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Computer algebra tools for Feynman integrals and related multi-sums

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Symbolic integration

- ▶ method of hyperlogarithms (F. Brown)
- ▶ multivariate [Almkvist-Zeilberger algorithm](#)

Multivariate Almkvist-Zeilberger algorithm (MultiIntegrate.m)

GIVEN an integral in the form

$$F(\varepsilon, N) = \int_0^1 dx_1 \cdots \int_0^1 dx_s f(\varepsilon, N, x_1, \dots, x_s)$$

where f is hyperexponential in x_i for $i = 1, 2, \dots, s$, i.e.,

$$\frac{D_{x_i} f(\dots, x_i, \dots)}{f(\dots, x_i, \dots)} \in \mathbb{K}(\varepsilon, N, x_1, \dots, x_s).$$

\mathbb{K} is a field containing the rational numbers.

M. Apagodu and D. Zeilberger, Adv. Appl. Math. (Special Regev Issue), **37** (2006) 139;

J. Ablinger, Ph.D. Thesis, J. Kepler University Linz, 2012, arXiv:1305.0687;

J. Ablinger, A. Behring, J. Blümlein, A. De Freitas, A. von Manteuffel, CS. Comput. Phys. Comm. 202(2016), arXiv:1509.08324

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where f is hyperexponential in x_i for $i = 1, 2, \dots, s$, i.e.,

$$\frac{D_{x_i} f(\dots, x_i, \dots)}{f(\dots, x_i, \dots)} \in \mathbb{K}(\varepsilon, N, x_1, \dots, x_s).$$

Compute a **linear recurrence** of the form

$$a_0(\varepsilon, N)F(\varepsilon, N) + a_1(\varepsilon, N)F(\varepsilon, N+1) + \cdots + a_d(\varepsilon, N)F(\varepsilon, N+d) = 0$$

with polynomials $a_i(\varepsilon, N) \in \mathbb{K}[\varepsilon, N]$

\mathbb{K} is a field containing the rational numbers.

Example: A master integral from Ladder and V -topologies

[arXiv:1509.08324]

$$F(\varepsilon, N) = \int_0^1 dx \int_0^1 dy \int_0^1 dz x^{\varepsilon/2} y^{\varepsilon/2} (1-z)^{-\frac{3\varepsilon}{2}-2} z^{\frac{\varepsilon}{2}+N+1} \\ (1-xz)^{\varepsilon/2} \times (1-yz)^{\varepsilon/2} (x+y-1)^N$$

Ablinger's
MultIntegrate.m \downarrow (9 hours)

$$a_0(\varepsilon, N)F(\varepsilon, N) + a_1(\varepsilon, N)F(\varepsilon, N+1) + \cdots + a_5(\varepsilon, N)F(\varepsilon, N+5) = 0$$

$$\begin{aligned} a_0(N, \varepsilon) = & (N + 1)(N + 2)(8\varepsilon^{10} + 104\varepsilon^9(N + 3) + 4\varepsilon^8(96N^2 + 601N + 887) \\ & + 4\varepsilon^7(12N^3 + 414N^2 + 1583N + 1393) \\ & - 8\varepsilon^6(264N^4 + 2436N^3 + 8643N^2 + 14518N + 9947) \\ & - 16\varepsilon^5(156N^5 + 1690N^4 + 6847N^3 + 12661N^2 + 9537N + 717) \\ & + 32\varepsilon^4(68N^6 + 1158N^5 + 8155N^4 + 30114N^3 + 61712N^2 + 67616N + 31693) \\ & + 64\varepsilon^3(40N^7 + 560N^6 + 2755N^5 + 3729N^4 - 14194N^3 - 61920N^2 - 89140N - 46600) \\ & - 128\varepsilon^2(N + 2)(12N^7 + 254N^6 + 2249N^5 + 10758N^4 + 30173N^3 + 50610N^2 \\ & + 49122N + 22706) \\ & + 256\varepsilon(N + 2)^2(N + 3)(N + 4)(44N^4 + 501N^3 + 2044N^2 + 3455N + 1976) \\ & - 512(N + 1)(N + 2)^3(N + 3)^2(N + 4)(6N^2 + 47N + 95)), \end{aligned}$$

$$\begin{aligned}
a_1(N, \varepsilon) = & (N + 2)(-22\varepsilon^{11} - 2\varepsilon^{10}(157N + 435) - \varepsilon^9(1500N^2 + 8611N + 11745) \\
& - \varepsilon^8(2548N^3 + 22936N^2 + 63597N + 54229) \\
& + 4\varepsilon^7(266N^4 + 1857N^3 + 6065N^2 + 14351N + 15987) \\
& + 8\varepsilon^6(994N^5 + 12961N^4 + 67246N^3 + 174692N^2 + 226821N + 116092) \\
& + 16\varepsilon^5(336N^6 + 5348N^5 + 33569N^4 + 104918N^3 + 165290N^2 + 108259N + 6100) \\
& - 16\varepsilon^4(404N^7 + 7578N^6 + 61778N^5 + 284762N^4 + 802660N^3 + 1382074N^2 \\
& + 1340455N + 560287) \\
& - 64\varepsilon^3(94N^8 + 1823N^7 + 14305N^6 + 55870N^5 + 96299N^4 - 37256N^3 \\
& - 447044N^2 - 704959N - 379338) \\
& + 128\varepsilon^2(N + 3)(30N^8 + 715N^7 + 7667N^6 + 48253N^5 + 194086N^4 + 507439N^3 \\
& + 835393N^2 + 785327N + 320382) \\
& - 256\varepsilon(N + 2)(N + 3)^2(107N^6 + 2070N^5 + 16342N^4 + 67226N^3 + 151557N^2 \\
& + 176932N + 83196) \\
& + 256(N + 2)^3(N + 3)^3(N + 4)(30N^3 + 331N^2 + 1193N + 1386)),
\end{aligned}$$

$$\begin{aligned}
a_2(N, \varepsilon) = & (12\varepsilon^{12} + 12\varepsilon^{11}(17N + 45) + 2\varepsilon^{10}(620N^2 + 3553N + 4795) \\
& + 2\varepsilon^9(1504N^3 + 14190N^2 + 41901N + 38907) \\
& + 4\varepsilon^8(172N^4 + 4983N^3 + 30942N^2 + 69119N + 50850) \\
& - 4\varepsilon^7(1996N^5 + 24056N^4 + 113313N^3 + 269119N^2 + 337198N + 185290) \\
& - 16\varepsilon^6(450N^6 + 8210N^5 + 59749N^4 + 227386N^3 + 486841N^2 + 563176N + 275664) \\
& + 16\varepsilon^5(340N^7 + 4314N^6 + 19137N^5 + 25532N^4 - 55105N^3 - 206516N^2 - 191528N \\
& - 23458) \\
& + 32\varepsilon^4(140N^8 + 2940N^7 + 26550N^6 + 139926N^5 + 493839N^4 + 1240186N^3 \\
& + 2161699N^2 + 2304248N + 1100084) \\
& + 64\varepsilon^3(4N^9 + 506N^8 + 8651N^7 + 63510N^6 + 236215N^5 + 395334N^4 - 105413N^3 \\
& - 1551017N^2 - 2362944N - 1217770) \\
& - 128\varepsilon^2(N + 3)(12N^9 + 314N^8 + 3782N^7 + 29105N^6 + 160727N^5 + 640273N^4 \\
& + 1750874N^3 + 3052505N^2 + 3017094N + 1276604) \\
& + 256\varepsilon(N + 2)(N + 3)^2(N + 4)(26N^6 + 825N^5 + 8967N^4 + 46529N^3 + 125411N^2 \\
& + 168628N + 88652) \\
& - 512(N + 1)(N + 2)^2(N + 3)^3(N + 4)^2(6N^3 + 98N^2 + 459N + 655)),
\end{aligned}$$

$$\begin{aligned}
a_3(N, \varepsilon) = & (-64\varepsilon^{12} - 8\varepsilon^{11}(113N + 298) - 8\varepsilon^{10}(519N^2 + 2948N + 3896) \\
& - 4\varepsilon^9(1444N^3 + 13839N^2 + 39746N + 34305) \\
& + 4\varepsilon^8(1948N^4 + 17868N^3 + 63837N^2 + 112966N + 84655) \\
& + 16\varepsilon^7(1456N^5 + 20460N^4 + 112365N^3 + 304963N^2 + 412258N + 221769) \\
& - 8\varepsilon^6(320N^6 + 2050N^5 + 4192N^4 + 27408N^3 + 174901N^2 + 411759N + 324872) \\
& - 16\varepsilon^5(1756N^7 + 33154N^6 + 265889N^5 + 1186719N^4 + 3218059N^3 + 5349388N^2 \\
& + 5071913N + 2113696) \\
& + 32\varepsilon^4(188N^8 + 4802N^7 + 59527N^6 + 439922N^5 + 2025336N^4 + 5813984N^3 \\
& + 10076450N^2 + 9621283N + 3878602) \\
& + 64\varepsilon^3(140N^9 + 2768N^8 + 22500N^7 + 99545N^6 + 287700N^5 + 723136N^4 \\
& + 1854572N^3 + 3714620N^2 + 4272517N + 2031600) \\
& - 128\varepsilon^2(24N^{10} + 830N^9 + 14362N^8 + 152630N^7 + 1053620N^6 + 4834279N^5 \\
& + 14824351N^4 + 29964399N^3 + 38244797N^2 + 27875896N + 8824032) \\
& + 256\varepsilon(N + 2)(N + 3)(N + 4)(118N^7 + 2639N^6 + 24247N^5 + 118311N^4 + 329565N^3 \\
& + 520306N^2 + 426076N + 136854) \\
& - 512(N + 1)(N + 2)^2(N + 3)^2(N + 4)^2(N + 5)(12N^3 + 97N^2 + 230N + 144)),
\end{aligned}$$

$$\begin{aligned}
a_4(N, \varepsilon) = & (64\varepsilon^{12} + 192\varepsilon^{11}(5N + 14) + 16\varepsilon^{10}(297N^2 + 1769N + 2451) \\
& + 16\varepsilon^9(453N^3 + 4462N^2 + 13094N + 11244) \\
& - 8\varepsilon^8(1084N^4 + 11117N^3 + 47258N^2 + 103981N + 94650) \\
& - 8\varepsilon^7(3304N^5 + 51138N^4 + 311957N^3 + 948722N^2 + 1440105N + 858544) \\
& + 16\varepsilon^6(420N^6 + 5507N^5 + 36275N^4 + 169650N^3 + 536911N^2 + 952507N + 694370) \\
& + 16\varepsilon^5(1828N^7 + 38868N^6 + 353301N^5 + 1801014N^4 + 5604391N^3 + 10664390N^2 \\
& + 11433064N + 5260048) \\
& - 32\varepsilon^4(316N^8 + 8356N^7 + 105800N^6 + 802421N^5 + 3836854N^4 + 11588223N^3 \\
& + 21401558N^2 + 22066744N + 9745752) \\
& - 64\varepsilon^3(116N^9 + 2424N^8 + 19923N^7 + 82966N^6 + 208191N^5 + 530980N^4 + 1847484N^3 \\
& + 4687014N^2 + 6120858N + 3111104) \\
& + 128\varepsilon^2(24N^{10} + 826N^9 + 14897N^8 + 172000N^7 + 1314686N^6 + 6710299N^5 \\
& + 22873183N^4 + 51298261N^3 + 72551278N^2 + 58573022N + 20544948) \\
& - 256\varepsilon(N + 2)(N + 3)(106N^8 + 3278N^7 + 42903N^6 + 310942N^5 + 1366350N^4 \\
& + 3729418N^3 + 6173159N^2 + 5657732N + 2191212) \\
& + 512(N + 1)(N + 2)^2(N + 3)^2(N + 4)(N + 5)(N + 6)(12N^3 + 121N^2 + 396N + 431)),
\end{aligned}$$

$$\begin{aligned}
a_5(N, \varepsilon) = & (N + 5)(-128\varepsilon^{11} - 128\varepsilon^{10}(11N + 26) - 32\varepsilon^9(115N^2 + 592N + 647) \\
& + 32\varepsilon^8(63N^3 + 430N^2 + 1665N + 2384) \\
& + 16\varepsilon^7(714N^4 + 7881N^3 + 33802N^2 + 66225N + 47654) \\
& - 16\varepsilon^6(234N^5 + 2444N^4 + 13989N^3 + 50862N^2 + 104083N + 87848) \\
& - 16\varepsilon^5(580N^6 + 10181N^5 + 76586N^4 + 319207N^3 + 772120N^2 + 1012046N + 547832) \\
& + 16\varepsilon^4(244N^7 + 5456N^6 + 61605N^5 + 401216N^4 + 1536277N^3 + 3408574N^2 \\
& + 4066436N + 2026928) \\
& + 64\varepsilon^3(26N^8 + 357N^7 + 583N^6 - 11139N^5 - 65193N^4 - 120264N^3 + 11864N^2 \\
& + 272830N + 222624) \\
& - 64\varepsilon^2(N + 3)(12N^8 + 298N^7 + 4684N^6 + 49024N^5 + 306907N^4 + 1122441N^3 \\
& + 2350650N^2 + 2607576N + 1185072) \\
& + 256\varepsilon(N + 2)(N + 3)(25N^7 + 743N^6 + 8856N^5 + 55358N^4 + 197497N^3 + 404131N^2 \\
& + 439902N + 196128) \\
& - 256(N + 1)(N + 2)^2(N + 3)^2(N + 4)(N + 6)(N + 7)(6N^2 + 35N + 54).
\end{aligned}$$

Example: A master integral from Ladder and V -topologies

[arXiv:1509.08324]

$$F(\varepsilon, N) = \int_0^1 dx \int_0^1 dy \int_0^1 dz x^{\varepsilon/2} y^{\varepsilon/2} (1-z)^{-\frac{3\varepsilon}{2}-2} z^{\frac{\varepsilon}{2}+N+1} \\ (1-xz)^{\varepsilon/2} \times (1-yz)^{\varepsilon/2} (x+y-1)^N$$

Ablinger's
MultIntegrate.m \downarrow (9 hours)

$$a_0(\varepsilon, N)F(\varepsilon, N) + a_1(\varepsilon, N)F(\varepsilon, N+1) + \cdots + a_5(\varepsilon, N)F(\varepsilon, N+5) = 0$$

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$$a_0(\varepsilon, N)F(\varepsilon, N) + a_1(\varepsilon, N)F(\varepsilon, N+1) + \cdots + a_5(\varepsilon, N)F(\varepsilon, N+5) = 0$$

recurrence solver \downarrow

$F(\varepsilon, N) =$ expression in terms of special functions

Recurrence solving

A recurrence solver (Sigma.m)

GIVEN a recurrence

$a_0(N), \dots, a_d(N)$: polynomials in N

$h(N)$: expression in **indefinite nested sums**
defined over hypergeometric products.

$$a_0(N)F(N) + \dots + a_d(N)F(N + d) = h(N);$$

together with initial values $F(0), \dots, F(d - 1) \in \mathbb{K}$

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DECIDE constructively if $F(N)$ can be expressed in terms **indefinite nested sums** defined over hypergeometric products.

A recurrence solver (Sigma.m)

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DECIDE constructively if $F(N)$ can be expressed in terms **indefinite nested sums** defined over hypergeometric products.

Special cases of indefinite nested sums over hypergeometric products:

$$S_{2,1}(N) = \sum_{i=1}^n \frac{1}{i^2} \sum_{j=1}^i \frac{1}{j} \quad (\text{harmonic sums})$$

J. Blümlein and S. Kurth, Phys. Rev. D **60** (1999) 014018 [arXiv:hep-ph/9810241];

J.A.M. Vermaseren, Int. J. Mod. Phys. A **14** (1999) 2037 [arXiv:hep-ph/9806280].

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DECIDE constructively if $F(N)$ can be expressed in terms **indefinite nested sums** defined over hypergeometric products.

Special cases of indefinite nested sums over hypergeometric products:

$$\sum_{k=1}^n \frac{2^k}{k} \sum_{i=1}^k \frac{2^{-i}}{i} \sum_{j=1}^i \frac{S_1(j)}{j} \quad (\text{generalized harmonic sums})$$

S. Moch, P. Uwer and S. Weinzierl, J. Math. Phys. **43** (2002) 3363 [hep-ph/0110083];

J. Ablinger, J. Blümlein and C. Schneider, J. Math. Phys. **54** (2013) 082301 [arXiv:1302.0378].

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Special cases of indefinite nested sums over hypergeometric products:

$$\sum_{k=1}^N \frac{1}{(1+2k)^2} \sum_{j=1}^k \frac{1}{j^2} \sum_{i=1}^j \frac{1}{1+2i} \quad (\text{cyclotomic harmonic sums})$$

J. Ablinger, J. Blümlein and C. Schneider, J. Math. Phys. **52** (2011) 102301 [arXiv:1105.6063].

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DECIDE constructively if $F(N)$ can be expressed in terms **indefinite nested sums** defined over hypergeometric products.

Special cases of **indefinite nested sums** over hypergeometric products:

$$\sum_{j=1}^N \frac{4^j S_1(j-1)}{\binom{2j}{j} j^2} \quad (\text{binomial sums})$$

J. Ablinger, J. Blümlein, C. G. Raab and C. Schneider, J. Math. Phys. **55** (2014) 112301 [arXiv:1407.1822].

A recurrence solver (Sigma.m)

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DECIDE constructively if $F(N)$ can be expressed in terms **indefinite nested sums** defined over hypergeometric products.

Special cases of indefinite nested sums over hypergeometric products:

$$\sum_{h=1}^N 2^{-2h} (1 - \eta)^h \binom{2h}{h} \sum_{k=1}^h \frac{2^{2k}}{k^2 \binom{2k}{k}} \quad (\text{generalized binomial sums})$$

see talks of J. Ablinger and K. Schönwald

Sigma.m is based on difference ring/field theory

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Example: A master integral from Ladder and V -topologies

[arXiv:1509.08324]

$$F(\varepsilon, N) = \int_0^1 dx \int_0^1 dy \int_0^1 dz x^{\varepsilon/2} y^{\varepsilon/2} (1-z)^{-\frac{3\varepsilon}{2}-2} z^{\frac{\varepsilon}{2}+N+1} (1-xz)^{\varepsilon/2} \times (1-yz)^{\varepsilon/2} (x+y-1)^N$$

Ablinger's
MultIntegrate.m \downarrow (9 hours)

$$a_0(\varepsilon, N)F(\varepsilon, N) + a_1(\varepsilon, N)F(\varepsilon, N+1) + \cdots + a_5(\varepsilon, N)F(\varepsilon, N+5) = 0$$

recurrence solver \downarrow

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Sigma \downarrow

$$F(\varepsilon, N) = F_{-3}(N)\varepsilon^{-3} + F_{-2}(N)\varepsilon^{-2} + \dots$$

A refined recurrence solver (Sigma.m)

GIVEN a recurrence

$a_0(\varepsilon, N), \dots, a_d(\varepsilon, N)$: polynomials in ε, N
 $h_l(N), h_{l+1}(N), \dots, h_r(N)$:
expressions in indefinite nested sums
defined over hypergeometric products.

$$a_0(\varepsilon, N)F(\varepsilon, N) + \dots + a_d(\varepsilon, N)F(\varepsilon, N + d) \\ = h_l(N)\varepsilon^l + h_{l+1}(N)\varepsilon^{l+1} + \dots h_r(N)\varepsilon^r + O(\varepsilon^{r+1});$$

together with ε -expansions of $F(0), \dots, F(d-1)$ up to a certain order.

A refined recurrence solver (Sigma.m)

GIVEN a recurrence

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 $h_l(N), h_{l+1}(N), \dots, h_r(N)$:
 expressions in indefinite nested sums
 defined over hypergeometric products.

$$a_0(\varepsilon, N)F(\varepsilon, N) + \dots + a_d(\varepsilon, N)F(\varepsilon, N + d) \\
= h_l(N)\varepsilon^l + h_{l+1}(N)\varepsilon^{l+1} + \dots + h_r(N)\varepsilon^r + O(\varepsilon^{r+1});$$

together with ε -expansions of $F(0), \dots, F(d-1)$ up to a certain order.

DECIDE constructively if the coefficients $F_i(N)$ of

$$F(N) = F_l(N)\varepsilon^l + F_{l+1}(N)\varepsilon^{l+1} + \dots + F_r(N)\varepsilon^r + O(\varepsilon^{r+1})$$

can be given in terms of indefinite nested sums defined over hypergeometric products.

Blümlein, Klein, CS, Stan, J. Symbol. Comput. 2012; arXiv:1011.2656[cs.SC]

Ablinger, Blümlein, Round, CS, LL2012, arXiv:1210.1685 [cs.SC]

$$F(N) = \int_0^1 dx \int_0^1 dy \int_0^1 dz x^{\varepsilon/2} y^{\varepsilon/2} (1-z)^{-\frac{3\varepsilon}{2}-2} z^{\frac{\varepsilon}{2}+N+1} (1-xz)^{\varepsilon/2} \times (1-yz)^{\varepsilon/2} (x+y-1)^N$$

↓ (package MultiIntegrate.m)

$$a_0(\varepsilon, N)F(N) + a_1(\varepsilon, N)F(N+1) + \dots + a_5(\varepsilon, N)F(N+5) = 0$$

$$F(2) = \frac{20}{27\varepsilon^3} - \frac{40}{27\varepsilon^2} + \frac{1}{\varepsilon} \left(\frac{1393}{486} + \frac{5\zeta_2}{18} \right) + \dots$$

⋮

$$F(6) = \frac{22}{147\varepsilon^3} - \frac{535}{2058\varepsilon^2} + \frac{1}{\varepsilon} \left(\frac{630043}{1234800} + \frac{11\zeta_2}{196} \right) + \dots$$

$$F(N) = \int_0^1 dx \int_0^1 dy \int_0^1 dz x^{\varepsilon/2} y^{\varepsilon/2} (1-z)^{-\frac{3\varepsilon}{2}-2} z^{\frac{\varepsilon}{2}+N+1} (1-xz)^{\varepsilon/2} \times (1-yz)^{\varepsilon/2} (x+y-1)^N$$

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↓ (summation package Sigma.m)

$$F(N) = F_{-3}(N)\varepsilon^{-3} + F_{-2}(N)\varepsilon^{-2} + F_{-1}(N)\varepsilon^{-1} + \dots$$

We get

$$F_{-3}(N) = \frac{8(-1)^N}{3(N+1)(N+2)} + \frac{8(2N+3)}{3(N+1)^2(N+2)}$$

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$$F_{-2}(N) = -\frac{4(-1)^N(3N^3+18N^2+31N+18)}{3(N+1)^3(N+2)^2} - \frac{4(6N^3+32N^2+51N+26)}{3(N+1)^3(N+2)^2}$$

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$$\begin{aligned} F_{-1}(N) &= (-1)^N \left(\frac{2(9N^5 + 81N^4 + 295N^3 + 533N^2 + 500N + 204)}{3(N+1)^4(N+2)^3} + \frac{\zeta_2}{(N+1)(N+2)} \right) \\ &+ \frac{2(18N^5 + 150N^4 + 490N^3 + 755N^2 + 536N + 132)}{3(N+1)^4(N+2)^3} + \frac{(2N+3)\zeta_2}{(N+1)^2(N+2)} \\ &+ \left(-\frac{4}{(N+1)^2(N+2)} + \frac{4(-1)^N}{(N+1)(N+2)} \right) S_2 \\ &+ \left(\frac{4(-1)^N}{3(N+1)(N+2)} - \frac{4(N+9)}{3(N+1)^2(N+2)} \right) S_{-2} \end{aligned}$$

Example: A master integral from Ladder and V -topologies

[arXiv:1509.08324]

$$F(\varepsilon, N) = \int_0^1 dx \int_0^1 dy \int_0^1 dz x^{\varepsilon/2} y^{\varepsilon/2} (1-z)^{-\frac{3\varepsilon}{2}-2} z^{\frac{\varepsilon}{2}+N+1} (1-xz)^{\varepsilon/2} \times (1-yz)^{\varepsilon/2} (x+y-1)^N$$

Ablinger's
MultiIntegrate.m \downarrow (9 hours)

$$a_0(\varepsilon, N)F(\varepsilon, N) + a_1(\varepsilon, N)F(\varepsilon, N+1) + \dots + a_5(\varepsilon, N)F(\varepsilon, N+5) = 0$$

Sigma \downarrow (2 hours)

$$F(\varepsilon, N) = F_{-3}(N)\varepsilon^{-3} + F_{-2}(N)\varepsilon^{-2} + \dots + F_4(N)\varepsilon^4 + O(\varepsilon^5)$$

Find a recurrence for the integral/sum

$$F(N) = \int_0^1 \cdots \int_0^1 \Phi(\varepsilon, N, x_1, x_2, \dots, x_7) dx_1 dx_2 \dots dx_7$$
$$\stackrel{?}{=} F_{-3}(N)\varepsilon^{-3} + F_{-2}(N)\varepsilon^{-2} + F_{-1}(N)\varepsilon^{-1} + \dots$$

multivariate
Almkvist/Zeilberger
(J. Ablinger)

$$a_0(\varepsilon, N)F(N) + \dots + a_d(\varepsilon, N)F(N + d) = h(\varepsilon, N)$$

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$$F(N) = \int_0^1 \cdots \int_0^1 \Phi(\varepsilon, N, x_1, x_2, \dots, x_7) dx_1 dx_2 \cdots dx_7$$

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 ε -recurrence solver

multivariate
Almkvist/Zeilberger
(J. Ablinger)

$$a_0(\varepsilon, N)F(N) + \dots + a_d(\varepsilon, N)F(N + d) = h(\varepsilon, N)$$

Symbolic summation

Find a recurrence for the integral/sum

$$F(N) = \int_0^1 \cdots \int_0^1 \Phi(\varepsilon, N, x_1, x_2, \dots, x_7) dx_1 dx_2 \cdots dx_7$$

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 ε -recurrence solver

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Almkvist/Zeilberger
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 ε -recurrence solverMellin-Barnes-
and ${}_pF_q$ -technologiesmultivariate
Almkvist/Zeilberger
(J. Ablinger)

$$\sum_{i_1} \cdots \sum_{i_7} f(\varepsilon, N, i_1, i_2, \dots, i_7)$$

$$a_0(\varepsilon, N)F(N) + \dots + a_d(\varepsilon, N)F(N+d) = h(\varepsilon, N)$$

Find a recurrence for the integral/sum

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 ε -recurrence solverMellin-Barnes-
and ${}_pF_q$ -technologiesmultivariate
Almkvist/Zeilberger
(J. Ablinger)

$$\sum_{i_1} \cdots \sum_{i_7} f(\varepsilon, N, i_1, i_2, \dots, i_7)$$

Wegschaider's MultiSum
Package (F. Star)

$$a_0(\varepsilon, N)F(N) + \dots + a_d(\varepsilon, N)F(N + d) = h(\varepsilon, N)$$

Find a recurrence for the integral/sum

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 ε -recurrence solverMellin-Barnes-
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Almkvist/Zeilberger
(J. Ablinger)

$$\sum_{i_1} \cdots \sum_{i_7} f(\varepsilon, N, i_1, i_2, \dots, i_7)$$

Wegschaider's MultiSum
Package (F. Star)Holonomic/difference field
approach (M. Round)

$$a_0(\varepsilon, N)F(N) + \dots + a_d(\varepsilon, N)F(N+d) = h(\varepsilon, N)$$

Sigma's full summation machinery

1. Creative telescoping (for the special case of hypergeometric terms see Zeilberger's algorithm (1991))

GIVEN a **definite** sum

$$F(N) = \sum_{k=0}^N f(N, k);$$

$f(N, k)$: indefinite nested product-sum in k ;
 N : extra parameter

FIND a **recurrence** for $F(N)$

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FIND **all solutions** expressible by indefinite nested products/sums

3. Find a "closed form"

$F(N)$ =combined solutions in terms of **indefinite nested** sums.

$$\sum_{j=0}^{n-2} \sum_{r=0}^{j+1} \sum_{s=0}^{n-j+r-2} \frac{(-1)^{r+s} \binom{j+1}{r} \binom{-j+n+r-2}{s} (-j+n-2)! r!}{(n-s)(s+1)(-j+n+r)!}$$

Simple sum

$$\sum_{j=0}^{n-2} \sum_{r=0}^{j+1} \sum_{s=0}^{n-j+r-2} \frac{(-1)^{r+s} \binom{j+1}{r} \binom{-j+n+r-2}{s} (-j+n-2)! r!}{(n-s)(s+1)(-j+n+r)!}$$

||

$$\sum_{j=0}^{n-2} \sum_{r=0}^{j+1} \boxed{\sum_{s=0}^{n-j+r-2} \frac{(-1)^{r+s} \binom{j+1}{r} \binom{-j+n+r-2}{s} (-j+n-2)! r!}{(n-s)(s+1)(-j+n+r)!}}$$

$$\sum_{j=0}^{n-2} \sum_{r=0}^{j+1} \sum_{s=0}^{n-j+r-2} \frac{(-1)^{r+s} \binom{j+1}{r} \binom{-j+n+r-2}{s} (-j+n-2)! r!}{(n-s)(s+1)(-j+n+r)!}$$

||

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||

$$\boxed{\binom{j+1}{r} \left(\frac{(-1)^r (-j+n-2)! r!}{(n+1)(-j+n+r-1)(-j+n+r)!} + \frac{(-1)^{n+r} (j+1)! (-j+n-2)! (-j+n-1)_r r!}{(n-1)n(n+1)(-j+n+r)! (-j-1)_r (2-n)_j} \right)}$$

$$\sum_{j=0}^{n-2} \sum_{r=0}^{j+1} \sum_{s=0}^{n-j+r-2} \frac{(-1)^{r+s} \binom{j+1}{r} \binom{-j+n+r-2}{s} (-j+n-2)! r!}{(n-s)(s+1)(-j+n+r)!}$$

$$\sum_{j=0}^{n-2} \left(\sum_{r=0}^{j+1} \binom{j+1}{r} \left(\frac{(-1)^r (-j+n-2)! r!}{(n+1)(-j+n+r-1)(-j+n+r)!} + \frac{(-1)^{n+r} (j+1)! (-j+n-2)! (-j+n-1)_r r!}{(n-1)n(n+1)(-j+n+r)! (-j-1)_r (2-n)_j} \right) \right)$$

$$\sum_{j=0}^{n-2} \sum_{r=0}^{j+1} \sum_{s=0}^{n-j+r-2} \frac{(-1)^{r+s} \binom{j+1}{r} \binom{-j+n+r-2}{s} (-j+n-2)! r!}{(n-s)(s+1)(-j+n+r)!}$$

||

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||

$$\left(\frac{n^2 - n + 1}{(n-1)^2 n^2 (n+1)(2-n)_j} + \frac{\sum_{i=1}^j \frac{(2-n)_i}{(-i+n-1)^2 (i+1)!}}{(n+1)(2-n)_j} + \frac{(-1)^{j+n} (-j-2)(-j+n-2)!}{(j-n+1)(n+1)^2 n!} \right) (j+1)! - \frac{1}{(n+1)^2 (-j+n-1)}$$

$$\sum_{j=0}^{n-2} \sum_{r=0}^{j+1} \sum_{s=0}^{n-j+r-2} \frac{(-1)^{r+s} \binom{j+1}{r} \binom{-j+n+r-2}{s} (-j+n-2)! r!}{(n-s)(s+1)(-j+n+r)!}$$

||

$$\sum_{j=0}^{n-2} \left(\left(\frac{n^2 - n + 1}{(n-1)^2 n^2 (n+1)(2-n)_j} + \frac{\sum_{i=1}^j \frac{(2-n)_i}{(-i+n-1)^2 (i+1)!}}{(n+1)(2-n)_j} + \frac{(-1)^{j+n} (-j-2)(-j+n-2)!}{(j-n+1)(n+1)^2 n!} \right) (j+1)! - \frac{1}{(n+1)^2 (-j+n-1)} \right)$$

$$\sum_{j=0}^{n-2} \sum_{r=0}^{j+1} \sum_{s=0}^{n-j+r-2} \frac{(-1)^{r+s} \binom{j+1}{r} \binom{-j+n+r-2}{s} (-j+n-2)! r!}{(n-s)(s+1)(-j+n+r)!}$$

||

$$\sum_{j=0}^{n-2} \left(\left(\frac{n^2 - n + 1}{(n-1)^2 n^2 (n+1)(2-n)_j} + \frac{\sum_{i=1}^j \frac{(2-n)_i}{(-i+n-1)^2 (i+1)!}}{(n+1)(2-n)_j} + \frac{(-1)^{j+n} (-j-2)(-j+n-2)!}{(j-n+1)(n+1)^2 n!} \right) (j+1)! - \frac{1}{(n+1)^2 (-j+n-1)} \right)$$

||

$$\frac{-n^2 - n - 1}{n^2 (n+1)^3} + \frac{(-1)^n (n^2 + n + 1)}{n^2 (n+1)^3} - \frac{2S_{-2}(n)}{n+1} + \frac{S_1(n)}{(n+1)^2} + \frac{S_2(n)}{-n-1}$$

Note: $S_a(n) = \sum_{i=1}^N \frac{\text{sign}(a)^i}{i^{|a|}}$, $a \in \mathbb{Z} \setminus \{0\}$.

In[1]:= << **Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[2]:= << **HarmonicSums.m**

HarmonicSums by Jakob Ablinger © RISC-Linz

In[3]:= << **EvaluateMultiSums.m**

EvaluateMultiSums by Carsten Schneider © RISC-Linz

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EvaluateMultiSums by Carsten Schneider © RISC-Linz

$$\text{In[4]:= mySum} = \sum_{j=0}^{n-2} \sum_{r=0}^{j+1} \sum_{s=0}^{n-j+r-2} \frac{(-1)^{r+s} \binom{j+1}{r} \binom{-j+n+r-2}{s} (-j+n-2)! r!}{(n-s)(s+1)(-j+n+r)!};$$

In[5]:= **EvaluateMultiSum**[mySum, {}, {N}, {1}, {∞}]

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Sigma - A summation package by Carsten Schneider © RISC-Linz

In[2]:= << **HarmonicSums.m**

HarmonicSums by Jakob Ablinger © RISC-Linz

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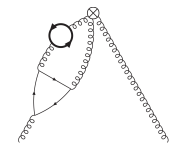
EvaluateMultiSums by Carsten Schneider © RISC-Linz

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$$\text{Out[5]=} \frac{-n^2 - n - 1}{n^2(n+1)^3} + \frac{(-1)^n (n^2 + n + 1)}{n^2(n+1)^3} - \frac{2S_{-2}[n]}{n+1} + \frac{S_1[n]}{(n+1)^2} + \frac{S_2[n]}{-n-1}$$

Example: a 2-mass 3-loop Feynman integral [arXiv:1804.02226]
(arose in the calculation of the gluonic operator matrix element $A_{gg,Q}^{(3)}$)

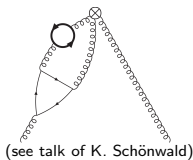


(see talk of K. Schönwald)

The diagram is produced with axodraw (J. Vermaseren).

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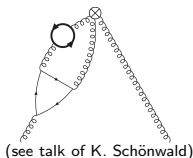
Mellin-Barnes-
and ${}_pF_q$ -technologies \rightarrow

expression (95 MB) with

- 150 single sums
- 1000 double sums
- 12160 triple sums
- 1555 quadruple sums

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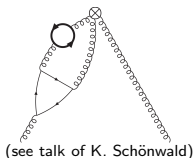
Typical triple sum:

$$\sum_{j=0}^N \sum_{i=0}^j \sum_{k=0}^i \frac{(4+\varepsilon)(-2+N)(-1+N)N\pi(-1)^{2-k}}{2+\varepsilon} \times 2^{-2+\varepsilon} e^{-\frac{3\varepsilon\gamma}{2}} \eta^k \times$$

$$\frac{\Gamma(1-\frac{\varepsilon}{2}-i+j+k)\Gamma(-1-\frac{\varepsilon}{2})\Gamma(2+\frac{\varepsilon}{2})\Gamma(1+N)\Gamma(1+\varepsilon+i-k)\Gamma(-\frac{3\varepsilon}{2}+k)\Gamma(1-\varepsilon+k)\Gamma(3-\varepsilon+k)\Gamma(-\frac{1}{2}-\frac{\varepsilon}{2}+k)}{\Gamma(-\frac{3}{2}-\frac{\varepsilon}{2})\Gamma(\frac{5}{2}+\frac{\varepsilon}{2})\Gamma(2+i)\Gamma(1+k)\Gamma(2-i+j)\Gamma(2-\varepsilon+k)\Gamma(\frac{5}{2}-\varepsilon+k)\Gamma(-\frac{\varepsilon}{2}+k)\Gamma(5+\frac{\varepsilon}{2}+N)}$$

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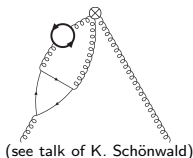
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ε^{-1} -contribution: 30 minutes for this sum

\sim 1 year for full expression

Example: a 2-mass 3-loop Feynman integral [arXiv:1804.02226]

(arose in the calculation of the gluonic operator matrix element $A_{gg,Q}^{(3)}$)



Mellin-Barnes-
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$$\frac{\Gamma(1-\frac{\epsilon}{2}-i+j+k)\Gamma(-1-\frac{\epsilon}{2})\Gamma(2+\frac{\epsilon}{2})\Gamma(1+N)\Gamma(1+\epsilon+i-k)\Gamma(-\frac{3\epsilon}{2}+k)\Gamma(1-\epsilon+k)\Gamma(3-\epsilon+k)\Gamma(-\frac{1}{2}-\frac{\epsilon}{2}+k)}{\Gamma(-\frac{3}{2}-\frac{\epsilon}{2})\Gamma(\frac{5}{2}+\frac{\epsilon}{2})\Gamma(2+i)\Gamma(1+k)\Gamma(2-i+j)\Gamma(2-\epsilon+k)\Gamma(\frac{5}{2}-\epsilon+k)\Gamma(-\frac{\epsilon}{2}+k)\Gamma(5+\frac{\epsilon}{2}+N)}$$

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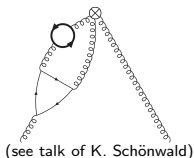
~ 1 year for full expression

ϵ^0 -contribution: sums are not expressible

in terms of indefinite nested sums

Example: a 2-mass 3-loop Feynman integral [arXiv:1804.02226]

(arose in the calculation of the gluonic operator matrix element $A_{gg,Q}^{(3)}$)



Mellin-Barnes-
and ${}_pF_q$ -technologies \rightarrow

expression (95 MB) with

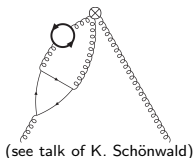
- 150 single sums
- 1000 double sums
- 12160 triple sums
- 1555 quadruple sums

\downarrow SumProduction.m (2 hours)

expression (377 MB)
consisting of 8 multi-sums

Example: a 2-mass 3-loop Feynman integral [arXiv:1804.02226]

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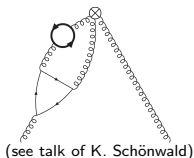
↓ EvaluateMultiSums.m

Example: a 2-mass 3-loop Feynman integral [arXiv:1804.02226]
 (arose in the calculation of the gluonic operator matrix element $A_{gg,Q}^{(3)}$)

sum	size of sum (with ε)	summand size of constant term	time of calculation	number of indef. sums
$\sum_{i_4=2}^{N-3} \sum_{i_3=0}^{i_4-2} \sum_{i_2=0}^{i_3} \sum_{i_1=0}^{\infty}$	17.7 MB	266.3 MB	177529 s (2.1 days)	1188
$\sum_{i_3=3}^{N-4} \sum_{i_2=0}^{i_3-1} \sum_{i_1=0}^{\infty}$	232 MB	1646.4 MB	980756 s (11.4 days)	747
$\sum_{i_2=3}^{N-4} \sum_{i_1=0}^{\infty}$	67.7 MB	458 MB	524485 s (6.1 days)	557
$\sum_{i_1=0}^{\infty}$	38.2 MB	90.5 MB	689100 s (8.0 days)	44
$\sum_{i_4=2}^{N-3} \sum_{i_3=0}^{i_4-2} \sum_{i_2=0}^{i_3} \sum_{i_1=0}^{i_2}$	1.3 MB	6.5 MB	305718 s (3.5 days)	1933
$\sum_{i_3=3}^{N-4} \sum_{i_2=0}^{i_3-1} \sum_{i_1=0}^{i_2}$	11.6 MB	32.4 MB	710576 s (8.2 days)	621
$\sum_{i_2=3}^{N-4} \sum_{i_1=0}^{i_2}$	4.5 MB	5.5 MB	435640 s (5.0 days)	536
$\sum_{i_1=3}^{N-4}$	0.7 MB	1.3 MB	9017s (2.5 hours)	68

Example: a 2-mass 3-loop Feynman integral [arXiv:1804.02226]

(arose in the calculation of the gluonic operator matrix element $A_{gg,Q}^{(3)}$)



Mellin-Barnes-
and pF_q -technologies \rightarrow

expression (95 MB) with

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- 1000 double sums
- 12160 triple sums
- 1555 quadruple sums

\downarrow SumProduction.m (2 hours)

expression (377 MB)
consisting of 8 multi-sums

\downarrow EvaluateMultiSums.m
(3 month)

expression (154 MB)
consisting of 4110 indefinite sums

Example: a 2-mass 3-loop Feynman integral [arXiv:1804.02226]
 (arose in the calculation of the gluonic operator matrix element $A_{gg,Q}^{(3)}$)

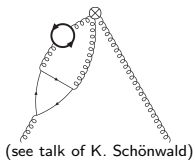
Most complicated objects: generalized binomial sums, like

$$\sum_{h=1}^N 2^{-2h} (1-\eta)^h \binom{2h}{h} \left(\sum_{i=1}^h \frac{2^{2i} (1-\eta)^{-i}}{i \binom{2i}{i}} \right) \left(\sum_{i=1}^h \frac{(1-\eta)^i \binom{2i}{i}}{2^{2i}} \right) \times$$

$$\times \left(\sum_{i=1}^h \frac{2^{2i} (1-\eta)^{-i} \sum_{j=1}^i \frac{\sum_{k=1}^j (1-\eta)^k}{k}}{i \binom{2i}{i}} \right).$$

Example: a 2-mass 3-loop Feynman integral [arXiv:1804.02226]

(arose in the calculation of the gluonic operator matrix element $A_{gg,Q}^{(3)}$)



Mellin-Barnes-
and pF_q -technologies

expression (95 MB) with

- 150 single sums
- 1000 double sums
- 12160 triple sums
- 1555 quadruple sums

↓ SumProduction.m (2 hours)

expression (377 MB)
consisting of 8 multi-sums

↓ EvaluateMultiSums.m
(3 month)

expression (8.3 MB)
consisting of
74 indefinite sums

← Sigma.m (32 days)

expression (154 MB)
consisting of 4110 indefinite sums

Solving coupled systems

The DE-REC approach

DE system

$$D\hat{I}(x) = A\hat{I}(x) + \hat{R}(x)$$

The DE-REC approach

DE system

$$D\hat{I}(x) = A\hat{I}(x) + \hat{R}(x)$$

OreSys package (S. Gerhold)

uncoupling algorithm

uncoupled DE system

$$\sum_i a_i(x) D^i \hat{I}_1(x) = r(x)$$
$$\hat{I}_k(x) = \text{expr}_k(D^i \hat{I}_1(x)), k > 1$$

The DE-REC approach

$$\begin{array}{c} \text{DE system} \\ D\hat{I}(x) = A\hat{I}(x) + \hat{R}(x) \end{array}$$

OreSys package (S. Gerhold)
uncoupling algorithm

$$\begin{array}{c} \text{uncoupled DE system} \\ \sum_i a_i(x) D^i \hat{I}_1(x) = r(x) \\ \hat{I}_k(x) = \text{expr}_k(D^i \hat{I}_1(x)), k > 1 \end{array}$$

$$\hat{I}_1(x) = \sum_{N=0}^{\infty} I_1(N)x^N$$

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coeff. comparison w.r.t. x^N

$$\text{linear recurrence} \\ \sum_i a'_i(N) I_1(N+i) = r'(N)$$

The DE-REC approach

DE system

$$D\hat{I}(x) = A\hat{I}(x) + \hat{R}(x)$$

$$\hat{I}_1(x) = \sum_{N=0}^{\infty} I_1(N)x^N$$

closed form solutions of $I_1(N)$
in the class of nested
sums and products
– if this is possible

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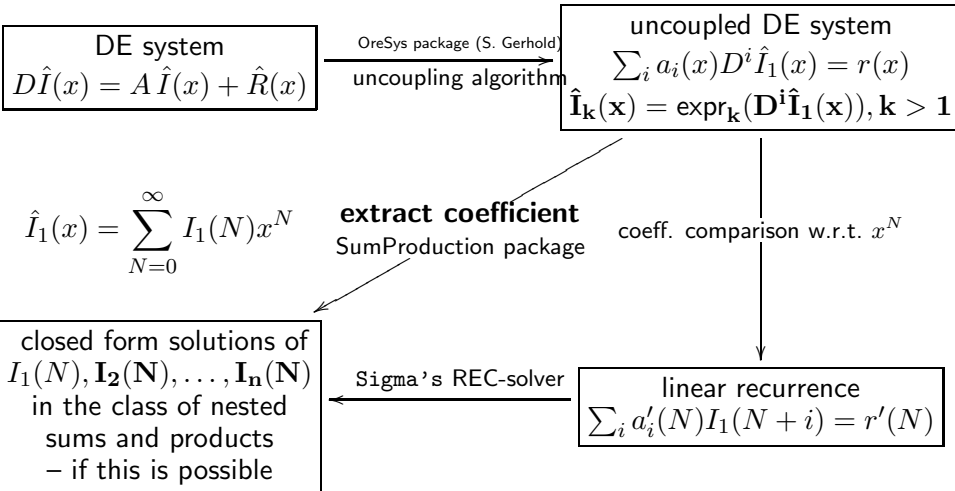
coeff. comparison w.r.t. x^N

Sigma's REC-solver

linear recurrence

$$\sum_i a'_i(N) I_1(N+i) = r'(N)$$

The DE-REC approach (SolveCoupledSystem package)



A coupled differential system for $\hat{I}_1(x)$, $\hat{I}_2(x)$, $\hat{I}_3(x)$

(produced by IBP [extension of REDUZE_2, A.v. Manteuffel])

$$D_x \begin{pmatrix} \hat{I}_1(x) \\ \hat{I}_2(x) \\ \hat{I}_3(x) \end{pmatrix} = \begin{pmatrix} -\frac{-1-\varepsilon+x}{(x-1)x} & -\frac{2}{(x-1)x} & 0 \\ \frac{\varepsilon(3\varepsilon+2)}{4(x-1)} & -\frac{-2-\varepsilon+3x+3\varepsilon x}{2(x-1)x} & -\frac{\varepsilon+1}{2(x-1)} \\ -\frac{\varepsilon(3\varepsilon+2)(x-2)}{4(x-1)x} & \frac{-2-5\varepsilon+x+3\varepsilon x}{2(x-1)x} & \frac{(-2\varepsilon-x+\varepsilon x)}{2(x-1)x} \end{pmatrix} \begin{pmatrix} \hat{I}_1(x) \\ \hat{I}_2(x) \\ \hat{I}_3(x) \end{pmatrix} + \begin{pmatrix} \hat{R}_1(x) \\ \hat{R}_2(x) \\ -\hat{R}_2(x) \end{pmatrix}$$

where

$$\hat{R}_1(x) = \frac{\hat{B}_4(x)}{(x-1)x},$$

$$\begin{aligned} \hat{R}_2(x) &= \frac{-(\varepsilon+2)^3}{16(\varepsilon+1)(x-1)x} \hat{B}_1(x) + \frac{(\varepsilon+2)(3\varepsilon+4)(19\varepsilon^2+36\varepsilon+16)}{16\varepsilon(5\varepsilon+6)(x-1)x} \hat{B}_2(x) \\ &+ \frac{(\varepsilon+1)^2(3\varepsilon+4)^2}{2\varepsilon(5\varepsilon+6)x} \hat{B}_3(x) + \frac{-24-50\varepsilon-25\varepsilon^2+8x+14\varepsilon x+6\varepsilon^2 x}{4(5\varepsilon+6)(x-1)x} \hat{B}_4(x) \end{aligned}$$

$\hat{B}_1(x)$, $\hat{B}_2(x)$, $\hat{B}_3(x)$ have been solved with symbolic summation.

Solving a coupled differential system

In[6]:= << OreSys.m

OreSys by Stefan Gerhold (optimized by C. Schneider) © RISC-Linz

In[7]:= << SolveCoupledSystem.m

SolveCoupledSystem by Carsten Schneider © RISC-Linz

In[8]:= **coupledDESys** = D[{ $\hat{I}_1(x)$, $\hat{I}_2(x)$, $\hat{I}_3(x)$ }, x] - A. $\{\hat{I}_1(x)$, $\hat{I}_2(x)$, $\hat{I}_3(x)\}$;

In[9]:= **rhs** = $\{\hat{R}_1(x)$, $\hat{R}_2(x)$, $-\hat{R}_2(x)\}$ in power series representation;

In[10]:= **SolveCoupledDESys**[coupledDESys, rhs, {**I**₁[x], **I**₂[x], **I**₃[x]}, ϵ , -3,
 {-2, -2, -2}, rhs, ...]

Solving a coupled differential system

In[6]:= << OreSys.m

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In[8]:= **coupledDESys** = D[{ $\hat{I}_1(x)$, $\hat{I}_2(x)$, $\hat{I}_3(x)$ }, x] - A. $\{\hat{I}_1(x)$, $\hat{I}_2(x)$, $\hat{I}_3(x)\}$;

In[9]:= **rhs** = { $\hat{R}_1(x)$, $\hat{R}_2(x)$, $-\hat{R}_2(x)$ } in power series representation;

In[10]:= **SolveCoupledDESys**[coupledDESys, rhs, {**I**₁[x], **I**₂[x], **I**₃[x]}, ϵ , -3,
 {-2, -2, -2}, rhs, ...]

Out[10]=
$$\left\{ \frac{1}{\epsilon^3} \left(\frac{4(3N^2 + 6N + 4)}{3(N+1)^2} + \frac{4S_1[N]}{3(N+1)} \right) + \frac{1}{\epsilon^2} \left(-\frac{2(20N^3 + 58N^2 + 57N + 22)}{3(N+1)^3} + \frac{2(N+2)(2N-1)S_1[N]}{3(N+1)^2} - \frac{S_1[N]^2}{N+1} - \frac{S_2[N]}{N+1} \right), \right.$$

$$\left. \frac{4}{3\epsilon^3} - \frac{2}{\epsilon^2}, \frac{8}{3\epsilon^3} + \frac{1}{\epsilon^2} \left(-\frac{4(4N^2 + 7N + 2)}{3(N+1)^2} + \frac{4(N+2)S_1[N]}{3(N+1)} \right) \right\}$$

Concrete calculations:

- ▶ J. Ablinger, A. Behring, J. Blümlein, A. De Freitas, A. Hasselhuhn, A. von Manteuffel, M. Round, CS, F. Wissbrock. The 3-Loop Non-Singlet Heavy Flavor Contributions and Anomalous Dimensions for the Structure Function $F_2(x, Q^2)$ and Transversity. Nuclear Physics B 886, 2014. arXiv:1406.4654 [hep-ph].
- ▶ J. Ablinger, J. Blümlein, A. De Freitas, A. Hasselhuhn, A. von Manteuffel, M. Round, CS. The $O(\alpha_s^3 T_F^2)$ Contributions to the Gluonic Operator Matrix Element. Nuclear Physics B 885, 2014. arXiv:1405.4259 [hep-ph].
- ▶ J. Ablinger, J. Blümlein, A. De Freitas, A. Hasselhuhn, A. von Manteuffel, M. Round, CS, F. Wissbrock. The Transition Matrix Element $A_{gq}(N)$ of the Variable Flavor Number Scheme at $O(\alpha_s^3)$. Nuclear Physics B 882, 2014. arXiv:1402.0359 [hep-ph].
- ▶ A. Behring, J. Blümlein, A. De Freitas, A. Hasselhuhn, A. von Manteuffel, CS. The $O(\alpha_s^3)$ Heavy Flavor Contributions to the Charged Current Structure Function $xF_3(x, Q^2)$ at Large Momentum Transfer. Physical Review D 92(114005), 2015. arXiv:1508.01449 [hep-ph].
- ▶ A. Behring, J. Blümlein, A. De Freitas, A. von Manteuffel, CS. The 3-Loop Non-Singlet Heavy Flavor Contributions to the Structure Function $g_1(x, Q^2)$ at Large Momentum Transfer. Nucl. Phys. B 897, 2015. arXiv:1504.08217 [hep-ph].
- ▶ J. Ablinger, A. Behring, J. Blümlein, A. De Freitas, A. von Manteuffel, CS. The 3-Loop Pure Singlet Heavy Flavor Contributions to the Structure Function $F_2(x, Q^2)$ and the Anomalous Dimension. Nuclear Physics B 890, 2015. arXiv:1409.1135.
- ▶ A. Behring, J. Blümlein, G. Falcioni, A. De Freitas, A. von Manteuffel, CS. The Asymptotic 3-Loop Heavy Flavor Corrections to the Charged Current Structure Functions $F_L^{W^+ - W^-}(x, Q^2)$ and $F_2^{W^+ - W^-}(x, Q^2)$. Physical Review D 94(11), 2016. arXiv:1609.06255 [hep-ph].
- ▶ J. Ablinger, A. Behring, J. Blümlein, A. De Freitas, A. von Manteuffel, CS. Calculating Three Loop Ladder and V-Topologies for Massive Operator Matrix Elements by Computer Algebra. Comput. Phys. Comm. 202, 2016. arXiv:1509.08324 [hep-ph].
- ▶ J. Ablinger, J. Blümlein, P. Marquard, N. Rana, CS. Heavy Quark Form Factors at Three Loops in the Planar Limit, 2018. arXiv:1804.07313 [hep-ph].
- ▶ J. Ablinger, A. Behring, J. Blümlein, G. Falcioni, A. De Freitas, P. Marquard, N. Rana, CS. The Heavy Quark Form Factors at Two Loops. Physical Review D, 2018. arXiv:1712.09889.

Computing large moments and guessing recurrences

The DE-REC approach (SolveCoupledSystem package)

$$\text{DE system} \\ D\hat{I}(x) = A\hat{I}(x) + \hat{R}(x)$$

OreSys package (S. Gerhold)
uncoupling algorithm

$$\text{uncoupled DE system} \\ \sum_i a_i(x) D^i \hat{I}_1(x) = r(x) \\ \hat{I}_k(x) = \text{expr}_k(D^i \hat{I}_1(x)), k > 1$$

$$\hat{I}_1(x) = \sum_{N=0}^{\infty} I_1(N) x^N$$

extract coefficient
SumProduction package

coeff. comparison w.r.t. x^N

closed form solutions of
 $I_1(N), I_2(N), \dots, I_n(N)$
in the class of nested
sums and products
– if this is possible

Sigma's REC-solver

$$\text{linear recurrence} \\ \sum_i a'_i(N) I_1(N+i) = r'(N)$$

The method of large moments (SolveCoupledSystem)

DE system

$$D\hat{I}(x) = A\hat{I}(x) + \hat{R}(x)$$

OreSys package (S. Gerhold)
uncoupling algorithm

uncoupled DE system

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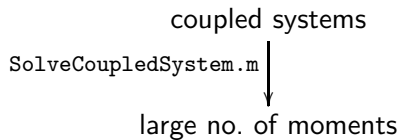
closed form solutions of $I_1(N)$
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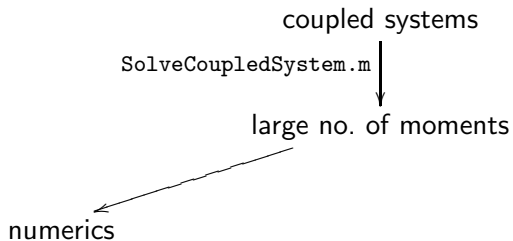
Sigma's REC-solver

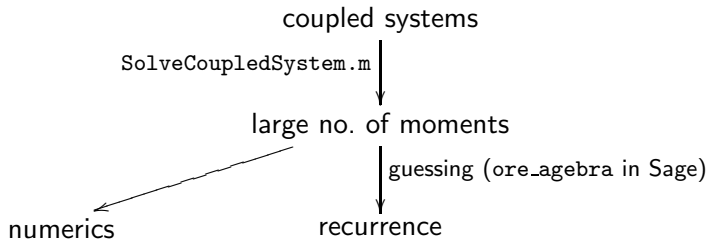
linear recurrence

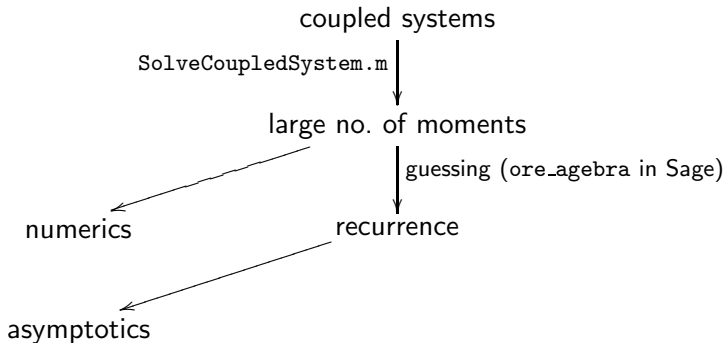
$$\sum_i a'_i(N) I_1(N+i) = r'(N)$$

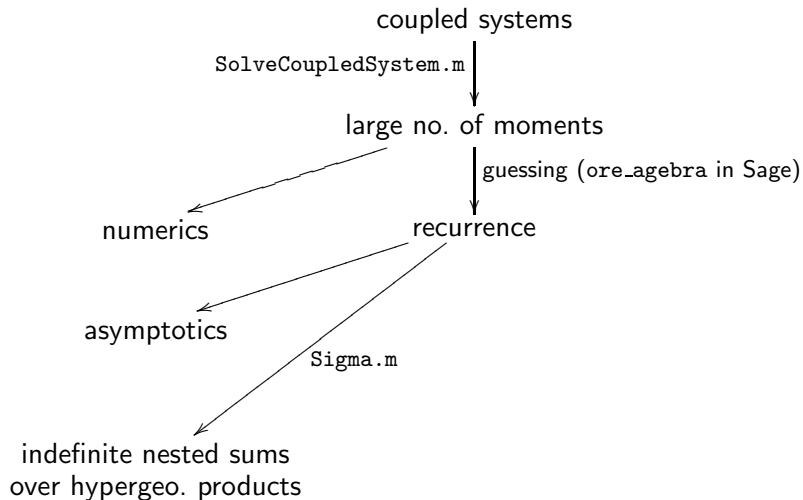
Given $r'(0), r'(1), \dots, r'(5114)$,
compute $I_1(0), I_1(1), \dots, I_1(5114)$

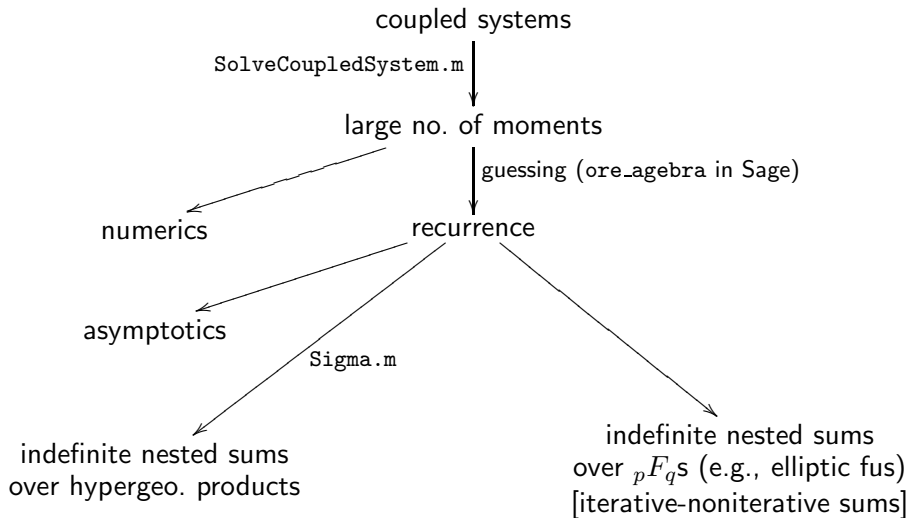


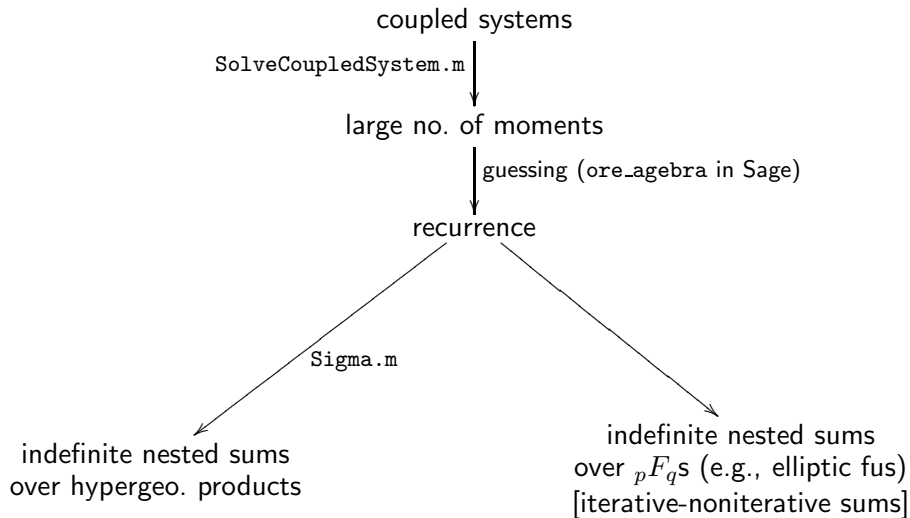












Concrete calculations of large moments:

- ▶ J. Ablinger, A. Behring, J. Blmlein, A. De Freitas, A. von Manteuffel, C. Schneider
The Three-Loop Splitting Functions $P_{qg}^{(2)}$ and $P_{gg}^{(2, N_F)}$. Nucl. Phys. B. 922, pp. 1-40. 2017. ISSN 0550-3213. arXiv:1705.01508 [hep-ph].
 1. computed ~ 2400 moments
 2. guessed all recurrences
 3. solved all recurrences in terms of harmonic sums.

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 1. computed 2000 moments
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The Three-Loop Splitting Functions $P_{qg}^{(2)}$ and $P_{gg}^{(2, \text{NF})}$. Nucl. Phys. B. 922, pp. 1-40. 2017. ISSN 0550-3213. arXiv:1705.01508 [hep-ph].
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- ▶ massive Wilson coefficient A_{Qg} :
 1. computed 2000 moments
 2. guessed and solved some recurrences.

- ▶ special case: $A_{Qg}^{T_F^2, (3)}$ (as case study)
 1. computed 8000 moments
 2. guessed all recurrences
 3. solved some recurrences

Special functions algorithms

(implemented in `HarmonicSums`)

Function Spaces

Sums

Harmonic Sums

$$\sum_{k=1}^N \frac{1}{k} \sum_{l=1}^k \frac{(-1)^l}{l^3}$$

gen. Harmonic Sums

$$\sum_{k=1}^N \frac{(1/2)^k}{k} \sum_{l=1}^k \frac{(-1)^l}{l^3}$$

Cycl. Harmonic Sums

$$\sum_{k=1}^N \frac{1}{(2k+1)} \sum_{l=1}^k \frac{(-1)^l}{l^3}$$

Binomial Sums

$$\sum_{k=1}^N \frac{1}{k^2} \binom{2k}{k} (-1)^k$$

Integrals

Harmonic Polylogarithms

$$\int_0^x \frac{dy}{y} \int_0^y \frac{dz}{1+z}$$

gen. Harmonic Polylogarithms

$$\int_0^x \frac{dy}{y} \int_0^y \frac{dz}{z-3}$$

Cycl. Harmonic Polylogarithms

$$\int_0^x \frac{dy}{1+y^2} \int_0^y \frac{dz}{1-z+z^2}$$

root-valued iterated integrals

$$\int_0^x \frac{dy}{y} \int_0^y \frac{dz}{z\sqrt{1+z}}$$

iterated integrals on CIS fct.

$$\int_0^z dx \frac{\ln(x)}{1+x} {}_2F_1 \left[\frac{4}{3}, \frac{5}{3}; \frac{x^2(x^2-9)^2}{(x^2+3)^3} \right]$$

Special Numbers

multiple zeta values

$$\int_0^1 dx \frac{\text{Li}_3(x)}{1+x} = -2\text{Li}_4(1/2) + \dots$$

gen. multiple zeta values

$$\int_0^1 dx \frac{\ln(x+2)}{x-3/2} = \text{Li}_2(1/3) + \dots$$

cycl. multiple zeta values

$$\mathbf{C} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2}$$

associated numbers

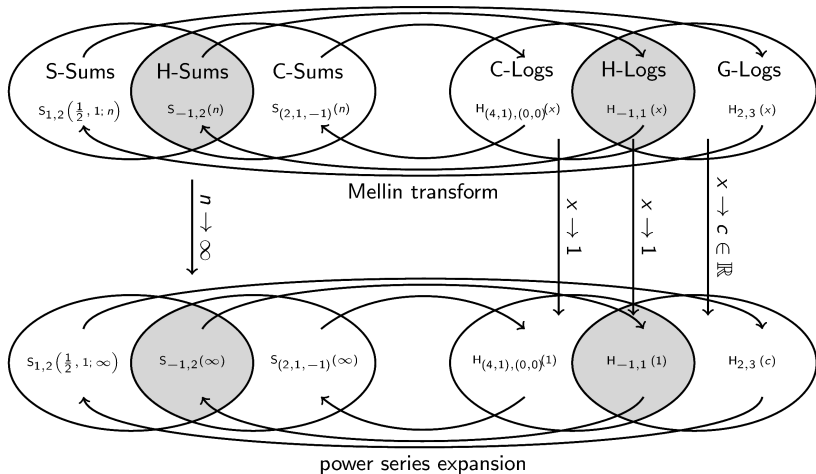
$$H_{8,w_3} = 2\text{arccot}(\sqrt{7})^2$$

associated numbers

$$\int_0^1 dx {}_2F_1 \left[\frac{4}{3}, \frac{5}{3}; \frac{x^2(x^2-9)^2}{(x^2+3)^3} \right]$$

shuffle, stuffle, and various structural relations \implies algebras

integral representation (inv. Mellin transform)



square-root valued letters \iff nested binomial sums $\binom{2i}{i}$

non-iterative iterative integrals \implies Iterate on CIS ${}_2F_1$'s: rat. argument

(special cases: complete elliptic integrals) [arXiv:1706.01299]

Conclusion (used techniques)

1. symbolic integration
(method of hyperlogarithms, Almkvist-Zeilberger algorithm)
2. generalized hypergeometric functions (and extensions)
3. Mellin-Barnes techniques
4. symbolic summation (WZ-, holonomic, difference ring methods)
5. recurrence solving
6. integration by parts technique
7. differential equation solving
8. method of large moments and guessing
9. special function algorithms