

ANALYTIC CALCULATION OF
ENERGY-ENERGY CORRELATIONS IN e^+e^- ANNIHILATION
AT NEXT-TO-LEADING ORDER



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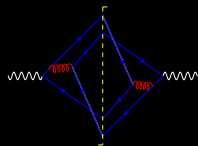
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Loops and Legs in Quantum Field Theory
1st of May 2018



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1 Motivation

- QCD Event Shape Observables

2 Energy-Energy Correlations

- State of the Art
- Fully Analytic Result at NLO

3 Summary and Outlook

- Event shape observables: IR- and collinear safe quantities that can be directly measured in experiments.
- Characterize the geometric structure of the final states (event topology) produced in strong interaction processes.
- Sensitivity to QCD radiation through soft gluon emissions influencing the shape of the energy flow.
- Examples:
 - Thrust [Brandt et al., 1964; Farhi, 1977].
 - C -parameter [Parisi, 1978; Donoghue et al., 1979; Ellis et al., 1981].
 - Jet broadening [Rakow & Webber, 1981; Ellis & Webber, 1986; Catani et al., 1992].
 - Jet masses [Clavelli, 1979].
 - Energy-energy correlations [Basham et al., 1978].
- Historically of crucial importance to verify the early predictions of QCD.
- Currently still very useful for many purposes:
 - Determination of α_s .
 - Tuning of Monte Carlo event generators.
 - New physics searches.
- Precise measurements done by the LEP experiments: ALEPH [Heister et al., 2004], DELPHI [Abdallah et al., 2004], L3 [Achard et al., 2004], OPAL [Abbiendi et al., 2005].
- Worthwhile task for the future e^+e^- colliders (CEPC in China, ILC in Japan and FCC-ee in Europe).

- The mathematical structure of the event shapes observables is still poorly understood.
- Full analytic results (fixed-order) are scarce, in most cases available only at LO.

Observable	Full analytic result at LO	Full analytic result at NLO
C -parameter	No ¹	No
Thrust	Yes [De Rujula et al., 1978]	No
Heavy jet mass	Yes ²	No
EEC	Yes [Basham et al., 1978]	Yes [THIS WORK]

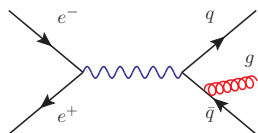
- NLO and often also NNLO results can be obtained using numerical methods.
- Nevertheless, full analytic results beyond LO are still useful and hence desirable.

¹Can be written as a single integral in a compact form [Ellis et al., 1981].

²Trivially follows from the result for Thrust.

- The original definition of the energy-energy correlation function (EEC) [Basham et al., 1978] uses e^+e^- annihilation in QCD at high energies

$$e^+e^- \rightarrow a + b + X.$$



- Two calorimeters at relative angle χ measure the energies of the hadrons a and b .
- Formal definition

$$\frac{1}{\sigma_{\text{tot}}} \frac{d\Sigma(\chi)}{d \cos \chi} = \sum_{a,b} \int \frac{E_a E_b}{Q^2} \delta(\cos \theta_{ab} - \cos \chi) d\sigma_{a+b+X}, \quad \cos \theta_{ab} = \hat{\mathbf{p}}_a \cdot \hat{\mathbf{p}}_b.$$

- EEC is a differential angular distribution of the energy flowing through the calorimeters.
- Can be computed in pQCD by the virtue of the momentum sum rule

$$\sum_h \int_0^1 dx x D_{h/q}(x, \mu_F^2) = 1.$$

- Nonperturbative corrections [Dokshitzer et al., 1999] are beyond the scope of this work.
- A good probe of QCD, as already the LO contribution starts with α_s .
- Also interesting for the determination of α_s [Kardos et al., 2018], [see today's talk of Z. Tulipánt].

- The analytic form of the EEC at LO is known since 40 years [Basham et al., 1978]

$$\frac{1}{\sigma_{\text{tot}}} \frac{d\Sigma(\chi)}{d \cos \chi} = \frac{\alpha_s(\mu)}{2\pi} C_F \frac{3-2z}{4(1-z)z^5} \left[3z(2-3z) + 2(2z^2-6z+3) \log(1-z) \right] + \mathcal{O}(\alpha_s^2), \quad \text{with } z = (1 - \cos \chi)/2.$$

- Emission of soft and collinear particles: The limits $z \rightarrow 0$ and $z \rightarrow 1$ require resummation of logarithms.
- Subsequent studies of higher order corrections relied on numerical methods.
- Different approaches to handle soft and collinear singularities from real radiation at NLO in the 80s and 90s:
 - Phase-space slicing [Schneider et al., 1984; Falck & Kramer, 1989; Glover & Sutton, 1995; Kramer & Spiesberger, 1997].
 - Subtraction methods [Ali & Barreiro, 1982, 1984; Richards et al., 1982, 1983; Kunszt et al., 1989; Glover & Sutton, 1995; Catani & Seymour, 1996, 1997].
 - Hybrid schemes [Glover & Sutton, 1995; Clay & Ellis, 1995; Kramer & Spiesberger, 1997].
- Some numerical discrepancies between the results obtained by different groups.

- Results of [Kunszt et al., 1989] (**N**), [Glover & Sutton, 1995] (**G**), [Catani & Seymour, 1996] (**S**) and [Clay & Ellis, 1995] (**C**) collected by P. Nason [Nason et al., 1996].
- The comparison is done between the moments of EEC

$$\int \sin^{2+m} \chi \cos^n \chi d\Sigma(\chi) d\cos \chi =$$

$$+ \frac{\alpha_s}{2\pi} A^{(m,n)} + C_F \left(C_A B_{C_A}^{(m,n)} + C_F B_{C_F}^{(m,n)} + T_F N_F B_{T_F}^{(m,n)} \right) + \mathcal{O}(\alpha_s^3).$$

Comparison of different computations of the $B_{C_A}^{(m,n)}$ coefficients

m	n	N	G	S	C
0	0	50.82 ± 0.05	50.54 ± 0.03	50.72 ± 0.02	46.4 ± 0.2
1	0	35.76 ± 0.04	35.53 ± 0.02	35.64 ± 0.02	32.09 ± 0.06
2	0	28.94 ± 0.03	28.75 ± 0.02	28.82 ± 0.02	25.73 ± 0.04
3	0	24.92 ± 0.03	24.75 ± 0.02	24.80 ± 0.02	22.03 ± 0.04
4	0	22.20 ± 0.03	22.05 ± 0.02	22.09 ± 0.02	19.54 ± 0.04
5	0	20.21 ± 0.03	20.07 ± 0.02	20.10 ± 0.02	17.74 ± 0.03
0	1	-6.468 ± 0.006	-6.50 ± 0.01	-6.455 ± 0.005	-6.0 ± 0.15
1	1	-2.356 ± 0.004	-2.365 ± 0.009	-2.344 ± 0.003	-2.15 ± 0.03
2	1	-1.189 ± 0.003	-1.194 ± 0.008	-1.177 ± 0.003	-1.06 ± 0.02
3	1	-0.714 ± 0.003	-0.718 ± 0.007	-0.702 ± 0.003	-0.62 ± 0.01
4	1	-0.478 ± 0.003	-0.479 ± 0.007	-0.466 ± 0.003	-0.41 ± 0.01
5	1	-0.344 ± 0.003	-0.344 ± 0.006	-0.331 ± 0.003	-0.28 ± 0.01

- Results of [Kunszt et al., 1989] (**N**), [Glover & Sutton, 1995] (**G**), [Catani & Seymour, 1996] (**S**) and [Clay & Ellis, 1995] (**C**) collected by P. Nason [Nason et al., 1996].
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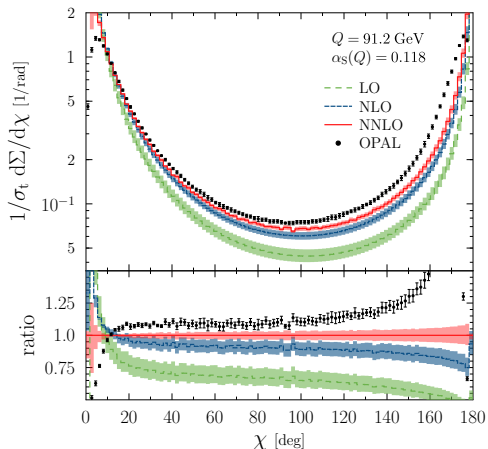
$$\int \sin^{2+m} \chi \cos^n \chi d\Sigma(\chi) d\cos \chi =$$

$$+ \frac{\alpha_s}{2\pi} A^{(m,n)} + C_F \left(C_A B_{C_A}^{(m,n)} + C_F B_{C_F}^{(m,n)} + T_F N_F B_{T_F}^{(m,n)} \right) + \mathcal{O}(\alpha_s^3).$$

Comparison of different computations of the $B_{C_F}^{(m,n)}$ coefficients

m	n	N	G	S	C
0	0	-13.29 ± 0.01	-13.94 ± 0.05	-13.40 ± 0.05	7.2 ± 0.2
1	0	-5.09 ± 0.01	-5.38 ± 0.04	-5.14 ± 0.04	9.98 ± 0.04
2	0	-2.98 ± 0.01	-3.20 ± 0.03	-3.02 ± 0.03	9.55 ± 0.03
3	0	-2.11 ± 0.01	-2.29 ± 0.03	-2.14 ± 0.03	8.86 ± 0.02
4	0	-1.65 ± 0.01	-1.81 ± 0.03	-1.67 ± 0.03	8.22 ± 0.02
5	0	-1.36 ± 0.01	-1.51 ± 0.03	-1.39 ± 0.03	7.69 ± 0.02
0	1	4.906 ± 0.002	4.92 ± 0.01	4.892 ± 0.006	2.6 ± 0.2
1	1	0.240 ± 0.002	0.259 ± 0.008	0.232 ± 0.004	-0.58 ± 0.02
2	1	-0.383 ± 0.002	-0.367 ± 0.006	-0.386 ± 0.004	-0.80 ± 0.01
3	1	-0.458 ± 0.002	-0.445 ± 0.005	-0.459 ± 0.003	-0.72 ± 0.01
4	1	-0.428 ± 0.002	-0.417 ± 0.005	-0.429 ± 0.003	-0.606 ± 0.007
5	1	-0.381 ± 0.002	-0.371 ± 0.004	-0.381 ± 0.003	-0.510 ± 0.006

- Dipole subtraction method is implemented in the publicly available code **EVENT2** [Catani & Seymour, 1996, 1997].
- The NNLO results using CoLoRfulNNLO subtraction are now also available [Del Duca et al., 2016; Tulipánt et al., 2017].
- NLO and NNLO corrections for the fixed-order predictions are not negligible.



Source: [Tulipánt et al., 2017].

- Impressive progress in the numerical calculations of EEC.
- Fully analytic fixed-order results: LO state of the art for almost 40 years!
- Important milestone: analytic NLO calculation in $\mathcal{N} = 4$ SYM [Belitsky et al., 2014]

$$\Sigma_{\mathcal{N}=4}(z) = \frac{1}{4z^2(1-z)} \left(aF_1(z) + a^2[(1-z)F_2(z) + F_3(z)] \right), \quad a = \frac{g_{\text{YM}}^2 N}{4\pi^2}.$$

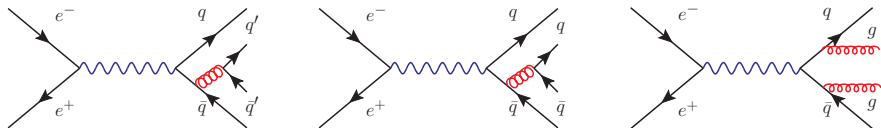
- Very simple and compact.
- Only classical polylogarithms.
- Highest transcendental weight 3.
- Alphabet contains \sqrt{z} .
- Maximal transcendental principle [Kotikov & Lipatov, 2003; Kotikov et al., 2004]: QCD result also simple?

$$F_1(z) = -\ln(1-z),$$

$$F_2(z) = 4\sqrt{z} \left[\text{Li}_2(-\sqrt{z}) - \text{Li}_2(\sqrt{z}) + \frac{\ln z}{2} \ln \left(\frac{1+\sqrt{z}}{1-\sqrt{z}} \right) \right] \\ + (1+z) \left[2\text{Li}_2(z) + \ln^2(1-z) \right] + 2\ln(1-z) \ln \left(\frac{z}{1-z} \right) + z \frac{\pi^2}{3},$$

$$F_3(z) = \frac{1}{4} \left\{ (1-z)(1+2z) \left[\ln^2 \left(\frac{1+\sqrt{z}}{1-\sqrt{z}} \right) \ln \left(\frac{1-z}{z} \right) - 8\text{Li}_3 \left(\frac{\sqrt{z}}{\sqrt{z}-1} \right) \right. \right. \\ \left. \left. - 8\text{Li}_3 \left(\frac{\sqrt{z}}{\sqrt{z}+1} \right) \right] - 4(z-4)\text{Li}_3(z) + 6(3+3z-4z^2)\text{Li}_3 \left(\frac{z}{z-1} \right) \right. \\ \left. - 2z(1+4z)\zeta_3 + 2 \left[2(2z^2-z-2) \ln(1-z) + (3-4z)z \ln z \right] \text{Li}_2(z) \right. \\ \left. + \frac{1}{3} \ln^2(1-z) \left[4(3z^2-2z-1) \ln(1-z) + 3(3-4z)z \ln z \right] \right. \\ \left. + \frac{\pi^2}{3} \left[2z^2 \ln z - (2z^2+z-2) \ln(1-z) \right] \right\}.$$

- Unfortunately, the Mellin-space techniques used in [Belitsky et al., 2014] are not easily applicable to QCD.
- Hardest part of the QCD calculation: Real corrections.



- The IBP-reduction [Chetyrkin & Tkachov, 1981] applied to the on-shell amplitudes appears more tractable.

- Use the reverse unitarity method [Anastasiou & Melnikov, 2002; Anastasiou et al., 2003]

$$\frac{d^D p}{(2\pi)^D} 2\pi \delta_+(p^2 - m^2) \rightarrow \frac{1}{i} \frac{d^D p}{(2\pi)^D} \left(\frac{1}{p^2 - m^2 - i\eta} - \frac{1}{p^2 - m^2 + i\eta} \right)$$

to rewrite the measurement function $\delta(\cos \theta_{ab} - \cos \chi)$ as

$$\frac{1}{(1 - \cos \chi)(p_a \cdot Q)(p_b \cdot Q) - Q^2(p_a \cdot p_b)} \Big|_{\text{cut}}.$$

- Nonstandard propagator, but no fundamental obstacles to use the IBPs.
- Important ingredient: Need to augment our set of IBP equations with

$$\int \left[\left((1 - \cos \chi)(p_a \cdot Q)(p_b \cdot Q) - Q^2(p_a \cdot p_b) \right) J_{ab}(n_1, \dots, n_{10}) - J_{ab}(n_1, \dots, n_{10} - 1) \right] = 0.$$

- Alternative approach [Gitiular & Moch, 2017]: Introduce an auxiliary variable to linearize the propagator

$$\delta \left[(1 - \cos \chi)(p_a \cdot Q)(p_b \cdot Q) - Q^2(p_a \cdot p_b) \right] = \int_0^1 dx \delta[x - (p_a \cdot Q)] \delta[x(1 - \cos \chi)(p_b \cdot Q) - Q^2(p_a \cdot p_b)].$$

- The full result using the method of [Gitiular & Moch, 2017] is not available yet.

- The actual calculation consists of mostly standard steps:
 - Generate LO and NLO Feynman diagrams with **QGRAF** [Nogueira, 1993].
 - Calculate $|\mathcal{M}|^2$ using **FORM** [Vermaseren, 2000] and **COLOR** [van Ritbergen et al., 1999].
 - Self-written *Mathematica* code to identify the topologies.
 - Derive the IBP-equations with **LITERED** [Lee, 2012, 2014].
 - Perform the IBP-reduction with **FIRE** [Smirnov, 2008, 2015].
- Some steps are, however, nonstandard:
 - **FIRE** and **LITERED** cannot work with our nonlinear propagator out-of-the box.
 - Minimal modifications in **LITERED** to be able to derive the IBP equations.
 - The equations are then imported into **FIRE**.
 - Very useful feature of **FIRE**: IBP reduction by employing an external (i. e. not derived within **FIRE**) set of IBP equations.
 - Since **FIRE** does not need to know to the exact form of our propagators, the nonlinear propagator does not cause any issues.
- The total number of loop integrals at NLO before the IBP reduction is 12375.
- The IBP-reduction with **FIRE** requires 3 days on a server equipped with Xeon E5-2696 (18 cores) and 128 GB RAM.

- To calculate the resulting 40 master integrals (3-loop) we employ the method of differential equations [Kotikov, 1991a, 1991b, 1991c; Bern et al., 1994; Remiddi, 1997; Gehrmann & Remiddi, 2000]

- The differential equations are derived using **LITERED** and **FIRE**.
- **FUCHSIA** [Gitiular & Magerya, 2017] provides a fast and convenient way to convert all systems of differential equations into a canonical form [Henn, 2013] using Lee's algorithm [Lee, 2015].
- Caveat: Some systems require a nonlinear change of variables

$$z \rightarrow \sqrt{z}, \quad z \rightarrow i\sqrt{z}/\sqrt{1-z}$$

- The transformations readily follow from the $\mathcal{N} = 4$ SYM result.
- Once in the canonical form, the differential equations are trivial to solve.
- Fixing the boundary conditions was the last, but also the most time consuming step:
 - Scaling of the master integrals in z in the limits $z \rightarrow 0$ and $z \rightarrow \infty$.
 - Integrate over z and match to inclusive 4-particle phase-space master integrals [Gehrmann-De Ridder et al., 2004], similar to the approach of [Gitiular & Moch, 2017].

- ✓ The single master integrals are divergent, but all the ϵ -poles cancel in $\Sigma(\chi)$.

- Example: Differential equation for one of the integral families contributing to the N_f piece of the full NLO result

$$\frac{d}{dy} F(y, \varepsilon) = \mathbb{A}(y, \varepsilon) F(y, \varepsilon), \quad y \equiv \cos \chi$$

$$\mathbb{A}(y, \varepsilon) = \begin{pmatrix} \frac{3-4\varepsilon}{1-y} - \frac{\varepsilon}{1+y} & -\frac{1-\varepsilon}{1-y} & 0 \\ -\frac{(2-3\varepsilon)^2}{(1-\varepsilon)(1+y)} & \frac{1-2\varepsilon}{1+y} + \frac{\varepsilon}{1-y} & 0 \\ \frac{(2-3\varepsilon)^2}{(1-\varepsilon)(1+y)} - \frac{2(2-3\varepsilon)^2}{(1-\varepsilon)(1+y)^2} & \frac{2(1-\varepsilon)}{(1+y)^2} - \frac{3(1-2\varepsilon)}{1+y} & \frac{\varepsilon}{1-y} - \frac{2\varepsilon}{1+y} \end{pmatrix}$$

- Canonical form easily obtained with **FUCHSIA**

$$\frac{d}{dy} G(y, \varepsilon) = \varepsilon \mathbb{B}(y) G(y, \varepsilon),$$

$$\mathbb{B}(y) = \begin{pmatrix} \frac{346}{25(1+y)} - \frac{577}{25(1-y)} & \frac{2709}{200(1-y)} - \frac{583}{50(1+y)} & 0 \\ \frac{504}{25(1+y)} - \frac{848}{25(1-y)} & \frac{502}{25(1-y)} - \frac{421}{25(1+y)} & 0 \\ -\frac{4840}{351(1+y)} - \frac{4840}{351(1-y)} & \frac{605}{78(1+y)} + \frac{605}{78(1-y)} & \frac{1}{1-y} - \frac{2}{1+y} \end{pmatrix}$$

- The transformation matrix $\mathbb{T}(y, \varepsilon) = (\mathbb{T}^1, \mathbb{T}^2, \mathbb{T}^3)$

$$\mathbb{T}^1(y, \varepsilon) = \begin{pmatrix} -\frac{64(1-5\varepsilon)(2-5\varepsilon)}{(6-11\varepsilon)(2-\varepsilon)(1-y)^3} - \frac{896\varepsilon^2-57\varepsilon+4}{(6-11\varepsilon)(2-\varepsilon)(1-y)} + \frac{16137\varepsilon^2-78\varepsilon+8}{(6-11\varepsilon)(2-\varepsilon)(1-y)^2} \\ \frac{32(1-5\varepsilon)(2-3\varepsilon)^2}{(6-11\varepsilon)(1-\varepsilon)(2-\varepsilon)(1-y)^2} - \frac{8(2-19\varepsilon)(2-3\varepsilon)^2}{(6-11\varepsilon)(1-\varepsilon)(2-\varepsilon)(1-y)} \\ 0 \end{pmatrix}$$

$$\mathbb{T}^2(y, \varepsilon) = \begin{pmatrix} \frac{36(1-5\varepsilon)(2-5\varepsilon)}{(6-11\varepsilon)(2-\varepsilon)(1-y)^3} + \frac{3169\varepsilon^2-98\varepsilon+6}{(6-11\varepsilon)(2-\varepsilon)(1-y)} - \frac{2679\varepsilon^2-376\varepsilon+36}{(6-11\varepsilon)(2-\varepsilon)(1-y)^2} \\ \frac{(9-98\varepsilon)(2-3\varepsilon)^2}{(6-11\varepsilon)(1-\varepsilon)(2-\varepsilon)(1-y)} - \frac{18(1-5\varepsilon)(2-3\varepsilon)^2}{(6-11\varepsilon)(1-\varepsilon)(2-\varepsilon)(1-y)^2} \\ 0 \end{pmatrix}$$

$$\mathbb{T}^3(y, \varepsilon) = \begin{pmatrix} 0 \\ 0 \\ \frac{351(3\varepsilon-2)^2(3\varepsilon-1)^2}{605(\varepsilon-2)(\varepsilon-1)\varepsilon(11\varepsilon-6)} \end{pmatrix}$$

- In the most complicated case $\mathbb{A}(y, \varepsilon)$ is given by a 19×19 matrix.

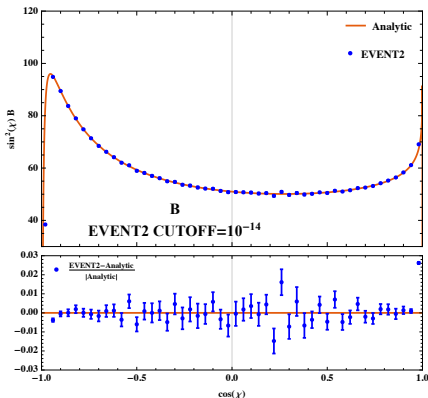
- Convenient notation for the full NLO result ($\beta_0 = 11C_A/3 - 4N_fT_f/3$)

$$\frac{1}{\sigma_{\text{tot}}} \frac{d\Sigma(\chi)}{d \cos \chi} = \frac{\alpha_s(\mu)}{2\pi} A(z) + \left(\frac{\alpha_s(\mu)}{2\pi} \right)^2 \left(\beta_0 \log \frac{\mu}{Q} A(z) + B(z) \right) + \mathcal{O}(\alpha_s^3).$$

- Color decomposition of the NLO coefficient $B(z)$

$$B(z) = C_F^2 B_{\text{lc}}(z) + C_F(C_A - 2C_F) B_{\text{nlc}}(z) + C_F N_f T_f B_{N_f}(z).$$

- Our result for $B(z)$ can be compared to the output of **EVENT2**.



- Sampling over 10^9 points.
- Perfect agreement over a large range of χ values.
- Rightmost bin discrepancy caused by the finite bin width in **EVENT2**.

- The analytic NLO result has a remarkably simple structure.
- Basis of pure functions $g_i^{(n)}$ of uniform transcendentality weight $n \leq 3$

$$g_1^{(1)} = \log(1-z), \quad g_2^{(1)} = \log(z), \quad g_1^{(2)} = 2(\text{Li}_2(z) + \zeta_2) + \log^2(1-z),$$

$$g_2^{(2)} = \text{Li}_2(1-z) - \text{Li}_2(z),$$

$$g_3^{(2)} = -2 \text{Li}_2(-\sqrt{z}) + 2 \text{Li}_2(\sqrt{z}) + \log\left(\frac{1-\sqrt{z}}{1+\sqrt{z}}\right) \log(z), \quad g_4^{(2)} = \zeta_2,$$

$$g_1^{(3)} = -6 \left[\text{Li}_3\left(-\frac{z}{1-z}\right) - \zeta_3 \right] - \log\left(\frac{z}{1-z}\right) (2(\text{Li}_2(z) + \zeta_2) + \log^2(1-z)),$$

$$g_2^{(3)} = -12 \left[\text{Li}_3(z) + \text{Li}_3\left(-\frac{z}{1-z}\right) \right] + 6 \text{Li}_2(z) \log(1-z) + \log^3(1-z),$$

$$g_3^{(3)} = 6 \log(1-z) (\text{Li}_2(z) - \zeta_2) - 12 \text{Li}_3(z) + \log^3(1-z),$$

$$g_4^{(3)} = \text{Li}_3\left(-\frac{z}{1-z}\right) - 3 \zeta_2 \log(z) + 8 \zeta_3,$$

$$g_5^{(3)} = -8 \left[\text{Li}_3\left(-\frac{\sqrt{z}}{1-\sqrt{z}}\right) + \text{Li}_3\left(\frac{\sqrt{z}}{1+\sqrt{z}}\right) \right] + 2 \text{Li}_3\left(-\frac{z}{1-z}\right) \\ + 4 \zeta_2 \log(1-z) + \log\left(\frac{1-z}{z}\right) \log^2\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right).$$

$$B(z) = C_F^2 B_{lc}(z) + C_F(C_A - 2C_F)B_{nlc}(z) + C_F N_f T_f B_{N_f}(z)$$

• Leading color coefficient $B_{lc}(z)$

$$\begin{aligned}
 &+ \frac{122400z^7 - 244800z^6 + 157060z^5 - 31000z^4 + 2064z^3 + 72305z^2 - 143577z + 63298}{1440(1-z)z^4} \\
 &- \frac{-244800z^9 + 673200z^8 - 667280z^7 + 283140z^6 - 48122z^5 + 2716z^4 - 6201z^3 + 11309z^2 - 9329z + 3007}{720(1-z)z^5} g_1^{(1)} \\
 &- \frac{244800z^8 - 550800z^7 + 422480z^6 - 126900z^5 + 13052z^4 - 336z^3 + 17261z^2 - 38295z + 19938}{720(1-z)z^4} g_2^{(1)} \\
 &+ \frac{4z^7 + 10z^6 - 17z^5 + 25z^4 - 96z^3 + 296z^2 - 211z + 87}{24(1-z)z^5} g_1^{(2)} \\
 &+ \frac{-40800z^8 + 61200z^7 - 28480z^6 + 4040z^5 - 320z^4 - 160z^3 + 1126z^2 - 4726z + 3323}{120z^5} g_2^{(2)} \\
 &- \frac{1 - 11z}{48z^{7/2}} g_3^{(2)} - \frac{120z^6 + 60z^5 + 160z^4 - 2246z^3 + 8812z^2 - 10159z + 4193}{120(1-z)z^5} g_4^{(2)} \\
 &- 2(85z^4 - 170z^3 + 116z^2 - 31z + 3)g_1^{(3)} + \frac{-4z^3 + 18z^2 - 21z + 5}{6(1-z)z^5} g_2^{(3)} + \frac{z^2 + 1}{12(1-z)} g_3^{(3)}.
 \end{aligned}$$

$$B(z) = C_F^2 B_{lc}(z) + C_F(C_A - 2C_F)B_{nlc}(z) + C_F N_f T_f B_{N_f}(z)$$

• Subleading color coefficient $B_{nlc}(z)$

$$\begin{aligned}
 & + \frac{57600z^7 - 115200z^6 + 75748z^5 - 17359z^4 + 902z^3 + 14966z^2 - 27552z + 9320}{720(1-z)z^4} \\
 & - \frac{-115200z^9 + 316800z^8 - 321680z^7 + 147846z^6 - 31035z^5 + 3225z^4 - 3571z^3 + 11322z^2 - 12412z + 4880}{360(1-z)z^5} g_1^{(1)} \\
 & - \frac{230400z^8 - 518400z^7 + 412960z^6 - 138600z^5 + 18696z^4 - 742z^3 + 10971z^2 - 25029z + 11424}{720(1-z)z^4} g_2^{(1)} \\
 & + \frac{-91z^7 + 235z^6 - 184z^5 + 15z^4 - 140z^3 + 721z^2 - 760z + 314}{120(1-z)z^5} g_1^{(2)} \\
 & + \frac{-19200z^8 + 28800z^7 - 14680z^6 + 2660z^5 - 340z^4 - 40z^3 + 315z^2 - 1431z + 952}{60z^5} g_2^{(2)} \\
 & + \frac{960z^4 - 160z^3 + 992z^2 + 547z + 1435}{480z^{7/2}} g_3^{(2)} - \frac{-120z^6 + 120z^5 - 130z^4 - 585z^3 + 2647z^2 - 3143z + 1266}{60(1-z)z^5} g_4^{(2)} \\
 & + \frac{640z^6 - 1920z^5 + 2196z^4 - 1196z^3 + 318z^2 - 42z + 3}{4(1-z)z} g_1^{(3)} + \frac{2z^7 - 3z^6 + 3z^5 - z^4 - z^3 + 9z^2 - 9z + 1}{12(1-z)z^5} g_2^{(3)} \\
 & - \frac{(1-2z)(z^2 - z + 1)}{2(1-z)z} g_4^{(3)} - \frac{2z^5 - z^4 + 2z^3 + z^2 + 3}{4z^4} g_5^{(3)}.
 \end{aligned}$$

$$B(z) = C_F^2 B_{lc}(z) + C_F(C_A - 2C_F)B_{nlc}(z) + C_F N_f T_f B_{N_f}(z)$$

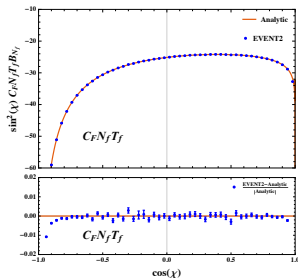
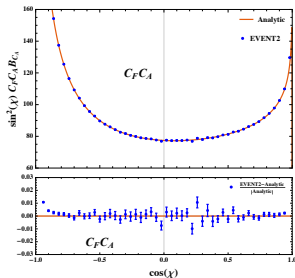
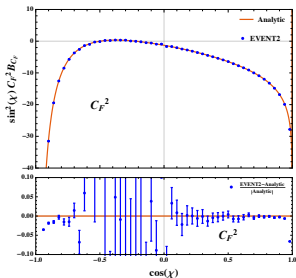
• N_f -piece $B_{N_f}(z)$

$$\begin{aligned} & - \frac{7200z^7 - 14400z^6 + 8852z^5 - 1568z^4 + 48z^3 + 1825z^2 - 4115z + 2050}{144(1-z)z^4} \\ & - \frac{72000z^9 - 198000z^8 + 193040z^7 - 77700z^6 + 10960z^5 - 100z^4 - 489z^3 + 3269z^2 - 4801z + 1801}{360(1-z)z^5} g_1^{(1)} \\ & + \frac{36000z^8 - 81000z^7 + 60520z^6 - 16650z^5 + 1190z^4 + 10z^3 + 428z^2 - 939z + 561}{180(1-z)z^4} g_2^{(1)} \\ & + \frac{-z^7 - 4z^3 + 18z^2 - 24z + 9}{6(1-z)z^5} g_1^{(2)} - \frac{-12000z^8 + 18000z^7 - 7840z^6 + 920z^5 + 72z^2 - 222z + 187}{60z^5} g_2^{(2)} \\ & + \frac{1-3z}{48z^{7/2}} g_3^{(2)} + \frac{8z^3 - 66z^2 + 71z + 7}{60(1-z)z^5} g_4^{(2)} + 2(50z^4 - 100z^3 + 66z^2 - 16z + 1) g_1^{(3)}. \end{aligned}$$

• This contribution was computed by Lance Dixon and Marc Schreiber already in 2004!

- We can also validate the individual color components of $B(z)$ against **EVENT2**.
- More convenient notation:

$$B(z) = C_F^2 B_{C_F}(z) + C_F C_A B_{C_A}(z) + C_F N_f T_f B_{N_f}(z).$$



- We can compare the asymptotic behavior of $B(z)$ to the literature.
- In the collinear limit ($z \rightarrow 0$), the $\log(z)/z$ term agrees with the jet calculus prediction [Konishi et al., 1978, 1979; Richards et al., 1983].

$$\begin{aligned}
 B(z) = & C_F \left\{ \frac{1}{z} \left[\log(z) \left(-\frac{107C_A}{120} + \frac{25C_F}{32} + \frac{53N_f T_f}{240} \right) \right. \right. \\
 & + C_A \left(-\frac{25\zeta_2}{12} + \frac{\zeta_3}{2} + \frac{17683}{2700} \right) \\
 & + C_F \left(\frac{43\zeta_2}{12} - \zeta_3 - \frac{8263}{1728} \right) - \frac{4913N_f T_f}{3600} \left. \right] + \log(z) \left[C_A \left(\frac{33\zeta_2}{2} - \frac{703439}{25200} \right) \right. \\
 & + C_F \left(\frac{42109}{1200} - 21\zeta_2 \right) + N_f T_f \left(\frac{86501}{12600} - 4\zeta_2 \right) \left. \right] \\
 & + C_A \left(\frac{213\zeta_2}{5} - \frac{101\zeta_3}{2} - \frac{26986007}{5292000} \right) + C_F \left(-\frac{1541\zeta_2}{30} + 65\zeta_3 + \frac{18563}{2700} \right) \\
 & \left. + N_f T_f \left(-\frac{46\zeta_2}{3} + 12\zeta_3 + \frac{2987627}{330750} \right) \right\} + \mathcal{O}(z).
 \end{aligned}$$

- In the back-to-back limit ($z \rightarrow 1$) terms enhanced by $1/(1-z)$ agree with the results from the NNLL soft gluon resummation [de Florian & Grazzini, 2005].
- N³LL factorization formula is already available [Moult & Zhu, 2018].

$$\begin{aligned}
 B(z) = & C_F \left\{ \frac{1}{1-z} \left[\frac{1}{2} C_F \log^3(1-z) + \log^2(1-z) \left(\frac{11C_A}{12} + \frac{9C_F}{4} - \frac{N_f T_f}{3} \right) \right. \right. \\
 & + \log(1-z) \left(C_A \left(\frac{\zeta_2}{2} - \frac{35}{72} \right) + C_F \left(\zeta_2 + \frac{17}{4} \right) + \frac{N_f T_f}{18} \right) \\
 & + C_A \left(\frac{11\zeta_2}{4} + \frac{3\zeta_3}{2} - \frac{35}{16} \right) + C_F \left(3\zeta_2 - \zeta_3 + \frac{45}{16} \right) + N_f T_f \left(\frac{3}{4} - \zeta_2 \right) \left. \right] \\
 & + \left(\frac{C_A}{2} + C_F \right) \log^3(1-z) + \log^2(1-z) \left(\frac{27C_A}{8} + \frac{13C_F}{2} - \frac{N_f T_f}{2} \right) \\
 & + \log(1-z) \left[C_A \left(22\zeta_2 - \frac{2011}{72} \right) + C_F (47 - 19\zeta_2) + N_f T_f \left(\frac{361}{36} - 4\zeta_2 \right) \right] \\
 & + C_A \left(\frac{6347\zeta_2}{80} - 21\zeta_2 \log(2) - \frac{137\zeta_3}{4} - \frac{3305}{72} \right) \\
 & + C_F \left(-\frac{1727\zeta_2}{20} + 42\zeta_2 \log(2) + \frac{121\zeta_3}{2} + \frac{3437}{96} \right) \\
 & \left. + N_f T_f \left(-\frac{1747\zeta_2}{120} + 12\zeta_3 + \frac{2099}{144} \right) \right\} + \mathcal{O}(1-z).
 \end{aligned}$$

- Notice that our σ_{tot} is LO, while de Florian and Grazzini used σ_{tot} at NLO.
- The relation $\sigma_{\text{tot, NLO}} = \sigma_{\text{tot, LO}} \left(1 + \frac{3}{4} C_F \frac{\alpha_s}{\pi} \right)$ shifts coefficients of C_F^2 .

- The physical values of $z = (1 - \cos \chi)/2$ are between 0 and 1.
- Analytic continuation: extend $\Sigma(\chi)$ beyond the physical region and consider the limit $z \rightarrow \infty$

$$A(z) = \frac{C_F}{z^3} \left[2 \log(-z) - \frac{9}{2} \right] + \mathcal{O}(1/z^4),$$

$$B_{lc}(z) = \frac{1}{z^3} \left[\left(4\zeta_2 + \frac{4699}{288} \right) \log(-z) - 8\zeta_3 + \frac{991}{84} \zeta_2 - \frac{85595}{1728} \right] \\ + \frac{i \operatorname{sign}(\operatorname{Im}(z))}{z^3} \left[\frac{11}{8} \zeta_2 \sqrt{-z} + \pi \left(-\frac{1459}{140} \log(-z) + \frac{466259}{19600} \right) \right] + \mathcal{O}(1/z^{7/2}),$$

$$B_{nlc}(z) = \frac{1}{z^3} \left[\left(\frac{3}{2} \zeta_2 + \frac{473}{72} \right) \log(-z) - \frac{9}{2} \zeta_3 + \frac{521}{70} \zeta_2 - \frac{32713}{1728} \right] + \\ \frac{i \operatorname{sign}(\operatorname{Im}(z))}{z^3} \left[-\frac{2059}{560} \zeta_2 \sqrt{-z} + \pi \left(-\frac{2407}{420} \log(-z) + \frac{3}{2} \zeta_2 + \frac{20518}{1225} \right) \right] + \mathcal{O}(1/z^{7/2}),$$

$$B_{N_f}(z) = \frac{1}{z^3} \left[-\frac{133}{36} \log(-z) - \frac{404}{105} \zeta_2 + \frac{51}{4} \right] \\ + \frac{i \operatorname{sign}(\operatorname{Im}(z))}{z^3} \left[-\frac{3}{8} \zeta_2 \sqrt{-z} + \pi \left(\frac{26}{21} \log(-z) - \frac{196003}{88200} \right) \right] + \mathcal{O}(1/z^{7/2}).$$

- The strong suppression in $1/z$ is remarkable.

Summary

- 🏆 After almost 40 years since the EEC has been computed at LO, we now finally have the fully analytic NLO result.
- 🏆 The result is amazingly simple and can be written in terms of functions with maximal transcendental weight 3.
- 🏆 New insights into the mathematical structure of event shape observables.
- 🏆 Testbench for new resummation and subtraction techniques.
- 🔧 This calculation would hardly be possible without recent advances in our understanding of phase space integrals, IBP-reduction and differential equations.
- 🔧 The availability of public software tools for dealing with multi-loop integrals was enormously helpful.

Outlook

- 🔍 Most event shape observables are lacking full analytic results beyond LO. Can this situation be improved?
- 🔍 Fully analytic result for EEC at NNLO would be highly desirable but also much more difficult to obtain. More powerful techniques and tools?

$$\cos \theta_{ab} = \frac{(\mathbf{p}_a \cdot \mathbf{p}_b)}{|\mathbf{p}_a||\mathbf{p}_b|} = \frac{E_a E_b - (p_a \cdot p_b)}{E_a E_b} = 1 - \frac{(p_a \cdot p_b)}{E_a E_b}.$$

In the CM frame with $Q^\mu = (Q, 0, 0, 0)^T$ we have

$$\cos \theta_{ab} = 1 - \frac{Q^2(p_a \cdot p_b)}{(Q \cdot p_a)(Q \cdot p_b)}.$$

Hence,

$$\begin{aligned} \delta(\cos \theta_{ab} - \cos \chi) &= \delta \left(1 - \cos \chi - \frac{Q^2(p_a \cdot p_b)}{(Q \cdot p_a)(Q \cdot p_b)} \right) \\ &= \frac{(Q \cdot p_a)(Q \cdot p_b)}{Q^2} \delta \left((1 - \cos \chi)(Q \cdot p_a)(Q \cdot p_b) - Q^2(p_a \cdot p_b) \right) \end{aligned}$$