Properties of Yang-Mills scattering forms

Leonardo de la Cruz

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collaboration with Alexander Kniss and Stefan Weinzierl

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Motivation

- Factorization of amplitudes may be encoded on an auxiliary space
- E.g., Moduli space of Riemann spheres with n marked points $\mathcal{M}_{0,n}$ (String theory amplitudes, Roiban-Spradlin-Volovich-Witten formalism, Cachazo-He-Yuan formalism)

$$\mathcal{A}_n(p,\varepsilon) = \mathrm{i}\oint_{\mathcal{O}} I(z,p,\varepsilon)\, \mathrm{d}\Omega_{\mathsf{CHY}}; \qquad f_i(z,p) \equiv \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{2p_i \cdot p_j}{z_i - z_j} = 0$$

- New approach by Arkani-Hamed, Bai, He, Lam, Yan, 2017 suggest rethink amplitudes as differential forms in positive kinematic space
- Differential forms can also be defined on auxiliary space (positive or full)

In this talk

- Tree-level n-point amplitudes, zero loops
- How the Yang-Mills scattering forms in auxiliary space looks like?
- What are properties of scattering forms?

Results

- Cyclic and polarization form on the full $\overline{\mathcal{M}}_{0,n}(\mathbb{C})$:

$$\Omega^{\text{cyclic}}(\sigma, z) \equiv C(\sigma, z)\frac{\mathrm{d}^n z}{\mathrm{d}\omega}; \qquad \Omega^{\text{pol}}(p, \varepsilon, z) \equiv E(\varepsilon, p, z)\frac{\mathrm{d}^n z}{\mathrm{d}\omega}$$

$$C(\sigma, z) = \frac{1}{z_{\sigma_1\sigma_2} z_{\sigma_2\sigma_3} \cdots z_{\sigma_n\sigma_1}}; \qquad E(p, \varepsilon, z) = \sum_{\kappa \in S_{n-2}^{(i,j)}} C(\kappa, z) N_{\text{comb}}^{\text{BCJ}}(\kappa)$$

$$\mathrm{d}\omega = (-1)^{p+q+r}\frac{\mathrm{d}z_p \mathrm{d}z_q \mathrm{d}z_r}{(z_p - z_q)(z_q - z_r)(z_r - z_p)}$$

- Properties:
  1. $\mathrm{PSL}(2, \mathbb{C})$ invariance
  2. Twisted intersection numbers give amplitudes [Mizera, 2017]

$$A_n = (\Omega^{\text{cyclic}}(\sigma, z), \Omega^{\text{pol}}(p, \varepsilon, z))_\eta \sim \text{CHY}$$

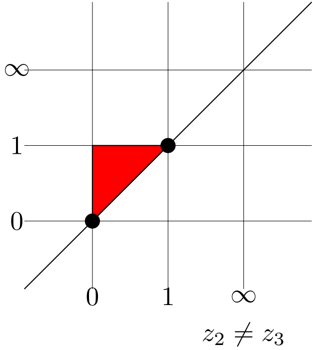
  3. Singularities on the $\overline{\mathcal{M}}_{0,n} \setminus \mathcal{M}_{0,n}$
  4. Logarithmic singularities
  5. Residues factorize into two scattering forms of lower points
- Analogous to $1, 3, 4, 5$ can be established on scattering forms on positive kinematic space [Arkani-Hamed, Bai, He, Lam, Yan, 2017]

Moduli space of genus zero curves

- $(n-3)$ dimensional affine variety

$$\mathcal{M}_{0,n}(\mathbb{C}) = \{(z_2, \dots, z_{n-2}) \in \mathbb{C}^{n-3} : z_i \neq z_j, z_i \neq 0, z_i \neq 1\}$$

- Example. $\mathbb{R}$eal part of $\mathcal{M}_{0,5}(\mathbb{C})$



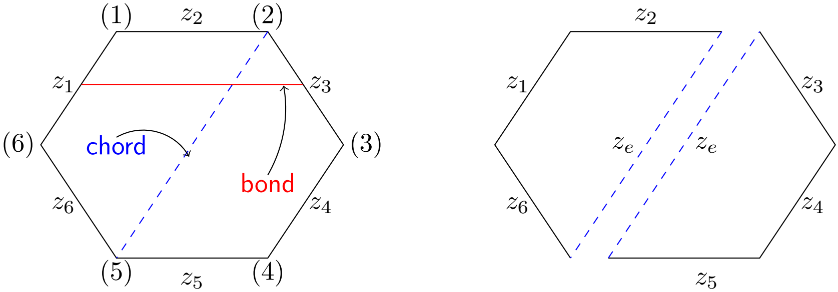
- Boundaries of red region do not cross *normally* at $(0,0)$, $(1,1)$ (left)
- We can blow them up so only two divisors meet at these points (right).

Dihedral coordinates and dihedral extension

- Dihedral structures (π, z) [Devadoss, 1998], [Brown, 2006]
- Chords$\{i,j\}$ define coordinates $u_{i,j} = \dfrac{(z_i - z_{j+1})(z_{i+1} - z_j)}{(z_i - z_j)(z_{i+1} - z_{j+1})}$
- Equations among $u_{i,j}$ define $\mathcal{M}^{\pi}_{0,z}$
- Gluing $\mathcal{M}^{\pi}_{0,z}$ gives [Deligne-Mumford-Knudsen] compactification

$$\overline{\mathcal{M}}_{0,z} = \bigcup_{\pi} \mathcal{M}^{\pi}_{0,z}$$

- Boundary $\mathcal{M}^{\pi}_{0,z} \backslash \mathcal{M}_{0,z}$ is a normal crossing divisor



Cyclic scattering form (I)

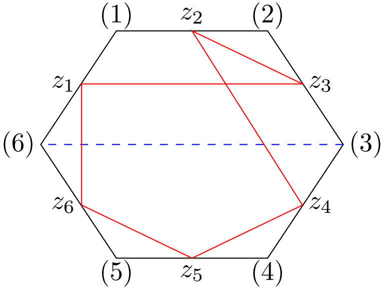
- Take $\pi = (1, 2, \ldots, n)$ and permutation σ

$$\Omega^{\text{cyclic}}(\sigma, z) = C(\sigma, z)\frac{\mathrm{d}^n z}{\mathrm{d}\omega}; \quad C(\sigma, z) = \frac{1}{z_{\sigma_1\sigma_2} z_{\sigma_2\sigma_3} \cdots z_{\sigma_n\sigma_1}}$$

- E.g., for $\pi = \sigma$

$$C(\pi, z)\frac{\mathrm{d}^n z}{\mathrm{d}\omega} = \prod_{j=2}^{n-2} \frac{1}{u_{j,n}(u_{j,n} - 1)} \mathrm{d}^{n-3} u$$

- What happens with $\Omega^{\text{cyclic}}(\sigma, z)$ when $u_{i_0,n} \to 0$?



we say $\sigma \sim_{(i_0,n)} \pi$
if two bonds cross the chord (i_0, n)

two bonds cross the chord $(3, 6)$

$(1, 3, 2, 4, 5, 6) \sim_{(3,6)} (1, 2, 3, 4, 5, 6)$

Cyclic scattering form (II)

- Number of bonds counts powers of $u_{i_0,n}$ in $\Omega^{\text{cyclic}}(\sigma,u)$
- Logarithmic singularities and residue factorization follows from bond analysis

1. Cyclic factor constructed of cross ratios $\Rightarrow$ PSL$(2,\mathbb{C})$ invariance
2. $m_n(\sigma,\tilde{\sigma}) = \mathrm{i}\oint_{\mathcal{C}} \mathrm{d}\Omega_{\text{CHY}} C(\sigma,z)C(\tilde{\sigma},z) = \mathrm{i}(\Omega,\tilde{\Omega})_\eta$, equivalent to CHY [Mizera, 2017]
3. $C(\sigma,z)$ singular when $z_{\sigma_i} = z_{\sigma_{i+1}}$, i.e., points on $\overline{\mathcal{M}}_{0,n} \setminus \mathcal{M}_{0,n}$
4. Analysis of bonds

$$
\begin{array}{cc}
\sigma \sim_{(i_0,n)} \pi & u_{i_0,n}^{1-i_0}\left(u_{i_0,n}^{i_0-2}\right) \\
\sigma \not\sim_{(i_0,n)} \pi & \text{fewer powers}
\end{array}
$$

5. Residue factorization(Hypersurface $u_{i_0,n}=0$ denoted by Y)

$$
\text{Res}_Y \Omega^{\text{cyclic}}(\sigma,z) = \begin{cases} (-1)^{i_0-1}\Omega^{\text{cyclic}}(\sigma',z)\wedge\Omega^{\text{cyclic}}(\sigma'',z), & \sigma \sim_{(i_0,n)} \pi, \\ 0, & \text{otherwise} \end{cases}
$$

Polarization scattering form

- Parke Taylor factor

$$\Omega^{\text{pol}}(p,\varepsilon,z) = \sum_{\kappa\in S_{n-2}^{(i,j)}} C(\kappa,z)\, N^{\text{BCJ}}_{\text{comb}}(\kappa)\, \frac{\mathrm{d}^n z}{\mathrm{d}\omega}$$

- BCJ numerator associated with comb graphs
- PSL$(2,\mathbb{C})$ invariant measure

$$C(\kappa,z) = \frac{1}{(z_{\kappa_1}-z_{\kappa_2})(z_{\kappa_2}-z_{\kappa_3})\cdots(z_{\kappa_n}-z_{\kappa_1})}$$

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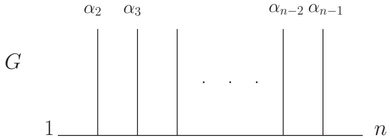
$$\kappa = (1,\alpha,n)$$



[Del Duca, Dixon, Maltoni, 1999]

Polarization scattering form

- Parke Taylor factor

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- BCJ numerator associated with comb graphs
- PSL$(2,\mathbb{C})$ invariant measure

$$\mathrm{d}^n z = \mathrm{d}z_1\cdots\mathrm{d}z_n, \qquad \mathrm{d}\omega = (-1)^{p+q+r}\frac{\mathrm{d}z_p\mathrm{d}z_q\mathrm{d}z_r}{(z_p-z_q)(z_q-z_r)(z_r-z_p)}$$

Comments on definition

- Polarization factor is different from the reduced Pfaffian on the full $\overline{\mathcal{M}}_{0,n}$

$$\frac{(-1)^{i+j}}{2z_{ij}}\mathrm{Pf}\Psi^{ij}_{ij} \quad \text{independent of } i,j \text{ on the support of scattering equations}$$

[CHY]

- Our definition
  1. Defined on the full $\overline{\mathcal{M}}_{0,n}$
  2. Factorization requires that in general

$$\varepsilon_i \cdot p_i \neq 0$$
$$p_j^2 \neq 0$$

- Factorization of residues requires factorization of $N^{\text{BCJ}}_{\text{comb}}(\kappa)$, what does it mean?

$$N(G) = \sum_{\substack{\text{particles} \\ \text{polarizations}}} N(G_1)N(G_2)$$

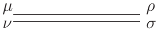
BCJ comb numerators

- $N^{\mathsf{BCJ}}(G)$ from effective lagrangian [Tolotti, Weinzierl, 2013]

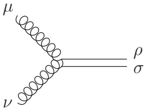
$$\mathscr{L} = \frac{1}{2g^2} \sum_{n=2}^{\infty} \mathscr{L}^{(n)}$$

$$\text{e.g. } \mathscr{L}^{(4)} = - g^{\mu_1 \mu_3} g^{\mu_2 \mu_4} g_{\nu_1 \nu_2} \frac{\partial_{12}^{\nu_1} \partial_{34}^{\nu_2}}{\Box_{12}} \mathrm{Tr}[\mathsf{A}_{\mu_1}, \mathsf{A}_{\mu_2}][\mathsf{A}_{\mu_3}, \mathsf{A}_{\mu_4}]$$

- Equivalent to introduce auxiliary particles [Draggiotis,Kleiss, Papadopoulos, 1998], [Duhr, Hoche, Maltoni, 2006]

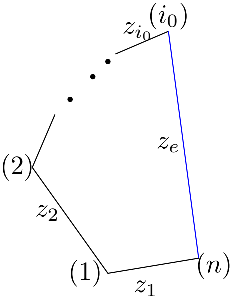


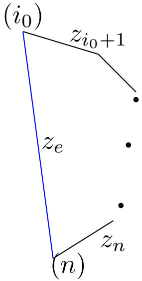
$$= -\frac{\mathrm{i}}{2}(g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho})$$



$$= \frac{\mathrm{i}}{\sqrt{2}}(g^{\mu\rho} g^{\nu\sigma} - g^{\mu\sigma} g^{\nu\sigma})$$

Factorization of data in a polygon





- $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{i_0}, \varepsilon_e)$
- $p' = (p_1, p_2, \ldots, p_{i_0}, p_q)$
- $\kappa' = (1, \kappa'_2, \ldots, \kappa'_{i_0}, e)$

- $\varepsilon' = (\varepsilon_e^*, \varepsilon_{i_0+1}, \ldots, \varepsilon_n)$
- $p'' = (\overline{p_q}, p_{i_0+1}, p_{i_0+2}, \ldots, p_n)$
- $\kappa'' = (e, \kappa''_{i_0+1}, \ldots, \kappa''_{n-1}, n)$

Introduce unphysical polarizations such that

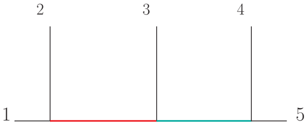
$$\sum_\lambda (\varepsilon_\mu^\lambda)^* \varepsilon_\nu^\lambda = -g_{\mu\nu}, \qquad \sum_\lambda (\varepsilon_{\mu\nu}^\lambda)^* \varepsilon_{\rho\sigma}^\lambda = -\frac{1}{2} p^2 (g_{\mu\rho} g_{\nu\rho} - g_{\mu\sigma} g_{\nu\rho})$$

Factorization of numerators

- With these definitions

$$N(G) = \sum_{f,\lambda} N(G_1) N(G_2)$$

- Similar story for $n > 4$



- $n = 4$ example

$$\Omega^{\mathsf{pol}}(p, \varepsilon, u) = \left[\frac{N^{\mathsf{BCJ}}_{\mathsf{comb}}((1,2,3,4))}{u_{2,4}(u_{2,4}-1)} - \frac{N^{\mathsf{BCJ}}_{\mathsf{comb}}((1,3,2,4))}{u_{2,4}-1} \right] \mathrm{d}u_{2,4}$$

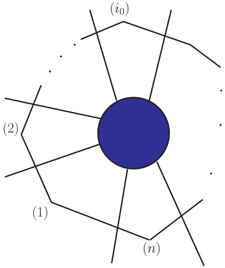
Properties of polarization scattering forms

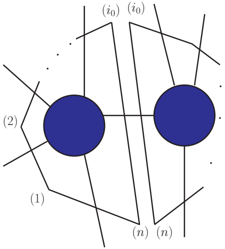
- Polarization factor

$$E(p,\varepsilon,z) = \sum_{\kappa\in S_{n-2}^{(i,j)}} C(\kappa,z)\mathsf{N}_{\mathsf{comb}}^{\mathsf{BCJ}}(\kappa)$$

gives good definition of a polarization factor.

1. Permutation invariant
2. $C(\kappa, z)$ gives properties 1,3,4
3. $\mathrm{i}\oint \mathrm{d}\Omega_{\mathsf{CHY}} C(\sigma,z)E(p,\varepsilon,z)$ has to be equal to Yang-Mills amplitude





Property 5. Residue factorization

1. Numerators: $N^{\text{BCJ}}_{\text{comb}}((\kappa',\kappa'')) = \sum_{f,\lambda} N^{\text{BCJ}}_{\text{comb}}(\kappa') N^{\text{BCJ}}_{\text{comb}}(\kappa'')$
2. Cyclic forms : $\text{Res}_Y \Omega_{\text{cyclic}}(\sigma, z) = (-1)^{i_0 - 1} \Omega_{\text{cyclic}}(\kappa', z) \wedge \Omega_{\text{cyclic}}(\kappa'', z)$

$\Rightarrow$

$$\text{Res}_Y \Omega_{\text{pol}}(p,\varepsilon,z) = (-1)^{i_0-1} \sum_{f,\lambda} \sum_{\kappa',\kappa''} N^{\text{BCJ}}_{\text{comb}}(\kappa')\Omega_{\text{cyclic}}(\kappa', z) \wedge N^{\text{BCJ}}_{\text{comb}}(\kappa'')\Omega_{\text{cyclic}}(\kappa'', z)$$

$$\text{Res}_Y \Omega_{\text{pol}}(p,\varepsilon,z) = \sum_{f,\lambda} (-1)^{i_0-1} \Omega_{\text{pol}}(p',\varepsilon',z) \wedge \Omega_{\text{pol}}(p'',\varepsilon'',z)$$

Example. $(n = 4,\ Y = u_{2,4} = 0)$

$$\text{Res}_Y \Omega^{\text{pol}}(p,\varepsilon,z) = -1 \sum_{f,\lambda} \Omega^{\text{pol}}(p_1,p_2,p_e,\varepsilon_1,\varepsilon_2,\varepsilon_e,z)\Omega^{\text{pol}}(p_e,p_3,p_4,\varepsilon^*_e,\varepsilon_3,\varepsilon_4,z)$$

Summary and Outlook

- We studied scattering forms Ω^{cyclic} and Ω^{pol} on the complete $(n-3)$ dimensional space $\overline{\mathcal{M}}_{0,n}$ away from solutions of scattering equations
- Polarization form away from solutions of scattering equations force us to introduce some non-physical polarizations
- Properties 1-5 builds bridge from CHY formalism to scattering forms on kinematic space
- Clear geometric picture of tree-level within bi-adjoint, Yang-Mills and gravity for any number of external particles
- Connection between scattering forms in kinematic space and forms in full $\overline{\mathcal{M}}_{0,n}$ yet to be understood...
- Extension to other theories with CHY representation ... loops

. . . Thanks