Loop calculations for models of graphene

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Graphene background

There has been a growing theoretical interest in recent years for describing the properties of phase transitions in graphene using continuum quantum field theories [Herbut et al]

Graphene is a sheet of carbon which is one atom thick; it is incredibly strong and is an excellent electrical conductor

Electrons are located at the corners of a honeycomb or hexagonal lattice and can be studied with lattice methods or spin models

It has several interesting properties such as stretching the sheet can effect a quantum phase change from a conductor to a Mott-insulating phase

Such properties have the potential to lead graphene based electronics

Graphene and particle physics

Equally the Mott transition could be a mimic of spontaneous symmetry breaking

One particular phase transition can be described by a 4-fermi theory known as the Gross-Neveu model with a Yukawa interaction

This phase transition connection is via the Wilson-Fisher fixed points of the continuum field theories

Versions of the *same* underlying theory have been used in the past for ideas in the Standard Model such as the composite Higgs construction of Kuti, A Hasenfratz, P Hasenfratz et al

The main graphene examples are based on the Gross-Neveu model but with variations in the interaction; specifically the $SU(4)$ theory relates to a particular electronic phase transition

Gross-Neveu model

The Gross-Neveu (GN) model is a renormalizable quantum field theory in two dimensions based on a 4-fermi interaction

The original Lagrangian is

$$L^{\rm GN} ~=~ i \bar{\psi}^i \partial\!\!\!/ \psi^i ~-~ m \bar{\psi}^i \psi^i ~+~ \frac{g^2}{2} \left(\bar{\psi}^i \psi^i \right)^2$$

or

$$L^{\rm GN} ~=~ i \bar{\psi}^i \partial\!\!\!/ \psi^i ~-~ m \bar{\psi}^i \psi^i ~+~ g \tilde{\sigma} \bar{\psi}^i \psi^i ~-~ \frac{1}{2} \tilde{\sigma} \tilde{\sigma}$$

which is sometimes called the Ising Gross-Neveu model

The σ field is an auxiliary and perturbatively has a unit propagator

Perturbative computations can be carried out in either Lagrangian formulation but neither is technically straightforward

Four loop $\overline{\mathrm{MS}}$ β-function now known [Gracey, Luthe, Schröder]

$$\begin{aligned}\beta(g) &= (d-2)g - (N-1)\frac{g^2}{\pi} + (N-1)\frac{g^3}{2\pi^2} + (N-1)(2N-7)\frac{g^4}{16\pi^3} \\ &\quad + (N-1)\left[-2N^2 - 19N + 24 - 6\zeta_3(11N-17)\right]\frac{g^5}{48\pi^4} + O(g^6)\end{aligned}$$

which required evaluating 1190 4-point graphs at four loops

Agrees with ϵ expansion of large N exponents of Vasil'ev et al and Gracey near two dimensions

Lagrangian is not multiplicatively renormalizable when dimensionally regularized due to generation of evanescent interactions; their effect in the renormalization is handled through the projection formalism of Bondi et al

The auxiliary field formulation allows one to connect the theory with higher dimensional Gross-Neveu-Yukawa (GNY) models through ultraviolet completion [Zinn-Justin]

Both theories have a common interaction $\tilde{\sigma}\bar{\psi}^i\psi^i$

Gross-Neveu-Yukawa models

The corresponding renormalizable four dimensional Gross-Neveu-Yukawa (GNY) model is

$$L^{\text{GNY}} = i\bar{\psi}^i \partial\!\!\!/ \psi^i - m\bar{\psi}^i\psi^i + \frac{1}{2}\partial_\mu\tilde{\sigma}\partial^\mu\tilde{\sigma} + \frac{1}{2}g_1\tilde{\sigma}\bar{\psi}^i\psi^i + \frac{1}{24}g_2^2\left(\tilde{\sigma}\tilde{\sigma}\right)^2$$

Both the GN and GNY models lie in the same universality class at the Wilson-Fisher fixed point in $2 < d < 4$

For graphene the main interest is the three dimensional theory and its critical exponents and specfically η, η_ϕ and $1/\nu$

In general estimates for critical exponents in three dimensions are provided by various methods such as Monte Carlo, functional renormalization group, summation of the ϵ expansion from two or four dimensional renormalization group functions or large N methods

For certain values of N the Yukawa theories have emergent supersymmetry; the critical couplings for g_1 and g_2 are equal

Extension of Gross-Neveu model

Specific graphene transitions are driven by a variation on the original Gross-Neveu model called the chiral Heisenberg-Gross-Neveu (cHGN) given by

$$L^{\mathrm{cHGN}} = i\bar{\psi}^i \partial\!\!\!/ \psi^i - m\bar{\psi}^i\psi^i + \frac{g^2}{2}\left(\bar{\psi}^i\sigma^a\psi^i\right)^2$$

or

$$L^{\mathrm{cHGN}} = i\bar{\psi}^i \partial\!\!\!/ \psi^i - m\bar{\psi}^i\psi^i + g\tilde{\pi}^a\bar{\psi}^i\sigma^a\psi^i - \frac{1}{2}\tilde{\pi}^a\tilde{\pi}^a$$

where σ^a are the Pauli matrices

Its ultraviolet completion is called the chiral Heisenberg-Gross-Neveu-Yukawa (cHGNY) model

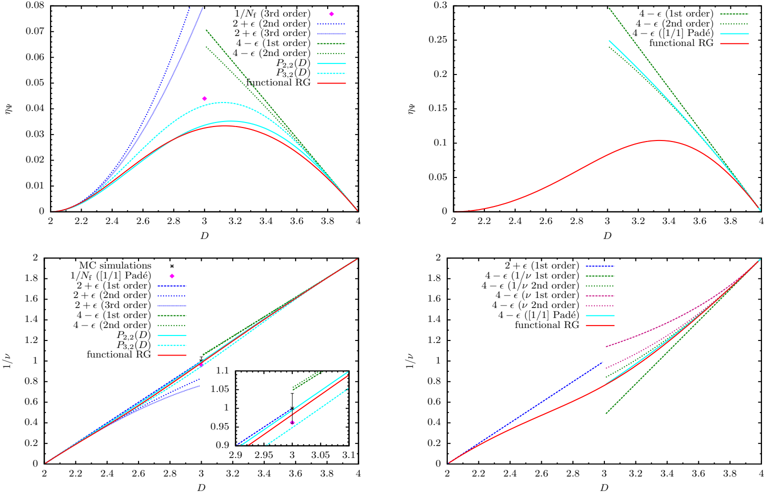
$$L^{\mathrm{cHGNY}} = i\bar{\psi}^i \partial\!\!\!/ \psi^i - m\bar{\psi}^i\psi^i + \frac{1}{2}\partial_\mu\tilde{\pi}^a\partial^\mu\tilde{\pi}^a + \frac{1}{2}g_1\tilde{\pi}^a\bar{\psi}^i\sigma^a\psi^i + \frac{1}{24}g_2^2\left(\tilde{\pi}^a\tilde{\pi}^a\right)^2$$

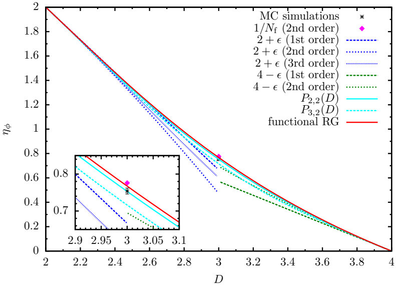
which also has emergent supersymmetry for low values of N

Perturbative and large N computations in all four cases GN, GNY, cHGN and cHGNY have been carried out in recent years

Current status

Summary of exponents η, $1/\nu$ and η_ϕ for Ising GN (left) and chiral Heisenberg GN (right) models [Janssen and Herbut, 2014]




MC simulations
$1/N_\mathrm{f}$ (2nd order)
$2+\epsilon$ (1st order)
$2+\epsilon$ (2nd order)
$2+\epsilon$ (3rd order)
$4-\epsilon$ (1st order)
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$P_{2,2}(D)$
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functional RG
η_ϕ
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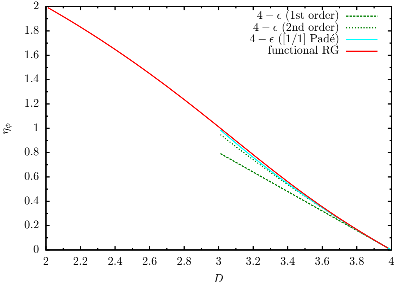

$4-\epsilon$ (1st order)
$4-\epsilon$ (2nd order)
$4-\epsilon$ ([1/1] Padé)
functional RG
η_ϕ
D

Summary of recent activity

GN critical exponents now available at four loops [Gracey, Luthe, Schröder]

GNY and cHGNY renormalization group functions computed at four loops [Zerf, Ihrig, Herbut, Scherer, Mihaila, Marquard]

Large N exponents $\omega_\pm$ computed at $O(1/N)$, [Gracey], and $O(1/N^2)$, [Manashov, Strohmaier], for GN universality class

Large N exponents η, η_ϕ and $1/\nu$ computed at $O(1/N^3)$, $O(1/N^2)$ and $O(1/N^2)$ respectively for cHGN universality class [Gracey]

Ising Gross-Neveu exponent estimates

Can use Padé approximants of ϵ expansion to obtain estimates for $SU(4)$ exponents in three dimensions

Exponent	2 loop	3 loop	4 loop	MC	FRG	$1/N$
η_ψ	0.097	0.083	0.082	–	0.032	0.044
η_ϕ	0.906	0.778	0.745	0.754(8)	0.756	0.776
$1/\nu$	0.857	0.784	0.931	1.00(4)	0.982	0.962

MC from Kärkkäinen et al, FRG from Hofling et al

Exponents η_ϕ and $1/\nu$ are competitive with other methods

Exponent η is significantly larger which is consistent with the overshoot evident in earlier plots in 2 $<$d$<$ 4

Large N expansion of cHGN universality class

Large N critical point method of Vasil'ev et al determines the critical exponents of the universality class as a function of $d = 2\mu$

Key ingredient is the universal interaction which is common in all the theories whose critical dimension is $2n$ where n is an integer

The ϵ expansion of the large N critical exponents are in complete agreement with the ϵ expansion of each theory in the tower where $d = 2n - 2\epsilon$

For the chiral Heisenberg Gross-Neveu universality class the critical point Lagrangian is

$$L = i\bar{\psi}^i \partial\!\!\!/ \psi^i + \pi^a \bar{\psi}^i \sigma^a \psi^i + f(\pi^a)$$

where the fermion kinetic term and the interaction define the canonical dimensions of the fields where $f(\pi^a)$ is a function of the auxiliary field and includes derivative interactions

For cHGN $f(\pi^a) \propto \pi^a \pi^a$ and $f(\pi^a)$ involves $(\pi^a \pi^a)^2$ for the cHGNY

In coordinate space the propagators in the asymptotic limit to the fixed point take the scaling forms, $(d = 2\mu)$,

$$\psi(x) \sim \frac{A\not{x}}{(x^2)^\alpha}\left[1 + A'(x^2)^\lambda\right] \quad , \quad \pi(x) \sim \frac{C}{(x^2)^\gamma}\left[1 + C'(x^2)^\lambda\right]$$

where corrections to scaling are included and

$$\alpha = \mu + \tfrac{1}{2}\eta \quad , \quad \gamma = 1 - \eta - \chi_\pi$$

The anomalous dimension of ψ is η and χ_π is the vertex anomalous dimension

They can be expanded in powers of $1/N$

$$\eta(\mu) = \sum_{n=1}^{\infty} \frac{\eta_n(\mu)}{N^n} \quad , \quad y(\mu) = \sum_{n=1}^{\infty} \frac{y_n(\mu)}{N^n}$$

where $y = A^2 C$

Explicit expressions for η and χ_π can be found by algebraically solving the skeleton Schwinger-Dyson equations in the approach to criticality

No self-energy corrections on propagators in skeleton Schwinger-Dyson equations as these are included in the exponents

$0 = \psi^{-1} +$  $+$  Σ_1

$0 = \pi^{-1} +$  $+$  Π_1

where

$$\psi^{-1}(x) \sim \frac{r(\alpha-1)\not{x}}{A(x^2)^{2\mu-\alpha+1}}\left[1 - A' s(\alpha-1)(x^2)^{\lambda}\right]$$

$$\pi^{-1}(x) \sim \frac{p(\gamma)}{C(x^2)^{2\mu-\gamma}}\left[1 - C' q(\gamma)(x^2)^{\lambda}\right]$$

and

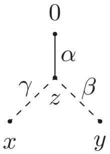
$$a(\alpha) = \frac{\Gamma(\mu-\alpha)}{\Gamma(\alpha)} \quad , \quad p(\gamma) = \frac{a(\gamma-\mu)}{a(\gamma)} \quad , \quad r(\alpha) = \frac{\alpha p(\alpha)}{(\mu-\alpha)}$$

$$q(\gamma) = \frac{a(\gamma-\mu+\lambda)a(\gamma-\lambda)}{a(\gamma-\mu)a(\gamma)} \quad , \quad s(\alpha) = \frac{\alpha(\alpha-\mu)q(\alpha)}{(\alpha-\mu+\lambda)(\alpha-\lambda)}$$

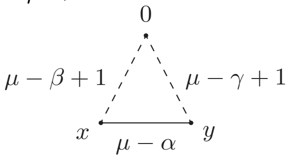
Integrals are evaluated using method of uniqueness or conformal integration for Yukawa vertex; in general when

$$\alpha + \beta + \gamma = 2\mu + 1$$

then



$$\equiv \quad \frac{a(\alpha)a(\beta-1)a(\gamma-1)}{(\beta-1)(\gamma-1)}$$



For the cHGN we have

$$2\alpha + \gamma = 2\mu + 1 - \chi_\pi$$

The $O(1/N^2)$ graphs of 2-point function are divergent and (analytic) regularization is introduced by the shift

$$\chi_\pi \to \chi_\pi + \Delta$$

Leading order results

From the first few orders in $1/N$ of 2-point functions find

$$\begin{aligned}
\eta_1 &= - \frac{3\Gamma(2\mu-1)}{\mu\Gamma(1-\mu)\Gamma(\mu-1)\Gamma^2(\mu)} \\
\chi_{\pi\,1} &= - \frac{\mu}{3(\mu-1)}\eta_1 \\
\eta_2 &= \left[\frac{(2\mu-3)}{3(\mu-1)}\Psi(\mu) ~+~ \frac{(4\mu^2-6\mu+1)}{2\mu(\mu-1)^2} \right] \eta_1^2
\end{aligned}$$

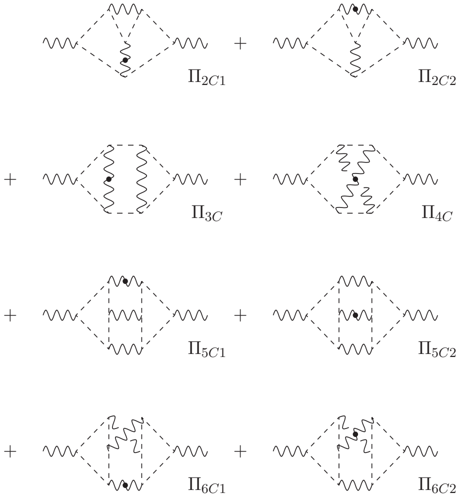
where

$$\Psi(\mu) ~=~ \psi(2\mu-1) ~-~ \psi(1) ~+~ \psi(2-\mu) ~-~ \psi(\mu)$$

ϵ expansion near four dimensions agrees exactly with recent four loop computation of Zerf et al

The exponent $1/\nu$ is determined from the correction to scaling part of the asymptotic propagators by setting $1/\nu = 2\lambda$ where $\lambda_0 = \mu - 1$

There is a reordering within the algebraic solution of the skeleton Dyson-Schwinger equations governing the correction to scaling which means that higher order diagrams are needed to find λ_2



At leading order

$$\lambda_1 \; = \; - \; (2\mu - 1)\eta_1$$

Solution at $O(1/N^2)$ gives

$$\begin{aligned}\lambda_2 \;\; = \;\; & \left[\frac{\mu^2[6\mu^2 - 3\mu - 8]}{6(\mu-1)(\mu-2)} \Theta(\mu) \; - \; \frac{2\mu^2(2\mu-3)}{3(\mu-1)(\mu-2)} \left[\Phi(\mu) + \Psi^2(\mu) \right] \right. \\ & + \; \frac{2\mu[8\mu^2 - 16\mu + 7]}{3(\mu-1)(\mu-2)^2\eta_1} \; + \; \frac{[72\mu^8 - 604\mu^7 + 1960\mu^6 - 3060\mu^5]}{18\mu(\mu-1)^3(\mu-2)^2} \\ & + \; \frac{[2151\mu^4 - 146\mu^3 - 621\mu^2 + 288\mu - 36]}{18\mu(\mu-1)^3(\mu-2)^2} \\ & \left. - \; \frac{(2\mu-3)[18\mu^5 - 95\mu^4 + 161\mu^3 - 86\mu^2 - 20\mu + 12]}{9(\mu-1)^2(\mu-2)^2} \Psi(\mu) \right] \eta_1^2\end{aligned}$$

where

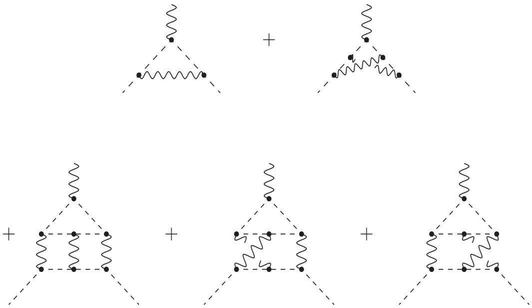
$$\begin{aligned}\Theta(\mu) \;\; &= \;\; \psi'(\mu) \; - \; \psi'(1) \\ \Phi(\mu) \;\; &= \;\; \psi'(2\mu-1) \; - \; \psi'(2-\mu) \; - \; \psi'(\mu) \; + \; \psi'(1)\end{aligned}$$

The ϵ of both exponents near four dimensions is in agreement with four loop perturbation theory of Zerf et al

Large N conformal bootstrap

To proceed to next order need to use another technique which is the large N conformal bootstrap of Vasil'ev et al motivated by early work by Parisi et al

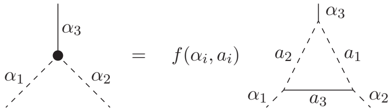
Same asymptotic scaling forms of the propagators are used but the 3-point function Schwinger-Dyson equation of primitive diagrams is solved



where each vertex is replaced by a conformal triangle

Conformal triangle

The conformal triangle vertex is



with all internal vertices unique and

$$\begin{aligned} a_1 + a_2 + \alpha_3 &= 2\mu + 1 \\ a_2 + a_3 + \alpha_1 &= 2\mu + 1 \\ a_3 + a_1 + \alpha_2 &= 2\mu + 1 \end{aligned}$$

where $f(\alpha_i, a_i)$ is the value of the vertex itself

Conformal transformation is

$$x_\mu \;\rightarrow\; \frac{x_\mu}{x^2}$$

which implies

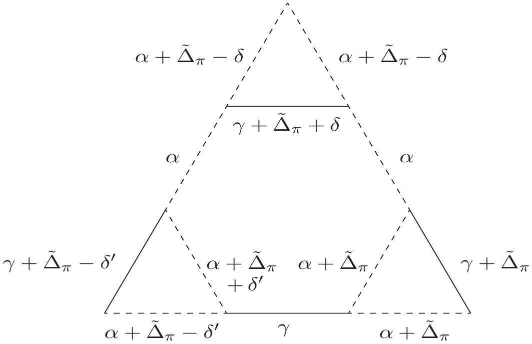
$$(\not{x} - \not{y}) \;\rightarrow\; - \;\frac{\not{y}(\not{x} - \not{y})\not{x}}{x^2 y^2} \;=\; - \;\frac{\not{x}(\not{x} - \not{y})\not{y}}{x^2 y^2}$$

This means that in effect γ-matrix structure is preserved throughout the graphs after a conformal transformation

$$(\not{x} - \not{y})(\not{y} - \not{z}) \;\rightarrow\; \frac{\not{x}(\not{x} - \not{y})(\not{y} - \not{z})\not{z}}{x^2 y^2 z^2}$$

Applying this transformation to 3-point vertex diagrams with conformal triangles as vertices reduces the diagram to d-dimensional 2-point functions

Regularization is required by shifting π^a and one ψ^i external leg dimension



where δ and δ' are the regulators and $\chi_\pi = 2\tilde{\Delta}_\pi$

In effect one is carrying out perturbation theory in the vertex anomalous dimension

η_3

Solving the large N conformal bootstrap equations in d-dimensions yields

$$\begin{aligned}\eta_3 &= \left[\frac{(2\mu-3)}{18(\mu-1)^2}\left[\Phi(\mu)+3\Psi^2(\mu)\right] - \frac{[\mu^3+18\mu^2-21\mu+9]}{36(\mu-1)^2}\Theta(\mu)\right.\\
&\quad - \frac{\mu^2}{3(\mu-1)}\Theta(\mu)\Psi(\mu) - \frac{\mu^2}{6(\mu-1)^3}\Xi(\mu) - \frac{\mu^2}{6(\mu-1)}\Xi(\mu)\Theta(\mu)\\
&\quad - \frac{[14\mu^7-15\mu^6-26\mu^5-77\mu^4+324\mu^3-297\mu^2+90\mu-9]}{18\mu^2(\mu-1)^4}\\
&\quad \left. - \frac{[14\mu^5-37\mu^4-50\mu^3+228\mu^2-183\mu+27]}{18\mu(\mu-1)^3}\Psi(\mu)\right]\eta_1^3\end{aligned}$$

$\Xi(\mu)$ does not have a closed form; its ϵ expansion near even dimensions involves multiple zeta values

η_3 agrees with four loop ϵ expansion of four dimensional cHGN model of Zerf et al

Three dimensional estimates for cHGN

Values for three dimensions are

$$\eta = \frac{4}{\pi^2 N} + \frac{64}{3\pi^4 N^2} + \frac{8[378\zeta_3 - 36\pi^2 \ln(2) - 45\pi^2 - 332]}{9\pi^6 N^3} + O\left(\frac{1}{N^4}\right)$$

$$\eta_\phi = 1 + \frac{16[3\pi^2 + 16]}{3\pi^4 N^2} + O\left(\frac{1}{N^3}\right)$$

$$\frac{1}{\nu} = 1 - \frac{16}{\pi^2 N} + \frac{16[144\pi^2 + 1664]}{3\pi^4 N^2} + O\left(\frac{1}{N^3}\right)$$

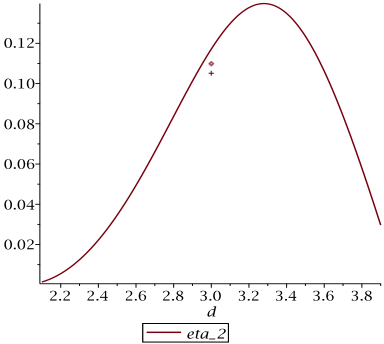
Comparison with other results

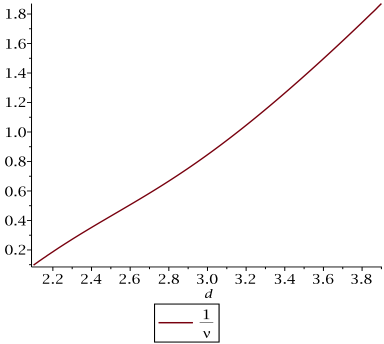
$N = 4$	$1/\nu$	η_ϕ	η	ν
ϵ expansion $[2, 2]$ [Zerf et al]	0.6426	0.9985	0.1833	–
ϵ expansion $[3, 1]$ [Zerf et al]	0.6447	0.9563	0.1560	1.2352
FRG [Knorr]	0.795	1.032	0.071	1.26
MC [Otsuka et al]	(0.98)		0.20(2)	1.02(1)
MC [Parisen Toldin et al]	(1.19)	0.70(15)		0.84(4)
Large N	0.8458	1.1849	0.1051	1.1823

Large N cHGN plots for $N = 4$

Can plot Padé approximants of the exponents as a function of d

As there is no analytic expression for $\Xi(\mu)$ as a function of μ only η_2 is plotted with points corresponding to the three dimensional values of [2, 1] and [1, 2] Padé approximants





Qualitatively similar to the FRG plots

Conclusions

More precise data on the Gross-Neveu universality classes in three dimensions from different methods is beginning to emerge; ϵ-expansions, large N, functional renormalization group and Monte Carlo

Ising Gross-Neveu model appears to have consistent exponents for the graphene case of interest

Clear that recent work on the chiral Heisenberg Gross-Neveu has provided a much better picture of exponents in three dimensions from the four dimensional four loop perturbative as well as recent large N computations

There appears to be reasonable agreement across methods and confirm the FRG initial analysis in d-dimensions but not in as good an order as GN case; fine detail needs to be resolved for graphene case