

PROGRESS ON TWO-LOOP FIVE-POINT MASTER INTEGRALS

Costas G. Papadopoulos & Chris Wever

INPP, NCSR “Demokritos”



Loops and Legs 2018, St. Goar, April 30, 2018

OUTLINE

- ① Introduction
- ② The two-loop five-point integrals: the first x
- ③ Yet another x : verifying physical region results
- ④ Summary - Discussion

FEYNMAN

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Space-Time Approach to Quantum Electrodynamics

R. P. FEYNMAN

Department of Physics, Cornell University, Ithaca, New York

(Received May 9, 1949)

In this paper two things are done. (1) It is shown that a considerable simplification can be attained in writing down matrix elements for complex processes in electrodynamics. Further, a physical point of view is available which permits them to be written down directly for any specific problem. Being simply a

and presumably consistent, method is therefore available for the calculation of all processes involving electrons and photons.

The simplification in writing the expressions results from an emphasis on the over-all space-time view resulting from a study of the solution of the equations of electrodynamics. The relation

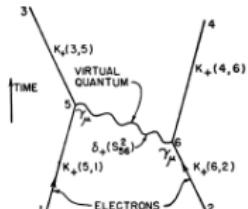
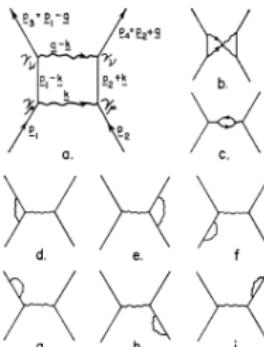


FIG. 1. The fundamental interaction Eq. (4). Exchange of one quantum between two electrons.

D. More Complex Problems

Matrix elements for complex problems can be set up in a manner analogous to that used for the simpler cases. We give three illustrations; higher order corrections to the Møller scatter-



BEST TODAY

C. G. Papadopoulos, D. Tommasini and C. Wever, JHEP **1604**, 078 (2016) [arXiv:1511.09404 [hep-ph]].

T. Gehrmann, J. M. Henn and N. A. Lo Presti, Phys. Rev. Lett. **116** (2016) no.6, 062001 Erratum: [Phys. Rev. Lett. **116** (2016) no.18, 189903] [arXiv:1511.05409 [hep-ph]].

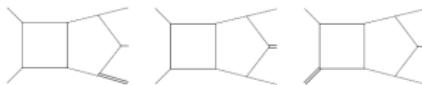


Figure 1. The three planar pentaboxes of the families P_1 (left), P_2 (middle) and P_3 (right) with one external massive leg.

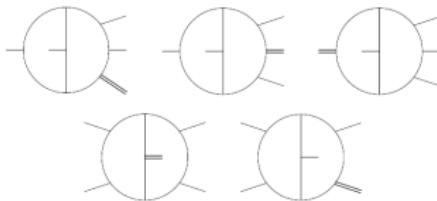


Figure 2. The five non-planar families with one external massive leg.

$$\begin{aligned} \mathbf{G} = & \varepsilon^{-2} \mathbf{b}_0^{(-2)} + \varepsilon^{-1} \left(\sum \mathcal{G}_a \mathbf{M}_a \mathbf{b}_0^{(-2)} + \mathbf{b}_0^{(-1)} \right) \\ & + \varepsilon^0 \left(\sum \mathcal{G}_{ab} \mathbf{M}_a \mathbf{M}_b \mathbf{b}_0^{(-2)} + \sum \mathcal{G}_a \mathbf{M}_a \mathbf{b}_0^{(-1)} + \mathbf{b}_0^{(0)} \right) \\ & + \varepsilon \left(\sum \mathcal{G}_{abc} \mathbf{M}_a \mathbf{M}_b \mathbf{M}_c \mathbf{b}_0^{(-2)} + \sum \mathcal{G}_{ab} \mathbf{M}_a \mathbf{M}_b \mathbf{b}_0^{(-1)} + \sum \mathcal{G}_a \mathbf{M}_a \mathbf{b}_0^{(0)} + \mathbf{b}_0^{(1)} \right) \\ & + \varepsilon^2 \left(\sum \mathcal{G}_{abcd} \mathbf{M}_a \mathbf{M}_b \mathbf{M}_c \mathbf{M}_d \mathbf{b}_0^{(-2)} + \sum \mathcal{G}_{abc} \mathbf{M}_a \mathbf{M}_b \mathbf{M}_c \mathbf{b}_0^{(-1)} \right. \\ & \quad \left. + \sum \mathcal{G}_{ab} \mathbf{M}_a \mathbf{M}_b \mathbf{b}_0^{(0)} + \sum \mathcal{G}_a \mathbf{M}_a \mathbf{b}_0^{(1)} + \mathbf{b}_0^{(2)} \right) \end{aligned} \quad (3.6)$$

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	ϵ^{-4}	ϵ^{-3}	ϵ^{-2}	ϵ^{-1}	ϵ^0
$\widehat{A}_{- - + + +}^{(2),[0]}$	12.5	27.7526	-23.773	-168.117	-175.207 ± 0.004
$P_{- - + + +}^{(2),[0]}$	12.5	27.7526	-23.773	-168.116	—
$\widehat{A}_{- + - + +}^{(2),[0]}$	12.5	27.7526	2.5029	-35.8094	69.661 ± 0.009
$P_{- + - + +}^{(2),[0]}$	12.5	27.7526	2.5028	-35.8086	—

TABLE II. The numerical evaluation of $\widehat{A}^{(2),[0]}(1, 2, 3, 4, 5)$ using $\{x_1 = -1, x_2 = 79/90, x_3 = 16/61, x_4 = 37/78, x_5 = 83/102\}$ in Eq.(6). The comparison with the universal pole structure, P , is shown. The **+++++** and **-++++** amplitudes vanish to $\mathcal{O}(\epsilon)$ for this $(d_s - 2)^0$ component.

S. Badger, C. BrG'Ennum-Hansen, H. B. Hartanto and T. Peraro, "A first look at two-loop five-gluon scattering in QCD,"

arXiv:1712.02229 [hep-ph].

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$\mathcal{A}^{(2)}/\mathcal{A}^{(0)}$	ϵ^{-4}	ϵ^{-3}	ϵ^{-2}	ϵ^{-1}	ϵ^0
(1 ⁻ , 2 ⁻ , 3 ⁺ , 4 ⁺ , 5 ⁺)	12.5000000	25.46246919	-1152.843107	-4072.938337	-3637.249567
(1 ⁻ , 2 ⁺ , 3 ⁻ , 4 ⁺ , 5 ⁺)	12.5000000	25.46246919	-6.121629624	-90.22184215	-115.7836685

TABLE II. Numeric results truncated to 10 significant figures for the two-loop split and alternating MHV amplitudes, normalized to the tree level, at the kinematic point of eq. (IV.1).

S. Abreu, F. Febres Cordero, H. Ita, B. Page and M. Zeng, “Planar Two-Loop Five-Gluon Amplitudes from Numerical Unitarity,” arXiv:1712.03946 [hep-ph].



LO - DYSON-SCHWINGER RECURSIVE EQUATIONS

From Feynman Diagrams to recursive equations: taming the $n!$

- 1999 HELAC: The first code to calculate recursively tree-order amplitudes for (practically) arbitrary number of particles

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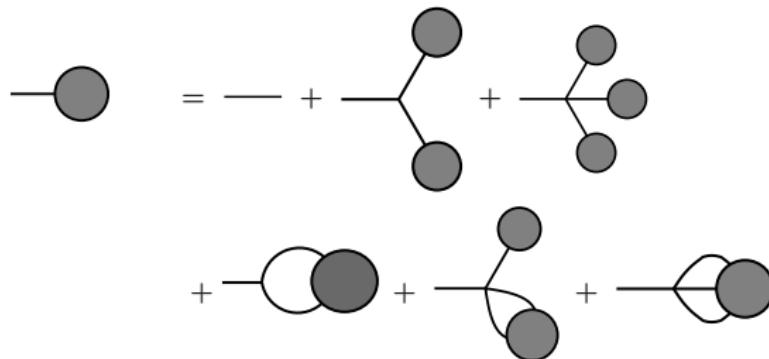
From Feynman Diagrams to recursive equations: taming the $n!$

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A. Kanaki and C. G. Papadopoulos, Comput. Phys. Commun. **132** (2000) 306 [arXiv:hep-ph/0002082].

F. A. Berends and W. T. Giele, Nucl. Phys. B **306** (1988) 759.

F. Caravaglios and M. Moretti, Phys. Lett. B **358** (1995) 332.



Unfortunately not so much on the second line !

TAMING THE BEAST ...

From Feynman graphs ...

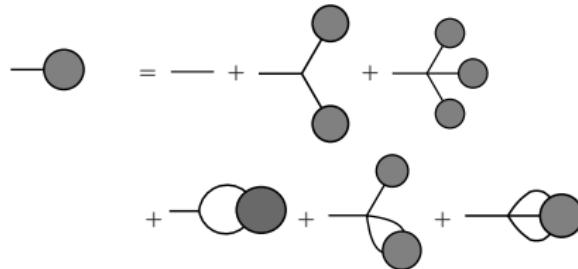
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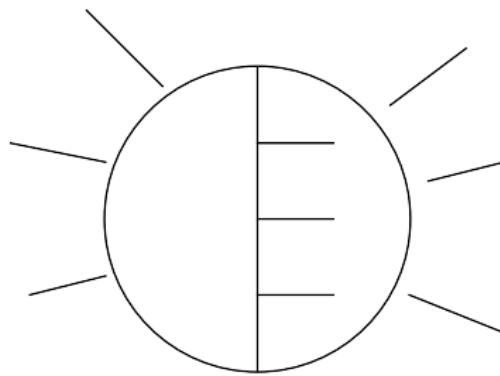
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to Dyson-Schwinger recursion! Helac-Phegas



$gg \rightarrow ng$	2	3	4	5	6	7	8	9
#	5	15	35	70	126	210	330	495

TWO-LOOP GRAPH



IBPI: THE CURRENT APPROACH

- m independent momenta / loops, $N = I(I+1)/2 + Im$ scalar products
- basis composed by $D_1 \dots D_N$, allows to express all scalar products

$$D_i = (\{k, l\} + p_i)^2 - M_i^2$$

-

$$F[a_1, \dots, a_N] = \int d^d k d^d l \frac{1}{D_1^{a_1} \dots D_N^{a_N}}$$

$$\int d^d k d^d l \frac{\partial}{\partial \{k^\mu, l^\mu\}} \left(\frac{\{k^\mu, l^\mu, v^\mu\}}{D_1^{a_1} \dots D_N^{a_N}} \right) = 0$$

- IBP Laporta: FIRE, AIR, Reduze, Kira reduce these to MI
- MI computed, Feynman parameters, Mellin-Barnes, Differential Equations
- Or numerical: SecDec, Weinzierl

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F. V. Tkachov, Phys. Lett. B 100 (1981) 65.

K. G. Chetyrkin and F. V. Tkachov, Nucl. Phys. B 192 (1981) 159.

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S. Laporta, Int. J. Mod. Phys. A 15 (2000) 5087

C. Anastasiou and A. Lazopoulos, JHEP 0407 (2004) 046

C. Studerus, Comput. Phys. Commun. 181 (2010) 1293

A. V. Smirnov, Comput. Phys. Commun. 189 (2014) 182

P. Maierhoefer, J. Usovitsch and P. Uwer, arXiv:1705.05610 [hep-ph].

- MI computed, Feynman parameters, Mellin-Barnes, Differential Equations
- Or numerical: See [Sect. 6.2](#), [Weinzierl](#)

IBPI: THE CURRENT APPROACH

- m independent momenta / loops, $N = l(l+1)/2 + lm$ scalar products
- basis composed by $D_1 \dots D_N$, allows to express all scalar products
- $$D_i = (\{k, l\} + p_i)^2 - M_i^2$$

$$F[a_1, \dots, a_N] = \int d^d k d^d l \frac{1}{D_1^{a_1} \dots D_N^{a_N}}$$
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Z. Bern, L. J. Dixon and D. A. Kosower, Phys. Lett. B **302** (1993) 299.

V. A. Smirnov, Phys. Lett. B **460** (1999) 397

T. Gehrmann and E. Remiddi, Nucl. Phys. B **580** (2000) 485 [[hep-ph/9912329](#)].

J. M. Henn, Phys. Rev. Lett. **110** (2013) 25, 251601 [[arXiv:1304.1806 \[hep-th\]](#)].

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S. Borowka, G. Heinrich, S. P. Jones, M. Kerner, J. Schlenk and T. Zirke, Comput. Phys. Commun. **196** (2015) 470

S. Becker, C. Reuschle and S. Weinzierl, JHEP **1012** (2010) 013

IBPI: THE CURRENT APPROACH

- Find a better IBP algorithm ... Generating function technique, Baikov ?

P. A. Baikov, Nucl. Instrum. Meth. A **389** (1997) 347

V. A. Smirnov and M. Steinhauser, Nucl. Phys. B **672** (2003) 199

$$F_{a_1 \dots a_N} = \sum_{i=\text{masters}} c_{a_1 \dots a_N}^{(i)} G_i$$

- Baikov polynomial \leftrightarrow LZ construction
- Sector \leftrightarrow cut

$$\delta((k+p)^2 - m^2) \leftrightarrow \oint_{z=0} dz \frac{1}{z^{n=1}}$$

- Cut with higher powers in denominator

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K. J. Larsen and Y. Zhang, Phys. Rev. D 93 (2016) no.4, 041701

A. Georgoudis, K. J. Larsen and Y. Zhang, arXiv:1612.04252 [hep-th].

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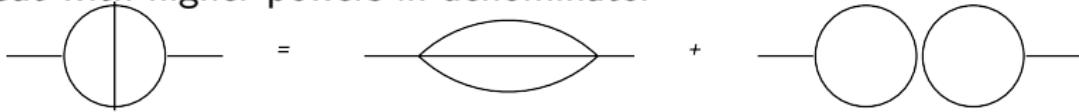
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$$F_{11111} = \frac{(3d-10)(3d-8)}{(d-4)^2(p^2)^2} F_{10011} + \frac{(3d-10)(3d-8)}{(d-4)^2(p^2)^2} F_{01101} - 2 \frac{(d-3)}{(d-4)p^2} F_{11110}$$

DIFFERENTIAL EQUATIONS APPROACH

The integral is a function of external momenta, so one can set-up differential equations by differentiating and using **IBP**

$$p_j^\mu \frac{\partial}{\partial p_i^\mu} G[a_1, \dots, a_n] \rightarrow \sum C_{b_1, \dots, b_n} F[b_1, \dots, b_n] \rightarrow \sum \textcolor{red}{C}_{a'_1, \dots, a'_n} G[a'_1, \dots, a'_n]$$

- **Find the proper parametrization:** Bring the system of equations in a form suitable to express the MI in terms of GPs

$$\begin{aligned}\partial_m f(\varepsilon, \{x_i\}) &= \varepsilon A_m(\{x_i\}) f(\varepsilon, \{x_i\}) \\ \partial_m A_n - \partial_n A_m &= 0 \quad [A_m, A_n] = 0\end{aligned}$$

★ f not MI!

J. M. Henn, Phys. Rev. Lett. 110 (2013) 25, 251601 [[arXiv:1304.1806 \[hep-th\]](https://arxiv.org/abs/1304.1806)].

- **Boundary conditions:** expansion by regions or regularity conditions.

B. Jantzen, A. V. Smirnov and V. A. Smirnov, Eur. Phys. J. C 72 (2012) 2139 [[arXiv:1206.0546 \[hep-ph\]](https://arxiv.org/abs/1206.0546)].

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DIFFERENTIAL EQUATIONS APPROACH

- Iterated Integrals

K. T. Chen, Iterated path integrals, Bull. Amer. Math. Soc. 83 (1977) 831

- Multiple Polylogarithms, Symbol algebra
- Goncharov Polylogarithms

$$\mathcal{G}(a_n, \dots, a_1, x) = \int_0^x dt \frac{1}{t - a_n} \mathcal{G}(a_{n-1}, \dots, a_1, t)$$

with the special cases, $\mathcal{G}(x) = 1$ and

$$\mathcal{G}\left(\underbrace{0, \dots, 0}_n, x\right) = \frac{1}{n!} \log^n(x)$$

- Shuffle algebra

DIFFERENTIAL EQUATIONS APPROACH

- Iterated Integrals
- Multiple Polylogarithms, Symbol algebra

A. B. Goncharov, M. Spradlin, C. Vergu and A. Volovich, Phys. Rev. Lett. **105** (2010) 151605.

C. Duhr, H. Gangl and J. R. Rhodes, JHEP **1210** (2012) 075 [arXiv:1110.0458 [math-ph]].

C. Bogner and F. Brown

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$$\mathcal{G}(a_1, a_2; x) \mathcal{G}(b_1; x) = \mathcal{G}(a_1, a_2, b_1; x) + \mathcal{G}(a_1, b_1, a_2; x) + \mathcal{G}(b_1, a_1, a_2; x)$$

THE SIMPLIFIED DIFFERENTIAL EQUATIONS APPROACH

C. G. Papadopoulos, JHEP 1407 (2014) 088

Making the whole procedure systematic (algorithmic) and straightforwardly expressible in terms of GPs.

- Introduce one parameter

$$G_{11\dots 1}(x) = \int \frac{d^d k}{i\pi^{d/2}} \frac{1}{(k^2)(k + \cancel{x} p_1)^2 (k + p_1 + p_2)^2 \dots (k + p_1 + p_2 + \dots + p_n)^2}$$

- Factorizing external momenta dependence:

$$x : (q_1 = xp_1, q_2 = p_{12} - xp_1, \dots) \rightarrow x \otimes (q_1 = p_1, q_2 = p_2, \dots)$$

- Now the integral as a function of x , allows to define a differential equation with respect to x , schematically given by

$$\frac{\partial}{\partial x} G_{11\dots 1}(x) = -\frac{1}{x} G_{11\dots 1}(x) + xp_1^2 G_{12\dots 1} + \frac{1}{x} G_{02\dots 1}$$

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Making the whole procedure systematic (algorithmic) and straightforwardly expressible in terms of GPs.

- Introduce one parameter

$$G_{11\dots 1}(x) = \int \frac{d^d k}{i\pi^{d/2}} \frac{1}{(k^2)(k + \cancel{x} p_1)^2 (k + p_1 + p_2)^2 \dots (k + p_1 + p_2 + \dots + p_n)^2}$$

- Factorizing external momenta dependence:

$$x : (q_1 = xp_1, q_2 = p_{12} - xp_1, \dots) \rightarrow x \otimes (q_1 = p_1, q_2 = p_2, \dots)$$

- Now the integral as a function of x , allows to define a differential equation with respect to x , schematically given by

$$\frac{\partial}{\partial x} G_{11\dots 1}(x) = -\frac{1}{x} G_{11\dots 1}(x) + xp_1^2 G_{12\dots 1} + \frac{1}{x} G_{02\dots 1}$$

5BOX - ONE LEG OFF-SHELL: ALL FAMILIES

C. G. Papadopoulos, D. Tommasini and C. Wever, arXiv:1511.09404 [hep-ph].

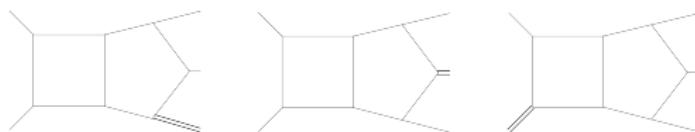


FIGURE : The three planar pentaboxes of the families P_1 (left), P_2 (middle) and P_3 (right) with one external massive leg.

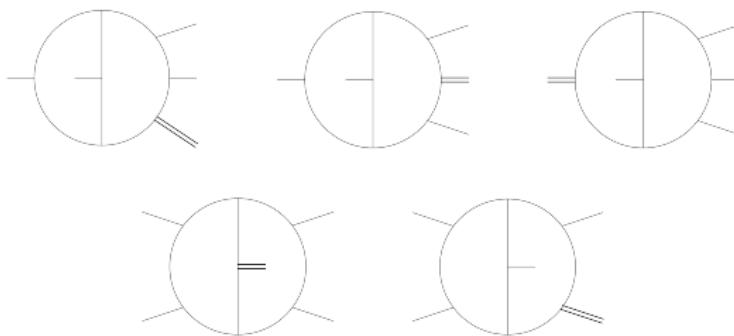


FIGURE : The five non-planar families with one external massive leg.

5BOX - ONE LEG OFF-SHELL: P1

$$p(q_1)p'(q_2) \rightarrow V(q_3)j_1(q_4)j_2(q_5), \quad q_1^2 = q_2^2 = 0, \quad q_3^2 = M_3^2, \quad q_4^2 = q_5^2 = 0.$$

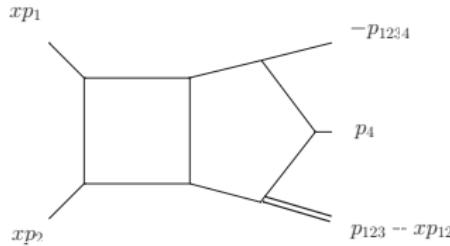


FIGURE : The parametrization of external momenta in terms of x for the planar pentabox of the family P_1 . All external momenta are incoming.

$$s_{12} := p_{12}^2, \quad s_{23} := p_{23}^2, \quad s_{34} := p_{34}^2, \quad s_{45} := p_{45}^2 = p_{123}^2, \quad s_{51} := p_{15}^2 = p_{234}^2,$$

$$q_1^2 = q_2^2 = q_4^2 = q_5^2 = 0 \quad q_3^2 = (s_{45} - s_{12}x)(1-x) \\ q_{12}^2 = s_{12}x^2 \quad q_{23}^2 = s_{45}(1-x) + s_{23}x \quad q_{34}^2 = (s_{34} - s_{12}(1-x))x \quad q_{45}^2 = s_{45} \quad q_{51}^2 = s_{51}x$$

5BOX - ONE LEG OFF-SHELL: P1

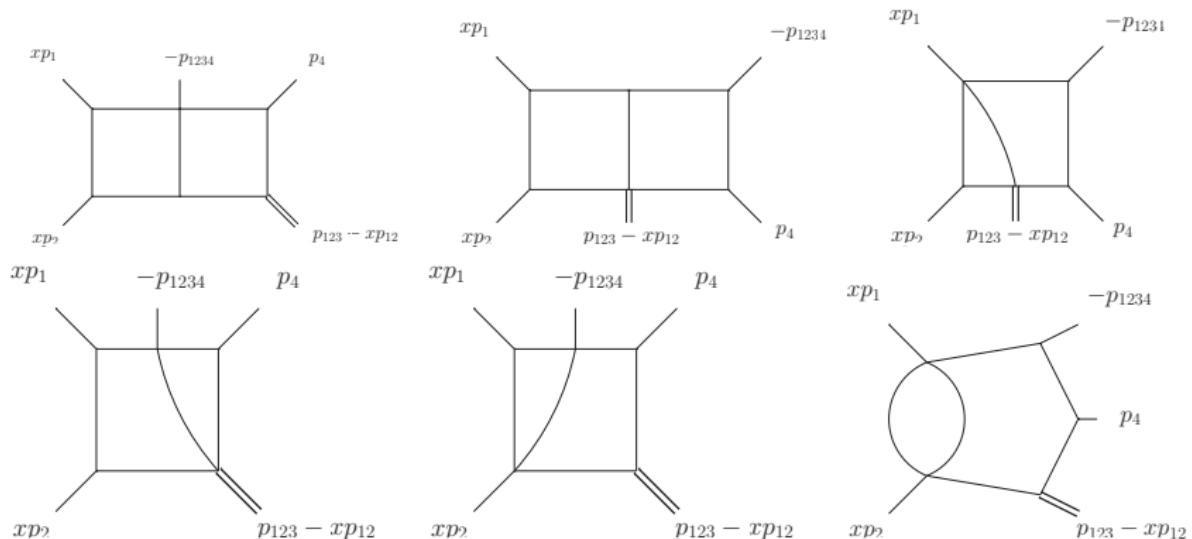


FIGURE : The five-point Feynman diagrams, besides the pentabox itself in Figure 5, that are contained in the family P_1 . All external momenta are incoming.

5BOX - ONE LEG OFF-SHELL: P1

$$G_{a_1 \dots a_{11}}^{P_1}(x, s, \epsilon) := e^{2\gamma_E \epsilon} \int \frac{d^d k_1}{i\pi^{d/2}} \frac{d^d k_2}{i\pi^{d/2}} \frac{1}{k_1^{2a_1} (k_1 + xp_1)^{2a_2} (k_1 + xp_{12})^{2a_3} (k_1 + p_{123})^{2a_4}} \\ \times \frac{1}{(k_1 + p_{1234})^{2a_5} k_2^{2a_6} (k_2 - xp_1)^{2a_7} (k_2 - xp_{12})^{2a_8} (k_2 - p_{123})^{2a_9} (k_2 - p_{1234})^{2a_{10}} (k_1 + k_2)^{2a_{11}}},$$

$P_1(74)$: {10000000101, 01000000101, 00100000101, 10000001001, 01000000011, 00100000011, 10100001100, 10100001010, 10100101000, 01000101001, 10100100100, 10100000102, 10100000101, 10100000011, 10000001102, 10000001101, 10000001011, 01000100101, 01000001101, 01000001011, 00100100102, 00100100101, 11100000101, 11100000011, 11000001102, 11000001101, 11000001012, 11000001011, 11000000111, 10100000112, 10000001111, 01100100102, 01100100101, 01100100011, 01100000111, 01000101102, 01000101101, 01000101011, 01000100111, 01000001111, 00100100111, 10100101100, 10100100101, 10100001101, 10100001011, 10100000111, 111m0000111, 110000m1111, 11000001111, 10100101110, 10100100111, 10100001111, 011001m0111, 01100100111, 010m0101111, 01000101111, 11100100101, 11100001101, 11100001011, 11100000111, 111m0101101, 111001m1101, 11100101101, 1110m1010111, 111001010111, 11100100111, 111000m1111, 111m0001111, 11100001111, 111001m01111, 11100101111, 111001m1111, 111m0101111},

$m = -1$

5BOX P1 - DE

$$\partial_x \mathbf{G} = \mathbf{M} (\{s_{ij}\}, \varepsilon, x) \mathbf{G}$$

$$\partial_x \mathbf{G}' = \mathbf{M}' \mathbf{G}' \quad \mathbf{M}' = \mathbf{T} \mathbf{M} \mathbf{T}^{-1} + (\partial_x \mathbf{T}) \mathbf{T}^{-1} \quad \mathbf{G}' = \mathbf{T} \mathbf{G}$$

$$(M_D)_{IJ} = \delta_{IJ} M_{II} (\varepsilon = 0), I, J = 1 \dots 74$$

$\mathbf{G} \rightarrow \mathbf{S}^{-1} \mathbf{G}$, $\mathbf{S} = \exp(\int dx \mathbf{M}_D)$ and $\mathbf{M} \rightarrow \mathbf{S}^{-1} (\mathbf{M} - \mathbf{M}_D) \mathbf{S}$.

$$M_{IJ} = N_{IJ}(\varepsilon) \left(\sum_{i=1}^{20} \sum_{j=1}^2 \sum_{k=0}^1 \frac{C_{IJ;ijk} \varepsilon^k}{(x - l_i)^j} + \sum_{j=0}^1 \sum_{k=0}^1 \tilde{C}_{IJ;jk} \varepsilon^k x^j \right).$$

Letters (20):

$$\begin{aligned}
& 0, \quad 1, \quad \frac{s_{45}}{s_{45}-s_{23}}, \quad \frac{s_{45}}{s_{12}}, \quad 1 - \frac{s_{34}}{s_{12}}, \quad 1 + \frac{s_{23}}{s_{12}}, \\
& 1 - \frac{s_{34}-s_{51}}{s_{12}}, \quad \frac{s_{45}-s_{23}}{s_{12}}, \quad -\frac{s_{51}}{s_{12}}, \quad \frac{s_{45}}{-s_{23}+s_{45}+s_{51}}, \quad \frac{s_{45}}{s_{34}+s_{45}}, \\
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$$\int_0^x dt \frac{1}{(t-a_n)^2} \mathcal{G}(a_{n-1}, \dots, a_1, t) \quad \quad \quad \int_0^x dt \ t^m \ \mathcal{G}(a_{n-1}, \dots, a_1, t)$$

Fuchsian

$$N_{IJ}(\varepsilon) = n_J(\varepsilon)/n_I(\varepsilon), \ G_I \rightarrow n_I(\varepsilon) \ G_I$$

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$$\mathbf{G} \rightarrow (\mathbf{I} - \mathbf{K}_i) \mathbf{G}, \quad \mathbf{M} \rightarrow (\mathbf{M} - \partial_x \mathbf{K}_i - \mathbf{K}_i \mathbf{M}) (\mathbf{I} - \mathbf{K}_i)^{-1} \quad i = 1, 2, 3$$

$$\partial_x \mathbf{G} = \left(\varepsilon \sum_{a=1}^{19} \frac{\mathbf{M}_a}{(x-l_a)} \right) \mathbf{G}$$

R. N. Lee, JHEP 1504 (2015) 108 [arXiv:1411.0911 [hep-ph]].

O. Gituliar and V. Magerya, Comput. Phys. Commun. 219 (2017) 329 [arXiv:1701.04269 [hep-ph]].

C. Meyer, Comput. Phys. Commun. 222 (2018) 295 [arXiv:1705.06252 [hep-ph]].



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$\mathbf{M}(\varepsilon = 0)$ contains $(x - l_i)^{-2}$ and x^0

$$(\mathbf{K}_1)_{IJ} = \begin{cases} \int dx (\mathbf{M}(\varepsilon = 0))_{IJ} & I, J \neq 69, 74 \\ 0 & I, J = 69, 74 \end{cases}$$

$$(\mathbf{K}_2)_{IJ} = \begin{cases} \int dx (\mathbf{M}(\varepsilon = 0))_{IJ} & I, J \neq 74 \\ 0 & I, J = 74 \end{cases}$$

$$(\mathbf{K}_3)_{IJ} = \int dx (\mathbf{M}(\varepsilon = 0))_{IJ}$$

M.A. Barkatou and E.Pflügel, Journal of Symbolic Computation, 44 (2009), 1017

$$\partial_x \mathbf{G} = \left(\varepsilon \sum_{a=1}^{19} \frac{\mathbf{M}_a}{(x - l_a)} \right) \mathbf{G}$$

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5BOX P1 - SOLUTION

- Solution:

$$\begin{aligned}\mathbf{G} = & \varepsilon^{-2} \mathbf{b}_0^{(-2)} + \varepsilon^{-1} \left(\sum \mathcal{G}_a \mathbf{M}_a \mathbf{b}_0^{(-2)} + \mathbf{b}_0^{(-1)} \right) \\ & + \varepsilon^0 \left(\sum \mathcal{G}_{ab} \mathbf{M}_a \mathbf{M}_b \mathbf{b}_0^{(-2)} + \sum \mathcal{G}_a \mathbf{M}_a \mathbf{b}_0^{(-1)} + \mathbf{b}_0^{(0)} \right) \\ & + \varepsilon \left(\sum \mathcal{G}_{abc} \mathbf{M}_a \mathbf{M}_b \mathbf{M}_c \mathbf{b}_0^{(-2)} + \sum \mathcal{G}_{ab} \mathbf{M}_a \mathbf{M}_b \mathbf{b}_0^{(-1)} + \sum \mathcal{G}_a \mathbf{M}_a \mathbf{b}_0^{(0)} + \mathbf{b}_0^{(1)} \right) \\ & + \varepsilon^2 \left(\sum \mathcal{G}_{abcd} \mathbf{M}_a \mathbf{M}_b \mathbf{M}_c \mathbf{M}_d \mathbf{b}_0^{(-2)} + \sum \mathcal{G}_{abc} \mathbf{M}_a \mathbf{M}_b \mathbf{M}_c \mathbf{b}_0^{(-1)} \right. \\ & \left. + \sum \mathcal{G}_{ab} \mathbf{M}_a \mathbf{M}_b \mathbf{b}_0^{(0)} + \sum \mathcal{G}_a \mathbf{M}_a \mathbf{b}_0^{(1)} + \mathbf{b}_0^{(2)} \right)\end{aligned}$$

$\mathbf{b}_0^{(k)}$, $k = -2, \dots, 2$ representing the x -independent boundary terms in the limit $x = 0$ at order ε^k

$$\mathbf{G} \underset{x \rightarrow 0}{\sim} \sum_{k=-2}^2 \varepsilon^k \sum_{n=0}^{k+2} \mathbf{b}_n^{(k)} \log^n(x) + \text{subleading terms.}$$

$\mathcal{G}_{a,b,\dots} = \mathcal{G}(l_a, l_b, \dots; x)$ with $a, b, c, d = 1, \dots, 19$.

- Uniform transcendental: UT multi- vs one-parameter DE

\mathbf{M}_a depend on kinematics, but eigenvalues not: $(x - l_a)^{-n_a \varepsilon}$, n_a positive integers, $x \rightarrow l_a$.

- Analytic continuation: F polynomial

C. G. Papadopoulos, D. Tommasini and C. Wever, JHEP 1501 (2015) 072 [arXiv:1409.6114 [hep-ph]].

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5BOX P1 - SOLUTION

- Solution:

$$\begin{aligned}\mathbf{G} = & \varepsilon^{-2} \mathbf{b}_0^{(-2)} + \varepsilon^{-1} \left(\sum \mathcal{G}_a \mathbf{M}_a \mathbf{b}_0^{(-2)} + \mathbf{b}_0^{(-1)} \right) \\ & + \varepsilon^0 \left(\sum \mathcal{G}_{ab} \mathbf{M}_a \mathbf{M}_b \mathbf{b}_0^{(-2)} + \sum \mathcal{G}_a \mathbf{M}_a \mathbf{b}_0^{(-1)} + \mathbf{b}_0^{(0)} \right) \\ & + \varepsilon \left(\sum \mathcal{G}_{abc} \mathbf{M}_a \mathbf{M}_b \mathbf{M}_c \mathbf{b}_0^{(-2)} + \sum \mathcal{G}_{ab} \mathbf{M}_a \mathbf{M}_b \mathbf{b}_0^{(-1)} + \sum \mathcal{G}_a \mathbf{M}_a \mathbf{b}_0^{(0)} + \mathbf{b}_0^{(1)} \right) \\ & + \varepsilon^2 \left(\sum \mathcal{G}_{abcd} \mathbf{M}_a \mathbf{M}_b \mathbf{M}_c \mathbf{M}_d \mathbf{b}_0^{(-2)} + \sum \mathcal{G}_{abc} \mathbf{M}_a \mathbf{M}_b \mathbf{M}_c \mathbf{b}_0^{(-1)} \right. \\ & \left. + \sum \mathcal{G}_{ab} \mathbf{M}_a \mathbf{M}_b \mathbf{b}_0^{(0)} + \sum \mathcal{G}_a \mathbf{M}_a \mathbf{b}_0^{(1)} + \mathbf{b}_0^{(2)} \right)\end{aligned}$$

$\mathbf{b}_0^{(k)}$, $k = -2, \dots, 2$ representing the x -independent boundary terms in the limit $x = 0$ at order ε^k

$$\mathbf{G} \underset{x \rightarrow 0}{\sim} \sum_{k=-2}^2 \varepsilon^k \sum_{n=0}^{k+2} \mathbf{b}_n^{(k)} \log^n(x) + \text{subleading terms.}$$

$\mathcal{G}_{a,b,\dots} = \mathcal{G}(I_a, I_b, \dots; x)$ with $a, b, c, d = 1, \dots, 19$.

- Uniform transcendental: UT multi- vs one-parameter DE

\mathbf{M}_a depend on kinematics, but eigenvalues not: $(x - I_a)^{-n_a \varepsilon}$, n_a positive integers, $x \rightarrow I_a$.

- Analytic continuation: F polynomial

THE $x = 1$ LIMIT

$$\mathbf{G} = \sum_{n \geq -2} \varepsilon^n \sum_{i=0}^{n+2} \frac{1}{i!} \mathbf{c}_i^{(n)} \log^i (1-x)$$

$\mathbf{c}_i^{(n)}$ are finite in the limit $x = 1$

$$\mathbf{c}_i^{(n)} = \mathbf{M}_2 \mathbf{c}_{i-1}^{(n-1)} \quad i \geq 1$$

$$\mathbf{G}_{reg} = \sum_{n \geq -2} \varepsilon^n \mathbf{c}_0^{(n)}.$$

$$\mathbf{G} = \mathbf{G}_{reg} + \frac{\left((1-x)^{-2\varepsilon} - 1 \right)}{(-2\varepsilon)} \mathbf{X} + \frac{\left((1-x)^{-\varepsilon} - 1 \right)}{(-\varepsilon)} \mathbf{Y}$$

$$\mathbf{X} = \sum_{n \geq -1} \varepsilon^n \mathbf{X}^{(n)} \quad \mathbf{Y} = \sum_{n \geq -1} \varepsilon^n \mathbf{Y}^{(n)}.$$

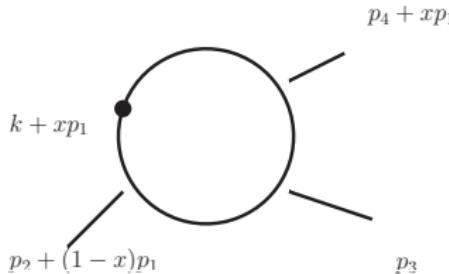
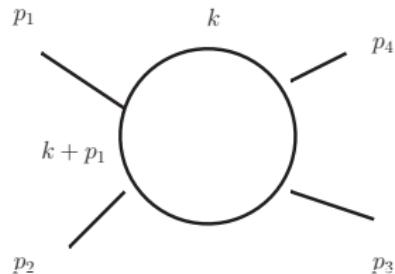
$$(-1)^n \mathbf{M}_2^n = \mathbf{M}_2^2 (2^{n-1} - 1) + \mathbf{M}_2 (2^{n-1} - 2), \quad n \geq 1.$$

minimal polynomial $x(x+1)(x+2)$ of the matrix \mathbf{M}_2

$$\mathbf{G}_{x=1} = \left(\mathbf{I} + \frac{3}{2} \mathbf{M}_2 + \frac{1}{2} \mathbf{M}_2^2 \right) \mathbf{G}_{trunc}$$

$\mathbf{G}_{trunc} \equiv \mathbf{G}_{reg}(x = 1).$

YET AN OTHER x ?



$$I = \int \frac{d^d k}{i\pi^{d/2}} \frac{1}{k^2(k + p_1)^2(k + p_{12})^2(k + p_{123})^2}$$

$$I = \underbrace{\int_0^1 dx \int \frac{d^d k}{i\pi^{d/2}} \frac{1}{((k + xp_1)^2)^2(k + p_{12})^2(k + p_{123})^2}}_{G_{211}}$$

YET AN OTHER x ?

$$G_{211} = -\frac{2(d-3)}{S_3(S_2-S_3)} G_{101} + \frac{2(d-3)}{S_2(S_2-S_3)} G_{110} = \frac{2}{\varepsilon} \left[\frac{(-s)^{-1-\varepsilon}(1-x)^{-1-\varepsilon}}{s(1-x)-tx} - \frac{(-t)^{-1-\varepsilon}x^{-1-\varepsilon}}{s(1-x)-tx} \right]$$

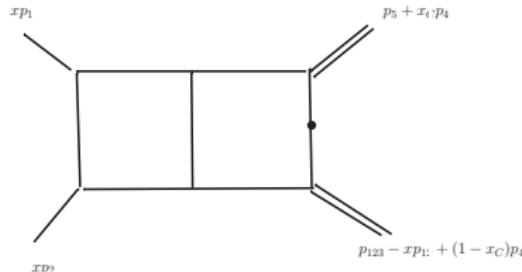
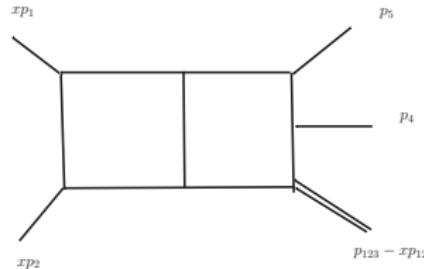
$$I = \int_0^1 dx G_{211}$$

$$= \frac{2}{\varepsilon^2} \frac{1}{st} \left[-(-s)^{-\varepsilon} \left(\frac{t}{s} \right) {}_2F_1 \left(1, 1; 1-\varepsilon; \frac{s+t}{s} \right) + (-t)^{-\varepsilon} {}_2F_1 \left(1, -\varepsilon; 1-\varepsilon; \frac{s+t}{s} \right) \right]$$

$$I = \frac{2}{\varepsilon^2} \frac{1}{st} \left[(-s)^{-\varepsilon} {}_2F_1 \left(1, -\varepsilon; 1-\varepsilon; \frac{s+t}{t} \right) + (-t)^{-\varepsilon} {}_2F_1 \left(1, -\varepsilon; 1-\varepsilon; \frac{s+t}{s} \right) \right]$$

Eq. (4.18) in Z. Bern, L. J. Dixon and D. A. Kosower, Nucl. Phys. B 412 (1994) 751 [hep-ph/9306240].

YET AN OTHER x ?



$$PB(x) = \int_0^1 dx_C DB(x, x_C)$$

Take care of the limits $x_C \rightarrow 1$ and $x_C \rightarrow 0$, as explained
At last, we verified 5box physical region result !

C. G. Papadopoulos & C. Wever in preparation

5BOX

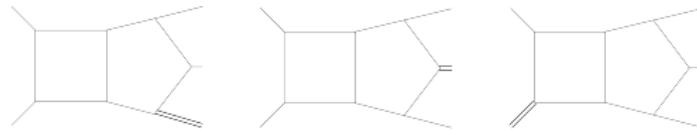


FIGURE : The three planar pentaboxes of the families P_1 (left), P_2 (middle) and P_3 (right) with one external massive leg.

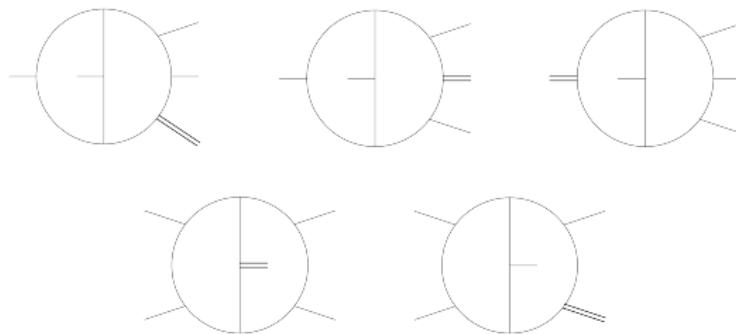


FIGURE : The five non-planar families with one external massive leg.

SUMMARY

- ➊ New IBP concepts and tools → reduce P+NP 5box MI and derive DE in x
- ➋ Baikov representation of cut integrals → establish the class of functions, MPL or EI
- ➌ Deriving canonical DE
- ➍ Solving and verifying by recycling known results → x_C
- ➎ All 5boxes with vanishing internal masses and up to one massive leg
- ➏ All 8-denominator MI, arbitrary internal masses, number of legs and external kinematics

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