}(x)$ and $R_{9b}(x)$.

In the case of the $\mathcal{O}(\varepsilon^0)$ terms of the master integrals $J_{10a}^{(3)}$ and $J_{10b}^{(3)}$, we have

$$\frac{d}{dx} f_{10a}(x) \quad = \quad K_{10a}(x) + \frac{4}{(x^2-1)^2 x} f_{8a}(x) + \frac{2(x^2+3)}{(x^2-1)^2 x} f_{9a}(x),$$

$$\frac{d}{dx} f_{10b}(x) \quad = \quad K_{10b}(x) + \frac{4}{3(x^2-1)^2 x^3} f_{8b}(x) + \frac{2(3x^2+1)}{3(x^2-1)^2 x^3} f_{9b}(x)$$

where $K_{10a}(x)$ and $K_{10b}(x)$ are functions that can be expressed in terms of rational functions and harmonic polylogarithms, defined by

$$H_{b,\vec{a}}(x) \quad = \quad \int_0^x dy f_b(y) H_{\vec{a}}(y); \quad f_b(x) \in \{f_0, f_1, f_{-1}\} \equiv \left\{ \frac{1}{x}, \frac{1}{1-x}, \frac{1}{1+x} \right\};$$

$$H_{\underbrace{0,\,\ldots,\,0}_{k}}(x) = \frac{1}{k!} \ln^k(x); \; H_{\emptyset}(x) \equiv 1 \; .$$

After deocupling the differential equations, we obtain second order differential equations for f_{8a} and f_{8b}:

$$\frac{d^2}{dx^2} f_{8a}(x) + \frac{9 - 30x^2 + 5x^4}{x(x^2-1)(9-x^2)} \frac{d}{dx} f_{8a}(x) - \frac{8(-3+x^2)}{(9-x^2)(x^2-1)} f_{8a}(x) = N_{8a}(x)$$

$$\frac{d^2}{dx^2} f_{8b}(x) - \frac{1 - 30x^2 + 45x^4}{x(9x^2-1)(x^2-1)} \frac{d}{dx} f_{8b}(x) + \frac{24(3x^2-1)}{(9x^2-1)(x^2-1)} f_{8b}(x) = N_{8b}(x)$$

where the inhomogeneous parts $N_{8a}(x)$ and $N_{8b}(x)$ are given in terms of rational functions and powers of logarithms.

Notice that in these equations we have four singular points in the variable x^2.

The functions f_{9a} and f_{9b} can be obtained from f_{8a} and f_{8b} and

$$f_{9a}(x) = K_{9a}(x) + \frac{x}{6} \frac{d}{dx} f_{8a}(x), \qquad f_{9b}(x) = K_{9b}(x) + \frac{x}{2} \frac{d}{dx} f_{8b}(x)$$

where, again, $K_{9a}(x)$ and $K_{9b}(x)$ are given in terms of rational functions and powers of logarithms.

The hom. diff. eqs. can be solved in terms of ${}_2F_1$ functions at rational argument. There are several algorithms that can be used for this purpose.

Two of them were present by E. Imamoglu and M. van Hoeij in J. Symbolic Comput. **83** (2017) 245 [arXiv:1606.01576[cs.SC]].

A `Maple` implementation of these two algorithms can be found at https://www.math.fsu.edu/∼eimamogl/hypergeometricsols/

In the case of the differential equation

$$\frac{d^2}{dx^2} f_{8a}(x) + \frac{9 - 30x^2 + 5x^4}{x(x^2-1)(9-x^2)} \frac{d}{dx} f_{8a}(x) - \frac{8(-3+x^2)}{(9-x^2)(x^2-1)} f_{8a}(x) = N_{8a}(x),$$

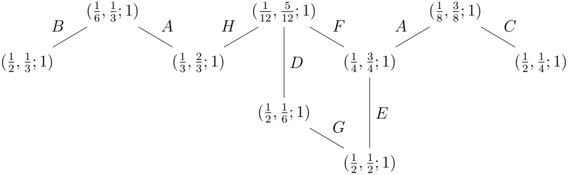
we get

$$\begin{aligned}
\psi^{(0)}_{1a}(x) &= \sqrt{2\sqrt{3}\pi \frac{x^2(x^2-1)^2(x^2-9)^2}{(x^2+3)^4}}\, {}_2F_1\left(\left. \begin{matrix} \frac{4}{3}, \frac{5}{3} \\ 2 \end{matrix} \right| z \right) \\
\psi^{(0)}_{2a}(x) &= \sqrt{2\sqrt{3}\pi \frac{x^2(x^2-1)^2(x^2-9)^2}{(x^2+3)^4}}\, {}_2F_1\left(\left. \begin{matrix} \frac{4}{3}, \frac{5}{3} \\ 2 \end{matrix} \right| 1-z \right),
\end{aligned}$$

with

$$z = z(x) = \frac{x^2(x^2-9)^2}{(x^2+3)^3} \ .$$

Equivalent solutions are found by applying contigous relations and relations due to triangle groups,



l	d	R	f
A	2	1	$4x(1-x)$
B	2	$(1-x)^{-1/6}$	$\frac{1}{4}x^2/(x-1)$
C	2	$(1-x)^{-1/8}$	$\frac{1}{4}x^2/(x-1)$
D	2	$(1-x)^{-1/12}$	$\frac{1}{4}x^2/(x-1)$
E	2	$(1-x/2)^{-1/2}$	$x^2/(x-2)^2$
F	3	$(1+3x)^{-1/4}$	$27x(1-x)^2/(1+3x)^3$
G	3	$(1+\omega x)^{-1/2}$	$1-(x+\omega)^3/(x+\overline{\omega})^3$
H	4	$(1-8x/9)^{-1/4}$	$64x^3(1-x)/(9-8x)^3$

$$ {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix} \middle| \, x\right) = R(x) \cdot {}_2F_1\left(\begin{matrix} a', b' \\ c' \end{matrix} \middle| \, f(x)\right) $$

This will allow us, whenever possible, to go from ${}_2F_1$'s to elliptic integrals.

We then get the following solutions for the homogeneous differential equation of f_{8a}.

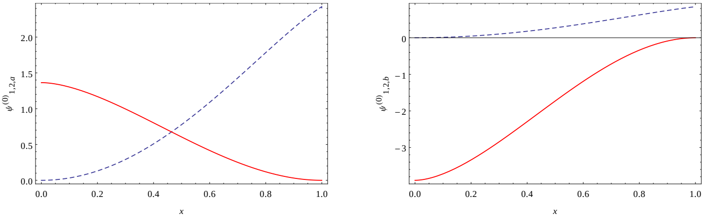
$$\psi_{1b}^{(0)}(x) = \frac{\sqrt{\pi}}{4\sqrt{6}}\sqrt{\frac{x+1}{9-3x}}\left\{-(x-1)(x-3)(x+3)^2 {}_2F_1\left(\begin{matrix}\frac{1}{2},\frac{1}{2}\\1\end{matrix}\,\middle|\, z\right)\right.$$
$$\left.+(x^2+3)(x-3)^2 {}_2F_1\left(\begin{matrix}\frac{1}{2},-\frac{1}{2}\\1\end{matrix}\,\middle|\, z\right)\right\}$$
$$\psi_{2b}^{(0)}(x) = \frac{2\sqrt{\pi}}{\sqrt{6}}\sqrt{(x+1)(9-3x)}\left\{x^2 {}_2F_1\left(\begin{matrix}\frac{1}{2},\frac{1}{2}\\1\end{matrix}\,\middle|\, 1-z\right)\right.$$
$$\left.+\frac{1}{8}(x-3)(x^2+3) {}_2F_1\left(\begin{matrix}\frac{1}{2},-\frac{1}{2}\\1\end{matrix}\,\middle|\, 1-z\right)\right\},$$

where

$$z(x) = -\frac{16x^3}{(x+1)(x-3)^3} \; .$$

The ratios of the homogeneous solutions are given by

$$\frac{\psi_{1a}^{(0)}(x)}{\psi_{1b}^{(0)}(x)} = 3^{3/4}\sqrt{\frac{\pi}{2}}, \qquad \frac{\psi_{2a}^{(0)}(x)}{\psi_{2b}^{(0)}(x)} = -\frac{1}{3^{3/4}}\sqrt{\frac{2}{\pi}} \; .$$



The hypergeometric functions appearing in $\psi^{(0)}_{1b}(x)$ and $\psi^{(0)}_{2b}(x)$ are given in terms of complete elliptic integrals

$$
{}_2F_1\left(\left.\begin{matrix}\frac{1}{2}, \frac{1}{2} \\ 1\end{matrix}\,\right|\, z\right) = \frac{2}{\pi}\mathbf{K}(z), \qquad {}_2F_1\left(\left.\begin{matrix}\frac{1}{2}, -\frac{1}{2} \\ 1\end{matrix}\,\right|\, z\right) = \frac{2}{\pi}\mathbf{E}(z) \; .
$$

Their integral representations in Legendre's normal form read

$$
\mathbf{K}(z) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-zt^2)}}, \qquad \mathbf{E}(z) = \int_0^1 dt \sqrt{\frac{1-zt^2}{1-t^2}} \; .
$$

In a similar way, we can find the homogeneous solutions of the differential equation for f_{8b}:

$$\frac{d^2}{dx^2}f_{8b}(x) - \frac{1-30x^2+45x^4}{x(9x^2-1)(x^2-1)}\frac{d}{dx}f_{8b}(x) + \frac{24(3x^2-1)}{(9x^2-1)(x^2-1)}f_{8b}(x) = N_{8b}(x)$$

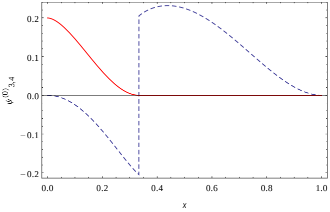
namely,

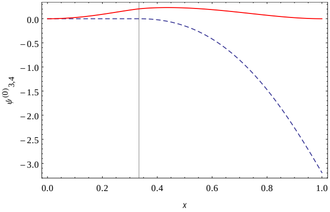
$$\psi_3^{(0)}(x) = -\frac{\sqrt{1-3x}\sqrt{x+1}}{2\sqrt{2\pi}}\left[(x+1)\left(3x^2+1\right)\mathbf{E}(z) - (x-1)^2(3x+1)\mathbf{K}(z)\right]$$

$$\psi_4^{(0)}(x) = -\frac{\sqrt{1-3x}\sqrt{x+1}}{2\sqrt{2\pi}}\left[8x^2\mathbf{K}(1-z) - (x+1)\left(3x^2+1\right)\mathbf{E}(1-z)\right],$$

with

$$z = \frac{16x^3}{(x+1)^3(3x-1)}$$





Inhomogeneous solutions

The corresponding Wronskians of the homogeneous solutions to the differential equations for f_{8a} and f_{8b} are

$$
\begin{aligned}
W_{8a}(x) &= x(9-x^2)(x^2-1) \quad \text{and} \\
W_{9a}(x) &= x(9x^2-1)(x^2-1),
\end{aligned}
$$

respectively, so the inhomogeneous solutions are

$$
f_{8a}(x) = \psi_{1b}^{(0)}(x)\left[C_1 - \int dx \psi_{2b}^{(0)}(x)\frac{N_{8a}(x)}{W_{8a}(x)}\right] + \psi_{2b}^{(0)}(x)\left[C_2 + \int dx \psi_{1b}^{(0)}(x)\frac{N_{8a}(x)}{W_{8a}(x)}\right],
$$

$$
f_{8b}(x) = \psi_{3}^{(0)}(x)\left[C_3 - \int dx \psi_{4}^{(0)}(x)\frac{N_{8b}(x)}{W_{8b}(x)}\right] + \psi_{4}^{(0)}(x)\left[C_4 + \int dx \psi_{3}^{(0)}(x)\frac{N_{8b}(x)}{W_{8b}(x)}\right]
$$

where C_1, C_2, C_3 and C_4 are determined from boundary conditions.

A New Class of Integrals in QFT:

$$\begin{aligned}\mathbb{H}_{a_1,\dots,a_{m-1};\{a_m;F_m(r(y_m))\},a_{m+1},\dots,a_q}(x) &= \int_0^x dy_1 f_{a_1}(y_1) \int_0^{y_1} dy_2 \dots \int_0^{y_{m-1}} dy_m f_{a_m}(y_m) \\ &\times F_m[r(y_m)] H_{a_{m+1},\dots,a_q}(y_{m+1}),\end{aligned}$$

Here the $f_{a_i}(y)$ are the usual letters considered in previous studies, while $F[r(y)]$ is a definite integral

$$F[r(y)] = \int_0^1 dz g(z, r(y)), \quad r(y) \in \mathbb{Q}[y],$$

where the y dependence cannot be completely transferred to one of the integration boundaries.

Specifically we have

$$\begin{aligned}F[r(y)] &= {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix} \middle| r(y)\right) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 dz\ z^b (1-z)^{c-b-1} (1-r(y)z)^{-a}, \\ & \qquad r(y) \in \mathbb{Q}[y], a, b, c \in \mathbb{Q}.\end{aligned}$$

Other functions are also possible.

Results in terms of definite integrals:

$$
\begin{aligned}
f_{8a}(x) \quad = \quad & \psi_{1b}(x)\left[C_1 - \int_0^x dy\left(\psi_{2b}(y)\frac{N_{8a}(y)}{W_{8a}(y)} - \sqrt{\frac{3}{2\pi}}\frac{3}{y}\left(43 - 24\ln(y) + 8\ln^2(y)\right)\right)\right. \\
& \left. -\frac{3}{4}\sqrt{\frac{3}{2\pi}}\left(43\ln(x) - 12\ln^2(x) + \frac{8}{3}\ln^3(x)\right)\right] \\
& +\psi_{2b}(x)\left[C_2 + \int_0^x dy\left(\psi_{1b}(y)\frac{N_{8a}(y)}{W_{8a}(y)}\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
C_1 \quad = \quad & \frac{1}{\sqrt{6\pi}}\left[-36\sqrt{3}\mathrm{Im}\left(\mathrm{Li}_3\left(\frac{e^{-\frac{i\pi}{6}}}{\sqrt{3}}\right)\right) - \frac{75}{8} - \frac{35\pi^3}{12\sqrt{3}} - \frac{3}{4}\sqrt{3}\pi\ln^2(3)\right. \\
& \left. + \frac{135\ln(3)}{4} - 4\pi^2\ln(3) + 6\ln(3)\psi^{(1)}\left(\frac{1}{3}\right)\right] \\
C_2 \quad = \quad & \sqrt{\frac{\pi}{6}}\left[-\frac{15}{4} + \frac{4\pi^2}{9} - \frac{2\psi^{(1)}\left(\frac{1}{3}\right)}{3}\right]
\end{aligned}
$$

Series Solutions

$$
\begin{aligned}
f_{8a}(x) \quad = \quad & -\sqrt{3}\Bigg[\pi^3\left(\frac{35x^2}{108} - \frac{35x^4}{486} - \frac{35x^6}{4374} - \frac{35x^8}{13122} - \frac{70x^{10}}{59049} - \frac{665x^{12}}{1062882}\right) + \Bigg(12x^2 - \frac{8x^4}{3} \\
& - \frac{8x^6}{27} - \frac{8x^8}{81} - \frac{32x^{10}}{729} - \frac{152x^{12}}{6561}\Bigg) \mathrm{Im}\left[\mathrm{Li}_3\left(\frac{e^{-\frac{i\pi}{6}}}{\sqrt{3}}\right)\right]\Bigg] - \pi^2\Bigg(1 + \frac{x^4}{9} - \frac{4x^6}{243} - \frac{46x^8}{6561} \\
& - \frac{214x^{10}}{59049} - \frac{5546x^{12}}{2657205}\Bigg) - \left(-\frac{3}{2} - \frac{x^4}{6} + \frac{2x^6}{81} + \frac{23x^8}{2187} + \frac{107x^{10}}{19683} + \frac{2773x^{12}}{885735}\right)\psi^{(1)}\left(\frac{1}{3}\right) \\
& -\sqrt{3}\pi\left(\frac{x^2}{4} - \frac{x^4}{18} - \frac{x^6}{162} - \frac{x^8}{486} - \frac{2x^{10}}{2187} - \frac{19x^{12}}{39366}\right)\ln^2(3) - \Bigg[33x^2 - \frac{5x^4}{4} - \frac{11x^6}{54} \\
& - \frac{19x^8}{324} - \frac{751x^{10}}{29160} - \frac{2227x^{12}}{164025} + \pi^2\left(\frac{4x^2}{3} - \frac{8x^4}{27} - \frac{8x^6}{243} - \frac{8x^8}{729} - \frac{32x^{10}}{6561} - \frac{152x^{12}}{59049}\right) \\
& + \left(-2x^2 + \frac{4x^4}{9} + \frac{4x^6}{81} + \frac{4x^8}{243} + \frac{16x^{10}}{2187} + \frac{76x^{12}}{19683}\right)\psi^{(1)}\left(\frac{1}{3}\right)\Bigg]\ln(x) + \frac{135}{16} + 19x^2 \\
& - \frac{43x^4}{48} - \frac{89x^6}{324} - \frac{1493x^8}{23328} - \frac{132503x^{10}}{5248800} - \frac{2924131x^{12}}{236196000} - \left(\frac{x^4}{2} - 12x^2\right)\ln^2(x) \\
& -2x^2\ln^3(x) + O\left(x^{14}\ln(x)\right).
\end{aligned}
$$

The solution can be easily extended to accuracies of $O(10^{-30})$ using Mathematica or Maple.

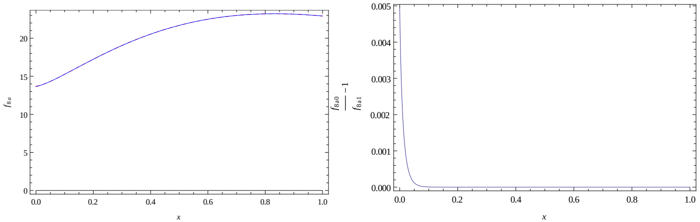


Figure: f_{8a} as a function of x. Left panel: Red dashed line: expansion around $x = 0$; blue line: expansion around $x = 1$. Right panel: illustration of the relative accuracy and overlap of the two solutions $f_{8a}(x)$ around 0 and 1.

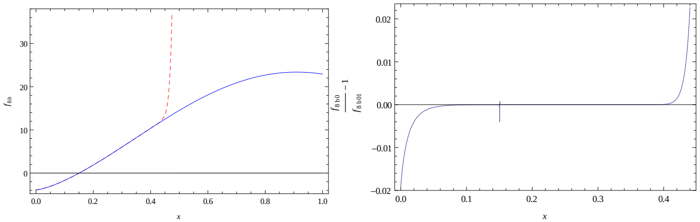
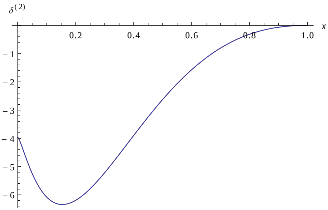


Figure: f_{8b} as a function of x. Left panel: Red dashed line: expansion around $x = 0$; blue line: expansion around $x = 1$. Right panel: illustration of the relative accuracy and overlap of the two solutions $f_{8b}(x)$ around 0 and 1.

$$\Delta\rho = \frac{3G_F m_t^2}{8\pi^2\sqrt{2}}\left(\delta^{(0)}(x) + \frac{\alpha_s}{\pi}\delta^{(1)}(x) + \left(\frac{\alpha_s}{\pi}\right)^2 \delta^{(2)}(x) + \mathcal{O}(\alpha_s^3)\right)$$

$$\begin{aligned}
\delta^{(2)}(x) \quad = \quad & \cdots + C_F\left(C_F - \frac{C_A}{2}\right)\left[\frac{11-x^2}{12(1-x^2)^2}f_{8a}(x) + \frac{9-x^2}{3(1-x^2)^2}f_{9a}(x) + \frac{1}{12}f_{10a}(x)\right. \\
& \left. + \frac{5-39x^2}{36(1-x^2)^2}f_{8b}(x) + \frac{1-9x^2}{9(1-x^2)^2}f_{9b}(x) + \frac{x^2}{12}f_{10b}(x)\right] \\
& + \frac{C_F T_F}{9(1-x^2)^3}\left[(5x^4 - 28x^2 - 9)f_{8a}(x) + \frac{1-3x^2}{3x^2}(9x^4 + 9x^2 - 2)f_{8b}(x)\right. \\
& \left. + (9-x^2)(x^4 - 6x^2 - 3)f_{9a}(x) + \frac{1-9x^2}{3x^2}(3x^4 + 6x^2 - 1)f_{9b}(x)\right]
\end{aligned}$$



For $x = 0$, this agrees with the result by Chetyrkin *et al* (and Avdeev *et al*), $\delta^{(2)}(0) = -3.9696$

η ratios and q-series representations

The appearance of elliptic solutions calls for the study of their representation in terms of related functions such as the Dedekind η function,

$$\eta(\tau) = q^{\frac{1}{12}} \prod_{k=1}^{\infty} \left(1 - q^{2k}\right),$$

the Jacobi ϑ_i functions

$$\vartheta_2(q) = \frac{2\eta^2(2\tau)}{\eta(\tau)} \quad \vartheta_3(q) = \frac{\eta^5(\tau)}{\eta^2\left(\frac{1}{2}\tau\right)\eta^2(2\tau)}, \quad \vartheta_4(q) = \frac{\eta^2\left(\frac{1}{2}\tau\right)}{\eta(\tau)}$$

with $q = \exp(i\pi\tau)$, as well as the corresponding q-series,

Applying a higher order Legendre-Jacobi transformation, one may transform the variable x in $\mathbf{K}(k^2) \equiv \mathbf{K}(r(x))$ into the nome q analytically by

$$k^2 = r(x) = \frac{\vartheta_2^4(q)}{\vartheta_3^4(q)}.$$

The integrands can then be written in terms of products of meromorphic modular forms, which can be expressed in terms of linear combinations of ratios of η functions.

One may express the elliptic integral of the first kind $\mathbf{K}$ appearing in the homogeneous solutions by

$$\mathbf{K}(k^2) = \frac{\pi\eta^{10}(\tau)}{2\eta^4\left(\frac{\tau}{2}\right)\eta^4(2\tau)}$$

and

$$\mathbf{E}(k^2) = \mathbf{K}(k^2) + \frac{\pi^2 q}{\mathbf{K}(k^2)}\frac{d}{dq}\ln\left(\vartheta_4(q)\right)$$

Other terms appearing, e.g., in $\psi_3(x)$ and $\psi_4(x)$, such as $\sqrt{(1-3x)(1+x)}$ can also be expressed in terms of η ratios:

$$\sqrt{(1-3x)(1+x)} = \frac{i}{\sqrt{3}}\left.\frac{\eta\left(\frac{\tau}{2}\right)\eta\left(\frac{3\tau}{2}\right)\eta(2\tau)\eta(3\tau)}{\eta(\tau)\eta^3(6\tau)}\right|_{q\to -q}.$$

All other ingredients in the homogeneous solutions can be expressed in similar ways.

In the case of one of the homogeneous solutions to the differential equation of $f_{8a}(x)$, namely

$$\psi_{1b}(x) = \frac{2}{\sqrt{3}} H(x)$$

with

$$H(x) = \frac{x^2(x^2-1)^2(x^2-9)^2}{(x^2+3)^4} \, {}_2F_1\left(\begin{matrix} \frac{4}{3}, \frac{5}{3} \\ 2 \end{matrix} \middle| \frac{x^2(x^2-9)^2}{(x^2+3)^3} \right)$$

Setting the kinematic variable

$$x = 3 \frac{\eta_1^2 \eta_6^4}{\eta_2^4 \eta_3^2},$$

Broadhurst has found the following modular representation:

$$\begin{aligned} H\left(3 \frac{\eta_1^2 \eta_6^4}{\eta_2^4 \eta_3^2} \right) &= \frac{1}{2} \left[\frac{\eta_1^{14} \eta_6^{10}}{\eta_2^{22} \eta_3^2} + \frac{\eta_1^6 \eta_6^4}{\eta_2^{12} \eta_3^2} \left(\frac{\eta_1^4 \eta_6^8}{\eta_2^8 \eta_3^4} + \frac{1}{3} \right) q \frac{d}{dq} \right] \frac{\eta_2 \eta_3}{\eta_1^3 \eta_6^2} \\ &= q - 6q^2 + 24q^3 - 74q^4 + 195q^5 - 474q^6 + 1100q^7 + \mathcal{O}(q^8) \end{aligned}$$

where $\eta_k = \eta(k\tau)$.

The inhomogeneities can be dealt with in a similar way. E.g., in the case of $f_{8b}(x)$, the inhomogeneous solution is of the form

$$I = \sum_{m=1}^{8} c_m \int \frac{dx}{x} H_0^n(x) \hat{f}_m(x) \psi_{3,4}(x), \quad n \in \{0,1,2,3\}, \quad c_m \in \mathbb{Q}$$

with

$$\hat{f}_m \in \left\{ \frac{1}{1 \pm x}, \frac{1}{(1 \pm x)^2}, \frac{1}{1 \pm 3x}, \frac{1}{(1 \pm 3x)^2} \right\}.$$

One obtains the following η ratios

$$\begin{aligned} \frac{1}{1-x} &= -3 \frac{\eta^2(\tau)\eta\left(\frac{3}{2}\tau\right)\eta^3(6\tau)}{\eta^3\left(\frac{1}{2}\tau\right)\eta(2\tau)\eta^2(3\tau)} \\ \frac{1}{1-3x} &= -\frac{\left[\eta(\tau)\eta\left(\frac{3}{2}\tau\right)\eta^2(6\tau)\right]^3}{\eta\left(\frac{1}{2}\tau\right)\eta^2(2\tau)\eta^9(3\tau)}, \end{aligned}$$

Summary

- We calculated 3-loop 2-mass corrections to the ρ parameter analytically. Here differential equations appear which only factorize to second order.
- The result can be given in terms of ${}_2F_1$-solutions, which are finally represented by the complete elliptic integrals of the first and second kind, K and E, at rational arguments.
- The corresponding closed form representation of the new terms can be given in terms of iterative non-iterative integrals. Series representations are obtained yielding precise overlapping representations in the physical region $x \in [0, 1]$.
- The elliptic integral of the second kind cannot be transformed away.
- Also q-series representations of the homogeneous solutions can be given in terms of Dedekind η-ratios. They are particularly compact, if we allow for differential operators as well. They are meromorphic modular forms.
- The q-integral over these structures is a higher transcendental function and not given completely in terms of elliptic polylogarithms.