

# Feynman integral relations from parametric annihilators

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Loops & Legs, St. Goar

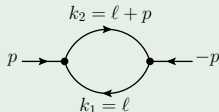
joint work with **Thomas Bitoun, Christian Bogner, René Pascal Klausen**  
[arXiv:1712.09215]

An **integral family** is defined by a set of denominators  $D_1, \dots, D_N$  that are quadratic (or linear) forms in loop momenta  $\ell_1, \dots, \ell_L$ :

$$\mathcal{I}(\nu_1, \dots, \nu_N; d) = \left( \prod_{k=1}^L \int \frac{d^d \ell_k}{i\pi^{d/2}} \right) \prod_{a=1}^N D_a^{-\nu_a}$$

### Example

$$\mathcal{I}(\nu_1, \nu_2; d) = \int \frac{d^d \ell}{i\pi^{d/2}} \frac{1}{(\ell^2)^{\nu_1} ((\ell + p)^2)^{\nu_2}}$$



A family is also described by a matrix  $\Lambda$ , vectors  $Q_i$  and a scalar  $J$  such that

$$\sum_{a=1}^N x_a D_a = - \sum_{i,j=1}^L \Lambda_{ij} (\ell_i \cdot \ell_j) + \sum_{i=1}^L 2(Q_i \cdot \ell_i) + J$$

Associated polynomials:  $\mathcal{U} := \det \Lambda$ ,  $\mathcal{F} := \mathcal{U} (Q^T \Lambda^{-1} Q + J)$

In terms of  $\omega := \nu_1 + \dots + \nu_N - L\frac{d}{2}$  and  $\mathcal{G} := \mathcal{U} + \mathcal{F}$  (Lee-Pomeransky),

$$\mathcal{I}(\nu) = \frac{\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{d}{2} - \omega\right)} \left( \prod_{i=1}^N \int_0^\infty \frac{x_i^{\nu_i-1} dx_i}{\Gamma(\nu_i)} \right) \mathcal{G}^{-d/2}$$

### Example

$$\mathcal{I}(\nu_1, \nu_2) = \frac{\Gamma\left(\frac{d}{2}\right)}{\Gamma(d - \nu_1 - \nu_2)} \int_0^\infty \frac{x_1^{\nu_1-1} dx_1}{\Gamma(\nu_1)} \int_0^\infty \frac{x_2^{\nu_2-1} dx_2}{\Gamma(\nu_2)} \left( \underbrace{x_1 + x_2}_{\mathcal{U}} - \underbrace{p^2 x_1 x_2}_{\mathcal{F}} \right)^{-\frac{d}{2}}$$

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The Mellin transform of a function  $f: \mathbb{R}_+^N \rightarrow \mathbb{C}$  is

$$\mathcal{M}\{f\}(\nu) := \left( \prod_{i=1}^N \int_0^\infty \frac{x_i^{\nu_i-1} dx_i}{\Gamma(\nu_i)} \right) f(x_1, \dots, x_N),$$

whenever this integral exists. Special case:

$$\mathcal{I}(\nu) = \frac{\Gamma(d/2)}{\Gamma(d/2 - \omega)} \tilde{\mathcal{I}}(\nu) \quad \text{for} \quad \tilde{\mathcal{I}}(\nu) = \mathcal{M}\{\mathcal{G}^{-d/2}\}(\nu).$$

## Speer

- Such integrals converge in a non-empty, open domain wrt  $(d, \nu)$ .
- They have a unique, meromorphic extension to  $\mathbb{C}^{1+N}$ .
- The poles are simple and located on rational hyperplanes.

⇒ To prove relations between regularized Feynman integrals, we may assume convergent values of the parameters.

## Example

$$\mathcal{I}(\nu_1, \nu_2) = (-p^2)^{d/2 - \nu_1 - \nu_2} \frac{\Gamma(d/2 - \nu_1)\Gamma(d/2 - \nu_2)\Gamma(\nu_1 + \nu_2 - d/2)}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(d - \nu_1 - \nu_2)}$$

Poles:  $\{d/2 - \nu_1 = k\} \cup \{d/2 - \nu_2 = k\} \cup \{\nu_1 + \nu_2 - d/2 = k\}; k \in \mathbb{Z}_{\leq 0}$

(If  $d$  is the only regulator ( $\nu \in \mathbb{Z}^N$ ), poles coalesce and cease to be simple.)

# Properties of the Mellin transform

$$\textcircled{1} \quad \mathcal{M}\{\alpha f + \beta g\}(\nu) = \alpha \mathcal{M}\{f\}(\nu) + \beta \mathcal{M}\{g\}(\nu) \quad (\alpha, \beta \in \mathbb{C})$$

$$\textcircled{2} \quad \mathcal{M}\{x_i f\}(\nu) = \nu_i \mathcal{M}\{f\}(\nu + \mathbf{e}_i)$$

$$\int_0^\infty \frac{x_i^{\nu_i-1} dx_i}{\Gamma(\nu_i)} (x_i f) = \int_0^\infty \frac{\nu_i x_i^{\nu_i} dx_i}{\nu_i \Gamma(\nu_i)} f = \int_0^\infty \frac{\nu_i x_i^{(\nu_i+1)-1} dx_i}{\Gamma(\nu_i+1)} f$$

$$\textcircled{3} \quad \mathcal{M}\{-\partial_i f\}(\nu) = \mathcal{M}\{f\}(\nu - \mathbf{e}_i)$$

$$\int_0^\infty \frac{x_i^{\nu_i-1} dx_i}{\Gamma(\nu_i)} (-\partial_i f) = - \left[ \frac{x_i^{\nu_i-1}}{\Gamma(\nu_i)} f \right]_{x_i=0}^\infty + \int_0^\infty \frac{x_i^{\nu_i-2} dx_i}{\Gamma(\nu_i-1)} f$$

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Shift operators:

$$(\mathbf{i}^- F)(\nu) := F(\nu - \mathbf{e}_i)$$

$$(\mathbf{n}_i F)(\nu) = \nu_i F(\nu) \text{ for}$$

$$(\hat{\mathbf{i}}^+ F)(\nu) := \nu_i F(\nu + \mathbf{e}_i)$$

$$\mathbf{n}_i := \hat{\mathbf{i}}^+ \mathbf{i}^-$$

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Given any differential operator  $P \in A^N[d]$  in the Weyl algebra

$$A^N[d] := \mathbb{C}[d] \langle x_1, \dots, x_N, \partial_1, \dots, \partial_N \mid [\partial_i, x_j] = \delta_{i,j} \rangle$$

such that  $P \bullet \mathcal{G}^{-d/2} = 0$  (**annihilator**), the substitutions

$$x_i \mapsto \hat{\mathbf{i}}^+, \quad \partial_i \mapsto -\mathbf{i}^-, \quad x_i \partial_i \mapsto -\mathbf{n}_i$$

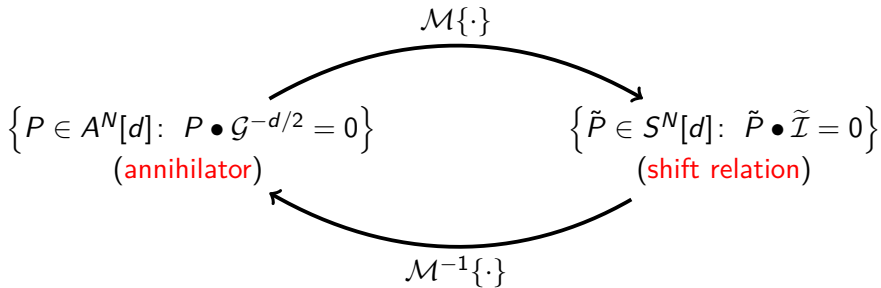
define a shift operator  $\mathcal{M}\{P\} \in S^N[d]$  in the shift algebra

$$S^N[d] := \mathbb{C}[d] \langle \hat{\mathbf{1}}^+, \dots, \hat{\mathbf{N}}^+, \mathbf{1}^-, \dots, \mathbf{N}^- \mid [-\mathbf{j}^-, \hat{\mathbf{i}}^+] = \delta_{i,j} \rangle$$

such that  $\mathcal{M}\{P\} \bullet \mathcal{M}\{\mathcal{G}^{-d/2}\} = \mathcal{M}\{P \bullet \mathcal{G}^{-d/2}\} = 0$  (**relation**).

**Example** ( $\mathcal{G} = x_1 + x_2 - p^2 x_1 x_2$ )

- ①  $[(-p^2)(-d/2 - x_1 \partial_1 + 1)x_1 + (-d/2 - x_1 \partial_1 - x_2 \partial_2)] \bullet \mathcal{G}^{-d/2} = 0$
- ②  $(-p^2)(-d/2 + \mathbf{n}_1 + 1)\hat{\mathbf{1}}^+ \tilde{\mathcal{I}} = -(-d/2 + \mathbf{n}_1 + \mathbf{n}_2)\tilde{\mathcal{I}}$
- ③  $(-p^2)\nu_1 \tilde{\mathcal{I}}(\nu_1 + 1, \nu_2) = -\frac{-d/2 + \nu_1 + \nu_2}{-d/2 + \nu_1 + 1} \tilde{\mathcal{I}}(\nu_1, \nu_2)$



The inverse Mellin transform of  $f^*(\nu) := \mathcal{M}\{f\}(\nu)$  is

$$f(x) = \mathcal{M}^{-1}\{f^*\}(x) = \left( \prod_{i=1}^N \int_{\sigma_i + i\mathbb{R}} \frac{\Gamma(\nu_i) d\nu_i}{(2\pi i) x_i^{\nu_i}} \right) f^*(\nu).$$

Therefore, **every** shift relation comes from an annihilator.

# Open problems

A finite list of generators for all annihilators (i.e. IBP relations) can sometimes be computed with SINGULAR.

## Question 1

Is the annihilator of  $\mathcal{G}^{-d/2}$  linearly generated?

For a full set of ISPs, the action of  $\frac{\partial}{\partial q_i} \cdot q_j$  on the *momentum space* integrand leads to IBP relations that map to linear annihilators  $\tilde{\mathbf{O}}_j^i$ .

## Question 2

Do the momentum space IBP's generate all annihilators (relations)?

No counterexamples found, but only few cases tested.

We define the **number of master integrals** of  $\tilde{\mathcal{I}}(\nu) = \mathcal{M}\{\mathcal{G}^{-d/2}\}(\nu)$  as

$$\mathfrak{e}(\mathcal{G}) := \dim_{\mathbb{C}(d,\nu)} \left( \sum_{n \in \mathbb{Z}^N} \mathbb{C}(d,\nu) \cdot \tilde{\mathcal{I}}(\nu + n) \right)$$

We define the **number of master integrals** of  $\tilde{\mathcal{I}}(\nu) = \mathcal{M}\{\mathcal{G}^{-d/2}\}(\nu)$  as

$$\mathfrak{C}(\mathcal{G}) := \dim_{\mathbb{C}(d,\nu)} \left( \sum_{n \in \mathbb{Z}^N} \mathbb{C}(d,\nu) \cdot \tilde{\mathcal{I}}(\nu + n) \right)$$

- $\mathfrak{C}(\mathcal{G}) \geq \dim_{\mathbb{C}(d)} \left( \sum_{n \in \mathbb{Z}^N} \mathbb{C}(d) \cdot \tilde{\mathcal{I}}(n) \right)$
- no symmetries
- not “modulo subtopologies”
- **exactly computable!**

Using the Mellin transform,  $\theta_i := x_i \partial_i = \mathcal{M}^{-1}\{\mathbf{n}_i\}$ ,

$$\mathfrak{C}(\mathcal{G}) = \dim_{\mathbb{C}(d,\theta)} \left( \underbrace{\mathbb{C}(d,\theta) \otimes_{\mathbb{C}[d,\theta]} A^N[d] \mathcal{G}^{-d/2}}_{\mathfrak{M}} \right)$$

Here,  $A^N(d) \mathcal{G}^{-d/2}$  is a holonomic  $D$ -module, and  $\mathfrak{M}$  is a holonomic system of finite difference equations [Loeser & Sabbah '91].

## Theorem

$$(-1)^N \mathfrak{C}(\mathcal{G}) = \chi\left(\mathbb{C}^N \setminus \{x_1 \cdots x_N \mathcal{G} = 0\}\right) = \chi\left((\mathbb{C}^*)^N \setminus \{\mathcal{G} = 0\}\right)$$

$\Rightarrow$  implies finiteness [Smirnov & Petukhov]

The Euler characteristic  $\chi(X) = \sum_i (-1)^i \dim H^i(X)$  is a fundamental invariant and can be computed with many different tools, for example:

- $\chi(X) = \chi(X \setminus Z) + \chi(Z)$
- $\chi(X \times Y) = \chi(X) \cdot \chi(Y)$
- $\chi(E) = \chi(B) \cdot \chi(F)$  for fibrations  $F \rightarrow E \rightarrow B$
- $D$ -modules and Groebner bases (e.g. SINGULAR) [Oaku & Takayama]
- algorithms by M. Helmer (CharacteristicClasses in Macaulay2)
- Kouchnirenko/Khovanskii's theorem: For *non-degenerate*  $\mathcal{G}$ ,

$$\mathfrak{C}(\mathcal{G}) = N! \cdot \text{Vol NP}(\mathcal{G})$$

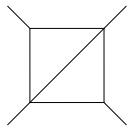
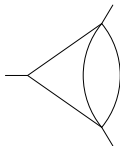
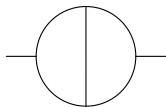
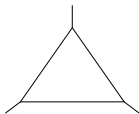
For some infinite families one can prove explicit formulas:

$$\mathfrak{e} \left( \text{---} \left( \text{---} \bigcirc \text{---} \right) \text{---} \right) = \mathfrak{e} \left( \text{---} \left( \text{---} \bigcirc \text{---} \right) \text{---} \right) = \frac{L(L+1)}{2}$$

$$\mathfrak{e} \left( \text{---} \left( \text{---} \bigcirc \text{---} \right) \text{---} \right) = 2^{L+1} - 1 \quad [\text{Kalmykov \& Kniehl}]$$

Plenty of further computations agreed with predictions by AZURITE, e.g.

Graph  $G$



$\mathfrak{C}(G)$  massless

4

3

4

20

$\mathfrak{C}(G)$  massive

7

30

19

55

# Massive one-loop sunrise

$$\mathcal{U} = x_1 + x_2 \quad \mathcal{F} = (x_1 + x_2)^2 + x_1 x_2$$

In *Macaulay2*, the Euler characteristic  $\mathfrak{C}(\mathcal{G}) = 3$  can be computed with

```
load "CharacteristicClasses.m2"  
R=QQ[x0,x1,x2]  
I=ideal(x0*x1*x2*((x1+x2)*x0+(x1+x2)^2+x1*x2))  
Euler(I)
```

The individual cohomology groups can also be obtained with

```
load "Dmodules.m2"  
R=QQ[x1,x2]  
f=x1*x2*(x1+x2+(x1+x2)^2+x1*x2)  
deRham f
```

$$\Rightarrow H^0(X) \cong \mathbb{Q}, \quad H^1(X) \cong \mathbb{Q}^3, \quad H^2(X) \cong \mathbb{Q}^5 \quad \Rightarrow \chi(X) = 5 - 3 + 1 = 3$$

The same can be done in SINGULAR.



# Thanks

Thank you for your attention!

- The Mellin transform translates IBP relations to annihilators [Tkachov, Baikov, Lee, Pomeransky].
- Algorithms for computations with  $D$ -modules are available.
- Application: The number of master integrals, for free  $\nu$ 's, is

$$\mathfrak{e}(\mathcal{G}) = (-1)^N \chi((\mathbb{C}^*)^N \setminus \{\mathcal{G} = 0\}) < \infty$$

- Goal: Extend IBP reduction from  $\nu \in \mathbb{Z}^N$  to free  $\nu$ .

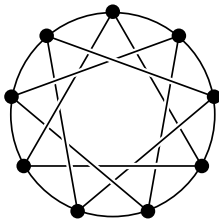
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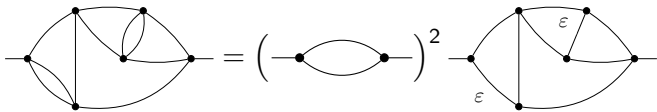
# Parametric representations

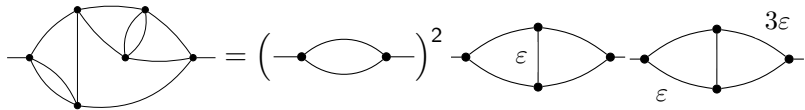
$$\omega := \nu_1 + \dots + \nu_N - L \frac{d}{2}$$

$$\mathcal{I}(\nu_1, \dots, \nu_N) = \left( \prod_{i=1}^N \int_0^\infty \frac{x_i^{\nu_i-1} dx_i}{\Gamma(\nu_i)} \right) \frac{e^{-\mathcal{F}/\mathcal{U}}}{\mathcal{U}^{d/2}},$$

$$\mathcal{I}(\nu_1, \dots, \nu_N) = \Gamma(\omega) \left( \prod_{i=1}^N \int_0^\infty \frac{x_i^{\nu_i-1} dx_i}{\Gamma(\nu_i)} \right) \frac{\delta(1 - \sum_{j=1}^N x_j)}{\mathcal{U}^{d/2-\omega} \mathcal{F}^\omega}$$

$$\mathcal{I}(\nu_1, \dots, \nu_N) = \frac{\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{d}{2} - \omega\right)} \left( \prod_{i=1}^N \int_0^\infty \frac{x_i^{\nu_i-1} dx_i}{\Gamma(\nu_i)} \right) \mathcal{G}^{-d/2}.$$





# Miscounting in Azurite

