Building bases for analytical fits of four-loop master integrals

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Summary

- 4-loop g-2 summary
- the master integrals of 4-loop g-2
- Families of constants
- Harmonic polylogarithms
- Elliptical constants
- New results

Mass-independent QED contribution, 1-3 loop

$$a_e^{QED} = C_1\left(\frac{\alpha}{\pi}\right) + C_2\left(\frac{\alpha}{\pi}\right)^2 + C_3\left(\frac{\alpha}{\pi}\right)^3 + C_4\left(\frac{\alpha}{\pi}\right)^4 + C_5\left(\frac{\alpha}{\pi}\right)^5 + \ldots$$

$$C_1 = \frac{1}{2}$$ (Schwinger 1948) *1 diagram*

$$C_2 = \frac{197}{144} + \frac{1}{12}\pi^2 - \frac{1}{2}\pi^2 \ln 2 + \frac{3}{4}\zeta(3)$$

$$= -0.328\,478\,965\,579\ldots,$$ (Petermann, Sommerfield 1957) *7 diagrams*

$$C_3 = \frac{83}{72}\pi^2\zeta(3) - \frac{215}{24}\zeta(5) + \frac{100}{3}\left[\left(a_4 + \frac{1}{24}\ln^4 2\right) - \frac{1}{24}\pi^2\ln^2 2\right]$$

$$- \frac{239}{2160}\pi^4 + \frac{139}{18}\zeta(3) - \frac{298}{9}\pi^2\ln 2 + \frac{17101}{810}\pi^2 + \frac{28259}{5184}$$

$$= 1.181\,241\,456\ldots,$$ (S.L., E.Remiddi 1996) *72 diagrams*

$\zeta(p)=\sum_{n=0}^{\infty} \frac{1}{n^p}$, $\quad a_4=\sum_{n=0}^{\infty} \frac{1}{2^n n^4}$, ...

Mass-independent QED contribution, 4-loop

$C_4 \rightarrow$ **891 diagrams** obtained by inserting a photon in 104 self-mass diagrams

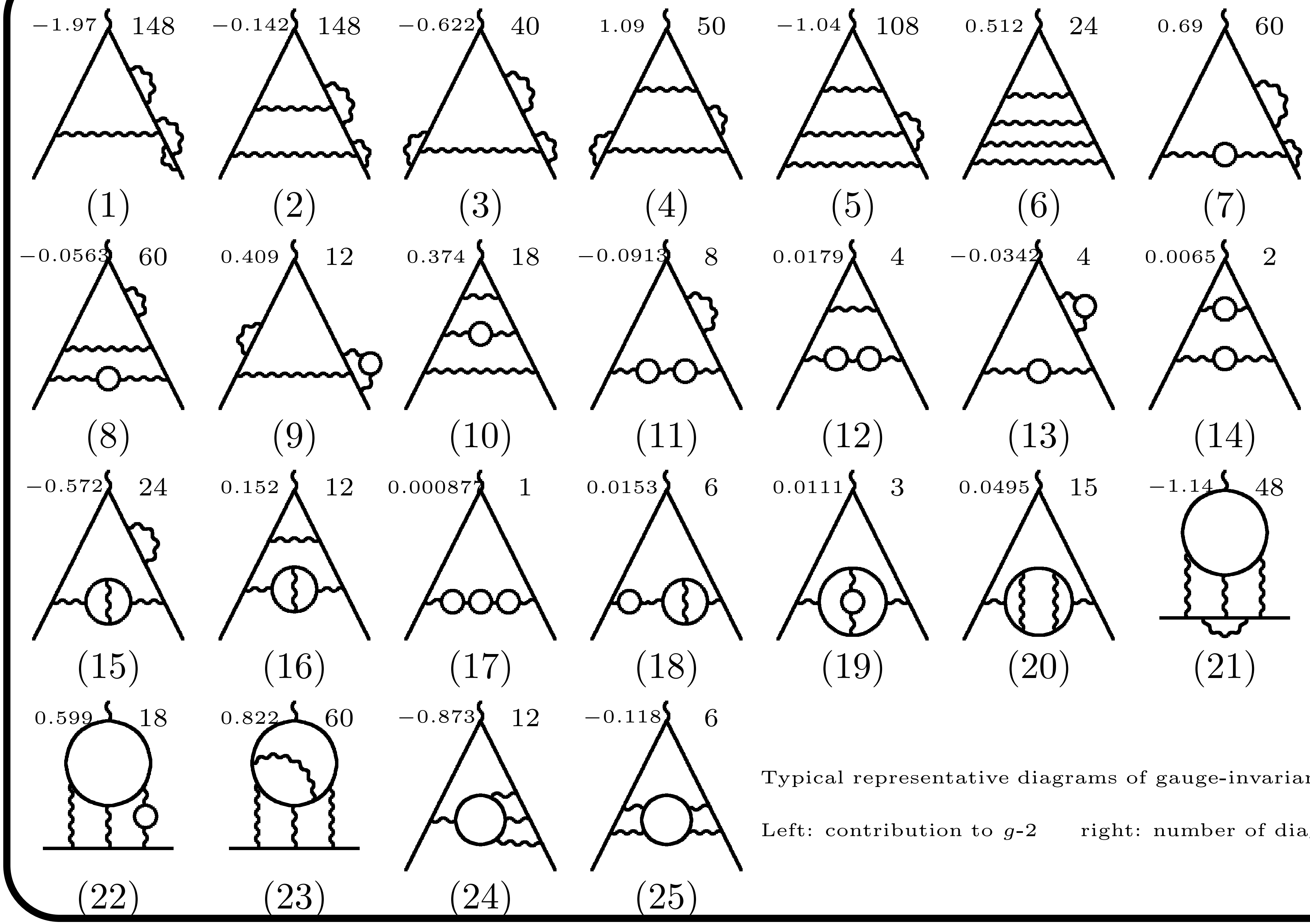
1100-digits result is (S.L., 2017):

$C_4 =$

```
-1.912245764926445574152647167439830054060873390658725345171329848006
0384439806517061427608927000036315837558415331473270056378514912854539
1902804327050273822304345578957045562729309941296699760277782211578472
0339064151908166527097970867438115012155147972274322164273431927975958
6074050057837384960701874328314024838025192249460742298558930463506140
4922526634310944240002356356881280620645494013224977594300429288836761
7488992369151808780869897052635785337537769641170245361960134975744943
6126848617516260683238718674730383150596274187801530551487940053697779
8369464278684326918431175889581159743566950433048349073613426586499531
1638781174347538542348836408558444188223721745670687104182330743051744
3055739459611715508589611489952612660612469940731184039274723400234649
6953173548258481799822409737371077365740464513521123091242528111137215
3021544537210148111211598489708842232798797204842014451228284515165852
3656178659459260099173303172130286546721234534050034910470072892448720
0616044261325449069000431915198230047488181494311038495378299406296758
6787538524978194698979313216219797575067670114290489796208505...
```

- For the sake of comparisons it is convenient to regroup the 891 diagrams in 25 gauge-invariant sets
- The contribution of gauge-invariant sets are I.R. finite and invariant under gauge transformations of the internal photons.

one diagram of each one of the 25 gauge-invariant sets



Typical representative diagrams of gauge-invariant sets

Left: contribution to g-2 right: number of diagrams

Steps for g-2

- reduction (exact) of the contribution to 334 master integrals
- high-precision numerical calculation of master integrals
- classification of master integrals $\Leftarrow$
- fit to an analytical basis

A posteriori the analytical expressions of the master integrals turn out to contain elements of different families of analytical constants:

- harmonic polylogarithms of 1 and 1/2.
- harmonic polylogarithms of arguments $e^{i\pi/3}$ and $e^{2i\pi/3}$
- harmonic polylogarithms of arguments $e^{i\pi/2}$
- elliptic constants with semi-analytic representation
- unknown elliptic constants

up to weight 7

See the colors in the next slide

the 104 4-loop electron self-masses

- reduction (exact) of the contribution to 334 master integrals
- high-precision numerical calculation of master integrals
- classification of master integrals
- fit to an analytical basis $\Leftarrow$

Recipe for analytical fits with PSLQ

1. compute an extremely high-precision value of a master integral
2. guess the right analytical ansatz for the basis
3. fit an analytical expression by using the "PSLQ algorithm"

- PSLQ algorithm (Ferguson and Bailey 1992)
- multi-integers extension of the Euclid algorithm for the calculation of the GCD of two integers
- it finds an integer relation between floating point numbers or bounds on size of coefficients.
- it requires high precision; at least number of digits of coefficients * number of real numbers
- there are various implementations of the algorithm publicy available; the fastest is a parallel modification of the algorithm by Bailey and Broadhurst 1999.

The simplest family: Harmonic polylogarithms of 1, 1/2

weight	number	constants
0	1	1
1	1	$\ln 2$
2	2	$\zeta(2),\ \ln^2 2$
3	3	$\zeta(3),\ \zeta(2)\ln 2,\ \ln^3 2$
4	5	$\zeta(4),\ \zeta(3)\ln 2,\ \zeta(2)\ln^2 2,\ \ln^4 2,\ a_4$
5	8	$\zeta(5),\ \zeta(4)\ln 2,\ \zeta(3)\ln^2 2,\ \zeta(2)\ln^3 2,\ \zeta(3)\zeta(2),\ a_4\ln 2,\ \ln^5 2,\ a_5$
6	13	$\zeta(6),\ \zeta(3)^2,\ \ldots,\ a_5\ln 2,\ \ln^6 2,\ a_6,\ b_6$
7	21	$\zeta(7),\ \zeta(4)\zeta(3),\ \ldots,\ b_6\ln 2,\ \ln^7 2,\ a_7,\ b_7,\ d_7$
n	F_{n+1}	

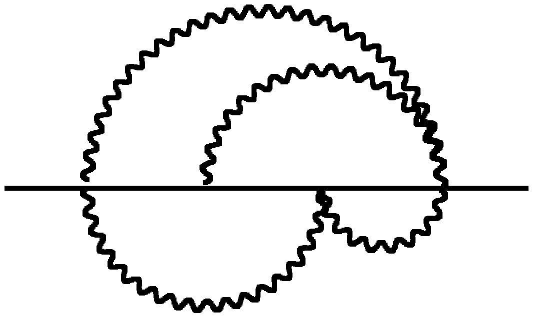
Fibonacci numbers $F_0 = 1,\ F_1 = 1,\ F_{n+1} = F_n + F_{n-1},$

$\zeta(n) = \mathrm{Li_n}(1),\ a_n = \mathrm{Li_n}(1/2)$ common polylogarithms

$b_6 = S_{4,2}(1/2),\ b_7 = S_{5,2}(1/2)$ Nielsen polylogarithms

$d_7 = H_{0,0,0,0,1,-1,-1}(1)$ only with harmonic polylogarithms

PSLQ example: analytical fit of a simple master integral



$$M = G_1\epsilon^{-4} + G_2\epsilon^{-3} + G_3\epsilon^{-2} + G_4\epsilon^{-1} + G_5 + G_6\epsilon + G_7\epsilon^2 + G_8\epsilon^3 + \ldots$$

$$\epsilon = (4-D)/2$$

G_i calculated numerically and fitted

$$G_1 = -\frac{1}{8} \qquad G_2 = -\frac{49}{48}$$

$$G_3 = -\frac{449}{96} - \frac{1}{6}\pi^2 - \frac{1}{2}\zeta(3) \qquad G_4 = -\frac{2429}{192} - \frac{7}{4}\pi^2 - \frac{11}{2}\zeta(3) - \frac{1}{120}\pi^4$$

$$G_5 = \frac{2687}{384} - \frac{277}{24}\pi^2 - \frac{125}{3}\zeta(3) + \frac{1}{24}\pi^4 + \frac{3}{2}\zeta(5) + \frac{2}{3}\pi^2\zeta(3)$$

$$G_6 = \frac{95689}{256} - \frac{2377}{48}\pi^2 - \frac{2999}{12}\zeta(3) - \frac{46}{45}\pi^4 + \frac{149}{2}\zeta(5) + \frac{15}{2}\zeta(3)^2 - \frac{58}{3}\pi^2\zeta(3) - \frac{29}{270}\pi^6$$

PSLQ example: analytical fit of a simple master integral

$$
\begin{aligned}
G_7 = -\,& 2342.20751410602307542352254059079270988532873205655947080 7\\
& 3594814835713846916806455916973185992614831948904197343569 86\\
& 6405364828391809277375993763069797378291106083117076717679 35\\
& 9831391259607669183299238838719305848684965160728687292431 83\\
& 3178005196947599399147517611412834358100307911368387937080 71\\
& 1573460997870203023575268524120954362873328464489262424305 03\\
& 2364495474744073075812911236379210785864186765175498779728 67\\
& \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
= & \frac{1671597}{512} - \frac{4381}{96}\pi^2 - \frac{22193}{24}\zeta(3) - 144\,\pi^2\ln 2 - \frac{3617}{240}\pi^4 - \frac{71}{2}\zeta(5)\\
& - \frac{393}{2}\pi^2\zeta(3) - \frac{869}{162}\pi^6 - 24\pi^4\ln^2 2 + 576\pi^2 a_4 + 24\pi^2\ln^4 2 - \frac{803}{2}\zeta(3)^2\\
& + 504\pi^2\zeta(3)\ln 2 - \frac{1735}{4}\zeta(7) + \frac{799}{6}\pi^2\zeta(5) - \frac{661}{180}\pi^4\zeta(3)
\end{aligned}
$$

black: ansatz brown: PSLQ result

example output pslq

```
PSLQM3 integer relation detection: n =   54
Iteration    532   updtmp: Min, max of y =   7.125852D  -124   8.335992D  -121
Iteration    532   Norm bound =   1.330171D    1   Max. bound =   1.330171D    1
Iteration   1210   updtmp: Min, max of y =   4.109989D  -247   8.231208D  -243
Iteration   1210   Norm bound =   2.868041D    3   Max. bound =   2.868041D    3
Iteration   1887   updtmp: Min, max of y =   2.529002D  -368   5.851086D  -365
Iteration   1887   Norm bound =   5.545033D    5   Max. bound =   5.545033D    5
Iteration   2174   updtmp: Min, max of y =   1.427865D -1193   4.367510D  -415
Iteration   2174   updtmp: Small value in y =   1.427865D -1193
Iteration   2174   Norm bound =   4.643888D    6   Max. bound =   4.643888D    6
Iteration   2174   Relation detected
Min, max of y =   1.427865D -1193   4.367510D  -415
Max. bound =   4.643888D     6
Index of relation =   5   Norm =    1.26788D    7   Residual =   1.427865D -1193
CPU times:
        0.17        0.02        2.88        2.28        5.75        0.00
Recovered relation: 0 =
+                                                        -5014791. * 1
+                                                          420576. * z2
+                                                         1420352. * z3
+                                                         1327104. * z2*lg2
+                                                         2083392. * z4
+                                                           54528. * z5
+                                                         1810944. * z3*z2
+                                                         7786240. * z6
+                                                         3317760. * z4*lg2^2
+                                                         -221184. * z2*lg2^4
+                                                          616704. * z3^2
+                                                        -4644864. * z3*z2*lg2
+                                                        -5308416. * a4*z2
+                                                          666240. * z7
+                                                        -1227264. * z2*z5
+                                                          507648. * z4*z3
+                                                            1536. * G7
CPU Time =     13.3480
```

publicy available PSLQ codes

1. Internal implementation in Mathematica
2. C code from Paul Zimmermann https://members.loria.fr/PZimmermann/software/pslq-1.0.c
3. Multiprecision fortran code from David Bailey, implementing sequential and parallel PSLQ http://crd-legacy.lbl.gov/ dhbailey/mpdist/

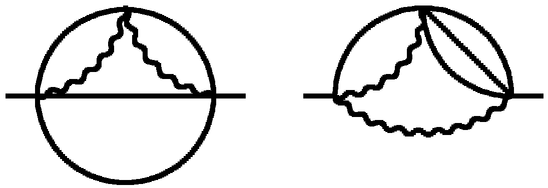
Only (3) works with basis of 500+ terms.

$$C_4 = T + \sqrt{3}V_a + V_b + W_b + \sqrt{3}E_a + E_b + U$$

46/54 terms

$$\begin{aligned}
T = & \frac{1243127611}{130636800} + \frac{30180451}{25920}\zeta(2) - \frac{255842141}{2721600}\zeta(3) - \frac{8873}{3}\zeta(2)\ln 2 + \frac{6768227}{2160}\zeta(4) \\
& + \frac{19063}{360}\zeta(2)\ln^2 2 + \frac{12097}{90}\left(a_4 + \frac{1}{24}\ln^4 2\right) - \frac{2862857}{6480}\zeta(5) - \frac{12720907}{64800}\zeta(3)\zeta(2) \\
& - \frac{221581}{2160}\zeta(4)\ln 2 + \frac{9656}{27}\left(a_5 + \frac{1}{12}\zeta(2)\ln^3 2 - \frac{1}{120}\ln^5 2\right) + \frac{191490607}{46656}\zeta(6) + \frac{10358551}{43200}\zeta^2(3) \\
& - \frac{40136}{27}a_6 + \frac{26404}{27}b_6 - \frac{700706}{675}a_4\zeta(2) - \frac{26404}{27}a_5\ln 2 + \frac{26404}{27}\zeta(5)\ln 2 - \frac{63749}{50}\zeta(3)\zeta(2)\ln 2 \\
& - \frac{40723}{135}\zeta(4)\ln^2 2 + \frac{13202}{81}\zeta(3)\ln^3 2 - \frac{253201}{2700}\zeta(2)\ln^4 2 + \frac{7657}{1620}\ln^6 2 + \frac{2895304273}{435456}\zeta(7) \\
& + \frac{670276309}{193536}\zeta(4)\zeta(3) + \frac{85933}{63}a_4\zeta(3) + \frac{7121162687}{967680}\zeta(5)\zeta(2) - \frac{142793}{18}a_5\zeta(2) - \frac{195848}{21}a_7 \\
& + \frac{195848}{63}b_7 - \frac{116506}{189}d_7 - \frac{4136495}{384}\zeta(6)\ln 2 - \frac{1053568}{189}a_6\ln 2 + \frac{233012}{189}b_6\ln 2 \\
& + \frac{407771}{432}\zeta^2(3)\ln 2 - \frac{8937}{2}a_4\zeta(2)\ln 2 + \frac{833683}{3024}\zeta(5)\ln^2 2 - \frac{3995099}{6048}\zeta(3)\zeta(2)\ln^2 2 \\
& - \frac{233012}{189}a_5\ln^2 2 + \frac{1705273}{1512}\zeta(4)\ln^3 2 + \frac{602303}{4536}\zeta(3)\ln^4 2 - \frac{1650461}{11340}\zeta(2)\ln^5 2 + \frac{52177}{15876}\ln^7 2
\end{aligned}$$

$a_n = \mathrm{Li}_n(1/2)$, $b_6 = H_{0,0,0,0,1,1}(1/2)$, $b_7 = H_{0,0,0,0,0,1,1}(1/2)$, $d_7 = H_{0,0,0,0,1,-1,-1}(1)$

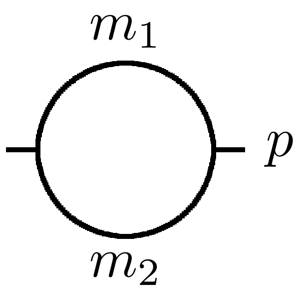


$$I(a) = \frac{7}{12\epsilon^4} + \frac{10}{3\epsilon^3} + \frac{121}{12\epsilon^2} + \left(\frac{1541}{72} + \frac{7}{6}\zeta(3)\right)\epsilon^{-1} + X_a^{(0)} + O(\epsilon)$$

$$I(b) = \frac{5}{8\epsilon^4} + \frac{59}{16\epsilon^3} + \left(\frac{1099}{36} + 3\zeta(2)\right)\epsilon^{-2} + \left(\frac{3781}{192} + \frac{33}{2}\zeta(2) + 6\zeta(3)\right)\epsilon^{-1} + X_b^{(0)} + O(\epsilon)$$

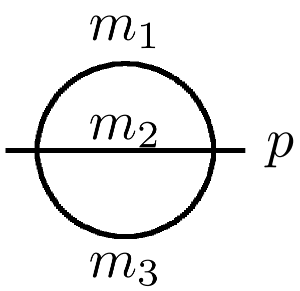
- $X_a^{(0)}$ and $X_b^{(0)}$ cannot be fitted with HPL of 1 and 1/2
- It must contain different analytical constants
- Closing the external line of I(a) or I(b)
  $\rightarrow$ same vacuum diagram
  $\rightarrow$ $X_a^{(0)}$ and $X_b^{(0)}$ must contain the same constants

$D = 2$



$$= \int \frac{kdk}{k^2 + m_1^2} \int \frac{d\Omega_k}{(p - k_1)^2 + m_2^2} = \int_0^\infty \frac{dl}{l + m_1^2} \frac{1}{R(l, p^2, -m_2^2)}$$

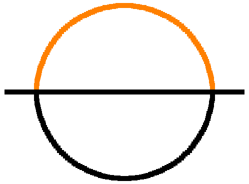
$R(x, y, z) = \sqrt{(x - y - z)^2 - 4yz}$ Källen function



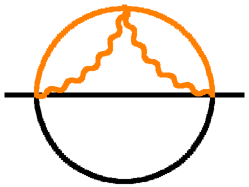
$$= \int_0^\infty \frac{dl}{l + m_1^2} \int_0^\infty \frac{dr}{R(l, r, -m_2^2) R(r, p^2, -m_3^2)}$$

disclaimer: in general analytic continuation to timelike moments may need a deformation of the contours of the radial integrals; not needed in this case

$m_i = 1$



$$= \int_0^\infty dl \frac{1}{l+1} \int_0^\infty \frac{dr}{R(l,r,-1)R(r,-1,-1)}$$



$$\to \int_0^\infty l\, dl \left(\frac{\ln(l+1)}{l}\right)^2 \int_0^\infty \frac{dr}{R(l,r,-1)R(r,-1,-1)}$$

Tentative basis: ad-hoc family of integrals

$$I(n_1, n_2) = \int_0^\infty \frac{dl}{l} \ln^{n_1} l \ln^{n_2}(l+1) \ldots \int_0^\infty \frac{dr}{R(l,r,-1)R(r,-1,-1)}$$

PSLQ gives

$$X_a^{(0)} = \frac{42155}{432} - \frac{380}{3}\zeta(2) + \frac{14}{3}\zeta(3) + \frac{3}{2}\zeta(4) - \frac{3}{2} I(0,2)$$

integrating analytically I(0,2) one finds (harmonic) polylogarithms of argument $e^{i\pi/3}$

$$X_a^{(0)} = \frac{42155}{432} - \frac{380}{3}\zeta(2) + \frac{14}{3}\zeta(3) + \frac{3}{2}\zeta(4) + \sqrt{3}\left(6\mathrm{Cl}_4\left(\frac{\pi}{3}\right) - 10\zeta(2)\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\right)$$

using this basis

$$X_b^{(0)} = \frac{25033}{1152} - \frac{47}{4}\zeta(2) + \frac{69}{2}\zeta(3) + \frac{411}{8}\zeta(4) - \sqrt{3}\left(9\mathrm{Cl}_4\left(\frac{\pi}{3}\right) + 9\zeta(2)\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\right)$$

Coefficients of $I(a)$ of power $>= 0$ have 6, 18, 61 and 199 terms, respectively.

Harmonic polylogarithms of 1, 1/2 and $e^{i\pi/3}$, $e^{2i\pi/3}$

weight	#re+im	constants
0	$1+0$	1
1	$2+1$	$\ln 2,\ \ln 3,\ \pi\sqrt{3}$
2	$5+3$	$\zeta(2),\ \ln^2 2,\ \ln 2 \ln 3,\ \ln^2 3,\ \mathrm{Re}(H_{1,-1}(e^{i\pi/3})),\ \pi \ln 3\sqrt{3},\ \mathrm{Cl}_2\left(\frac{\pi}{3}\right)\sqrt{3}, \ldots$
3	$12+9$	$\zeta(3),\ \mathrm{Re}(H_{1,1,-1}(e^{i\pi/3})),\ \mathrm{Im}(H_{0,1,1}(e^{2i\pi/3}))\sqrt{3},\ \ldots$
4	$30+25$	$\zeta(4),\ \mathrm{Re}(H_{0,0,1,1}(e^{2i\pi/3})),\ \mathrm{Im}(H_{0,1,1,1}(e^{2i\pi/3}))\sqrt{3},\ \mathrm{Im}(H_{0,1,1,-1}(e^{i\pi/3}))\sqrt{3},\ \ldots$
5	$76+68$	$\zeta(5),\ \mathrm{Re}(H_{0,0,1,1,1}(e^{2i\pi/3})),\ \mathrm{Im}(H_{0,0,0,1,1}(e^{i\pi/3}))\sqrt{3},\ \ldots$
6	$195+82$	$\zeta(6),\ \mathrm{Re}(H_{0,0,1,1,0,1}(e^{i\pi/3})),\ \mathrm{Im}(H_{0,0,0,1,1,1}(e^{2i\pi/3}))\sqrt{3},\ \ldots$
7	$504+483$	$\zeta(7),\ \mathrm{Re}(H_{0,0,0,0,1,1,1}(e^{2i\pi/3})),\ \ldots$
n	F_{2n+2}	

$F_{2n+2} \sim 2.618^n$ grows rapidly,
for large n, a challenge!

$$C_4 = T + \sqrt{3}V_a + V_b + W_b + \sqrt{3}E_a + E_b + U$$

22/671 terms

terms of weight 5 cancel out

$$\begin{aligned}
V_a =& -\frac{14101}{480}\mathrm{Cl}_4\left(\frac{\pi}{3}\right) - \frac{169703}{1440}\zeta(2)\mathrm{Cl}_2\left(\frac{\pi}{3}\right) \\
&+\frac{494}{27}\mathrm{Im}H_{0,0,0,1,-1,-1}\left(e^{i\frac{\pi}{3}}\right) + \frac{494}{27}\mathrm{Im}H_{0,0,0,1,-1,1}\left(e^{i\frac{2\pi}{3}}\right) + \frac{494}{27}\mathrm{Im}H_{0,0,0,1,1,-1}\left(e^{i\frac{2\pi}{3}}\right) \\
&+19\mathrm{Im}H_{0,0,1,0,1,1}\left(e^{i\frac{2\pi}{3}}\right) + \frac{437}{12}\mathrm{Im}H_{0,0,0,1,1,1}\left(e^{i\frac{2\pi}{3}}\right) + \frac{29812}{297}\mathrm{Cl}_6\left(\frac{\pi}{3}\right) \\
&+\frac{4940}{81}a_4\mathrm{Cl}_2\left(\frac{\pi}{3}\right) - \frac{520847}{69984}\zeta(5)\pi - \frac{129251}{81}\zeta(4)\mathrm{Cl}_2\left(\frac{\pi}{3}\right) \\
&-\frac{892}{15}\mathrm{Im}H_{0,1,1,-1}\left(e^{i\frac{2\pi}{3}}\right)\zeta(2) - \frac{1784}{45}\mathrm{Im}H_{0,1,1,-1}\left(e^{i\frac{\pi}{3}}\right)\zeta(2) + \frac{1729}{54}\zeta(3)\mathrm{Im}H_{0,1,-1}\left(e^{i\frac{\pi}{3}}\right) \\
&+\frac{1729}{36}\zeta(3)\mathrm{Im}H_{0,1,1}\left(e^{i\frac{2\pi}{3}}\right) + \frac{837190}{729}\mathrm{Cl}_4\left(\frac{\pi}{3}\right)\zeta(2) + \frac{25937}{4860}\zeta(3)\zeta(2)\pi \\
&-\frac{223}{243}\zeta(4)\pi\ln 2 + \frac{892}{9}\mathrm{Im}H_{0,1,-1}\left(e^{i\frac{\pi}{3}}\right)\zeta(2)\ln 2 + \frac{446}{3}\mathrm{Im}H_{0,1,1}\left(e^{i\frac{2\pi}{3}}\right)\zeta(2)\ln 2 \\
&-\frac{7925}{81}\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\zeta(2)\ln^2 2 + \frac{1235}{486}\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\ln^4 2
\end{aligned}$$

$$\mathrm{Cl}_n(\theta) = \mathrm{ImLi}_n(e^{i\theta})$$

analytical fit part 2...

$$C_4 = T + \sqrt{3}V_a + V_b + W_b + \sqrt{3}E_a + E_b + U$$

17/825 terms

$$\begin{aligned}
V_b =& \frac{13487}{60}\mathrm{Re}H_{0,0,0,1,0,1}\left(e^{i\frac{\pi}{3}}\right) + \frac{13487}{60}\mathrm{Cl}_4\left(\frac{\pi}{3}\right)\mathrm{Cl}_2\left(\frac{\pi}{3}\right) + \frac{136781}{360}\mathrm{Cl}_2^2\left(\frac{\pi}{3}\right)\zeta(2) \\
&+ \frac{651}{4}\mathrm{Re}H_{0,0,0,1,0,1,-1}\left(e^{i\frac{\pi}{3}}\right) + 651\mathrm{Re}H_{0,0,0,0,1,1,-1}\left(e^{i\frac{\pi}{3}}\right) - \frac{17577}{32}\mathrm{Re}H_{0,0,1,0,0,1,1}\left(e^{i\frac{2\pi}{3}}\right) \\
&- \frac{87885}{64}\mathrm{Re}H_{0,0,0,1,0,1,1}\left(e^{i\frac{2\pi}{3}}\right) - \frac{17577}{8}\mathrm{Re}H_{0,0,0,0,1,1,1}\left(e^{i\frac{2\pi}{3}}\right) \\
&+ \frac{651}{4}\mathrm{Cl}_4\left(\frac{\pi}{3}\right)\mathrm{Im}H_{0,1,-1}\left(e^{i\frac{\pi}{3}}\right) + \frac{1953}{8}\mathrm{Cl}_4\left(\frac{\pi}{3}\right)\mathrm{Im}H_{0,1,1}\left(e^{i\frac{2\pi}{3}}\right) + \frac{31465}{176}\mathrm{Cl}_6\left(\frac{\pi}{3}\right)\pi \\
&+ \frac{211}{4}\mathrm{Re}H_{0,1,0,1,-1}\left(e^{i\frac{\pi}{3}}\right)\zeta(2) + \frac{211}{2}\mathrm{Re}H_{0,0,1,1,-1}\left(e^{i\frac{\pi}{3}}\right)\zeta(2) \\
&+ \frac{1899}{16}\mathrm{Re}H_{0,1,0,1,1}\left(e^{i\frac{2\pi}{3}}\right)\zeta(2) + \frac{1899}{8}\mathrm{Re}H_{0,0,1,1,1}\left(e^{i\frac{2\pi}{3}}\right)\zeta(2) \\
&+ \frac{211}{4}\mathrm{Im}H_{0,1,-1}\left(e^{i\frac{\pi}{3}}\right)\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\zeta(2) + \frac{633}{8}\mathrm{Im}H_{0,1,1}\left(e^{i\frac{2\pi}{3}}\right)\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\zeta(2)
\end{aligned}$$

Harmonic polylogarithms of 1, 1/2 and $e^{i\pi/2}$

weight	#re+im	constants
0	$1+0$	1
1	$1+1$	$\ln 2,\ \pi$
2	$2+2$	$\zeta(2),\ \ln^2 2,\ \pi \ln 2,\ \beta_2$
3	$4+4$	$\zeta(3),\ \pi\beta_2, \mathrm{Im}(H_{0,1,1}(e^{i\pi/2})),\ \ldots$
4	$8+8$	$\zeta(4),\ \beta_2^2,\ \beta_4,\ \mathrm{Im}(H_{0,1,1,1}(e^{i\pi/2}))\sqrt{3},\ \ldots$
5	$16+16$	$\zeta(5),\ \mathrm{Re}(H_{0,1,0,1,1}(e^{i\pi/2})),\ \mathrm{Im}(H_{0,0,0,1,1}(e^{i\pi/2})),\ \ldots$
6	$32+32$	$\zeta(6),\ \mathrm{Re}(H_{0,0,0,1,0,1}(e^{i\pi/2})),\ \mathrm{Im}(H_{0,0,0,1,1,1}(e^{i\pi/2})),\ \ldots$
7	$64+64$	$\zeta(7),\ \mathrm{Re}(H_{0,0,0,1,0,1,1}(e^{i\pi/2})),\ \mathrm{Im}(H_{0,0,0,0,0,1,1}(e^{i\pi/2})),\ \ldots$
n	2^n	

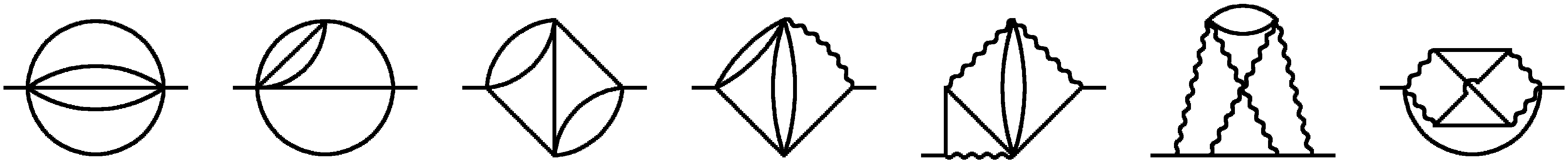
$\beta_2 = \frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \ldots$ is the Catalan's constant

$$C_4 = T + \sqrt{3}V_a + V_b + W_b + \sqrt{3}E_a + E_b + U$$

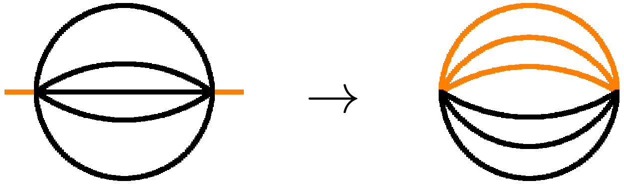
5/256 terms

$$W_b = -\frac{28276}{25}\zeta(2)\mathrm{Cl}_2\left(\frac{\pi}{2}\right)^2$$
$$+104\Bigg(4\mathrm{Re}H_{0,1,0,1,1}\left(e^{i\frac{\pi}{2}}\right)\zeta(2) + 4\mathrm{Im}H_{0,1,1}\left(e^{i\frac{\pi}{2}}\right)\mathrm{Cl}_2\left(\frac{\pi}{2}\right)\zeta(2)$$
$$-2\mathrm{Cl}_4\left(\frac{\pi}{2}\right)\zeta(2)\pi + \mathrm{Cl}_2^2\left(\frac{\pi}{2}\right)\zeta(2)\ln 2\Bigg)$$

Elliptic master integrals



(a) (b) (c) (d) (e) (f) (g)



$$= \int_0^\infty dl \left(\int_0^\infty \frac{dr}{R(l,r,-1)R(r,-1,-1)} \right)^2$$

$$\to \int_9^\infty ds \left(\int_9^\infty \frac{db}{R(s,b,1)R(s,1,1)} \right)^2$$

$$= \int_9^\infty ds D_1^2(s)$$

$$D_1(s) = \frac{2}{\sqrt{(\sqrt{s}+3)(\sqrt{s}-1)^3}} K\left(\frac{(\sqrt{s}-3)(\sqrt{s}+1)^3}{(\sqrt{s}+3)(\sqrt{s}-1)^3} \right) ,$$

$$D_2(s) = \frac{2}{\sqrt{(\sqrt{s}+3)(\sqrt{s}-1)^3}} K\left(1 - \frac{(\sqrt{s}-3)(\sqrt{s}+1)^3}{(\sqrt{s}+3)(\sqrt{s}-1)^3} \right) ;$$

$K(x)$ is the complete elliptic integral of the first kind; $D_1(s) \sim$ discontinuity of the 2-loop sunrise diagram with equal masses in $D = 2$ dimensions.

$$f_1(i,j,k) = \int_1^9 ds\, D_1^2(s)\left(s-\frac{9}{5}\right)\ln^i(9-s)\ln^j(s-1)\ln^k(s)\ ,$$

$$f_2(i,j,k) = \int_1^9 ds\, D_1(s)\mathrm{Re}\left(\sqrt{3}D_2(s)\right)\left(s-\frac{9}{5}\right)\ln^i(9-s)\ln^j(s-1)\ln^k(s)\ ,$$

$(s-9/5)$ ad-hoc factor

constant term of $I(a)$ fitted with integrals of elliptic integrals squared (SL,2008).

$$I(a, D{=}4\text{-}2\epsilon) = -\frac{5}{2\epsilon^4} - \frac{45}{4\epsilon^3} - \frac{4255}{144\epsilon^2} - \frac{106147}{1728\epsilon} + \frac{\pi\sqrt{3}}{240}(297B_3 - 1477C_3) - \frac{2320981}{20736} + O(\epsilon)$$

$$I(a, D{=}2\text{-}2\epsilon) = \sqrt{3}\pi B_3 + M_{501}\epsilon + M_{602}\epsilon^2 - M_{702}\epsilon^3 + \ldots$$

hypergeometric constants

$$A_3 = \int_0^1 dx \frac{K_c(x) K_c(1\text{-}x)}{\sqrt{1-x}} = \frac{2\pi^{\frac{3}{2}}}{3} \left(\frac{\Gamma^2(\frac{7}{6})\Gamma(\frac{1}{3})}{\Gamma^2(\frac{2}{3})\Gamma(\frac{5}{6})} {}_4F_3\left(\begin{matrix} \frac{1}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{2} \\ \frac{5}{6} & \frac{5}{6} & \frac{2}{3} & \end{matrix} ; 1 \right) - \frac{\Gamma^2(\frac{5}{6})\Gamma(-\frac{1}{3})}{\Gamma^2(\frac{1}{3})\Gamma(\frac{1}{6})} {}_4F_3\left(\begin{matrix} \frac{1}{2} & \frac{2}{3} & \frac{2}{3} & \frac{5}{6} \\ \frac{7}{6} & \frac{7}{6} & \frac{4}{3} & \end{matrix} ; 1 \right) \right)$$

$$B_3 = \int_0^1 dx \frac{K_c^2(x)}{\sqrt{1-x}} = \frac{4\pi^{\frac{3}{2}}}{3} \left(\frac{\Gamma^2(\frac{7}{6})\Gamma(\frac{1}{3})}{\Gamma^2(\frac{2}{3})\Gamma(\frac{5}{6})} {}_4F_3\left(\begin{matrix} \frac{1}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{2} \\ \frac{5}{6} & \frac{5}{6} & \frac{2}{3} & \end{matrix} ; 1 \right) + \frac{\Gamma^2(\frac{5}{6})\Gamma(-\frac{1}{3})}{\Gamma^2(\frac{1}{3})\Gamma(\frac{1}{6})} {}_4F_3\left(\begin{matrix} \frac{1}{2} & \frac{2}{3} & \frac{2}{3} & \frac{5}{6} \\ \frac{7}{6} & \frac{7}{6} & \frac{4}{3} & \end{matrix} ; 1 \right) \right)$$

$$C_3 = \int_0^1 dx \frac{E_c^2(x)}{\sqrt{1-x}} = \frac{486\pi^2}{1925} \, {}_7F_6\left(\begin{matrix} \frac{7}{4} & -\frac{1}{3} & \frac{1}{3} & \frac{2}{3} & \frac{4}{3} & \frac{3}{2} & \frac{3}{2} \\ \frac{3}{4} & 1 & \frac{7}{6} & \frac{11}{6} & \frac{13}{6} & \frac{17}{6} & \end{matrix} ; 1 \right) ,$$

$$K_c(x) = \frac{2\pi}{\sqrt{27}} {}_2F_1\left(\begin{matrix} \frac{1}{3} & \frac{2}{3} \\ 1 & \end{matrix} ; x \right) , \qquad E_c(x) = \frac{2\pi}{\sqrt{27}} {}_2F_1\left(\begin{matrix} \frac{1}{3} & -\frac{1}{3} \\ 1 & \end{matrix} ; x \right) .$$

A_3 cancels out in the diagram contributions

A_3 and B_3 seem to be both irreducible

$$C_4 = T + \sqrt{3}V_a + V_b + W_b + \sqrt{3}E_a + E_b + U$$

$$\begin{aligned}
E_a =& \pi \left(-\frac{28458503}{691200} B_3 + \frac{250077961}{18662400} C_3 \right) + \frac{483913}{77760} \pi f_2(0,0,1) \\
& + \pi \left(\frac{4715}{1944} \ln 2 \; f_2(0,0,1) + \frac{270433}{10935} f_2(0,2,0) - \frac{188147}{4860} f_2(0,1,1) + \frac{188147}{12960} f_2(0,0,2) \right) \\
& + \pi \left(\frac{826595}{248832} \zeta(2) f_2(0,0,1) - \frac{5525}{432} \ln 2 \; f_2(0,0,2) + \frac{5525}{162} \ln 2 \; f_2(0,1,1) \right. \\
& - \frac{5525}{243} \ln 2 \; f_2(0,2,0) + \frac{526015}{248832} f_2(0,0,3) - \frac{4675}{768} f_2(0,1,2) + \frac{1805965}{248832} f_2(0,2,1) \\
& - \frac{3710675}{1119744} f_2(0,3,0) - \frac{75145}{124416} f_2(1,0,2) - \frac{213635}{124416} f_2(1,1,1) + \frac{168455}{62208} f_2(1,2,0) \\
& \left. + \frac{69245}{124416} f_2(2,1,0) \right)
\end{aligned}$$

$$\begin{aligned}
E_b =& -\frac{4715}{1458} \zeta(2) f_1(0,0,1) \\
& + \zeta(2) \left(\frac{2541575}{82944} f_1(0,0,2) - \frac{556445}{6912} f_1(0,1,1) + \frac{54515}{972} f_1(0,2,0) - \frac{75145}{20736} f_1(1,0,1) \right) .
\end{aligned}$$

analytical fit part 4

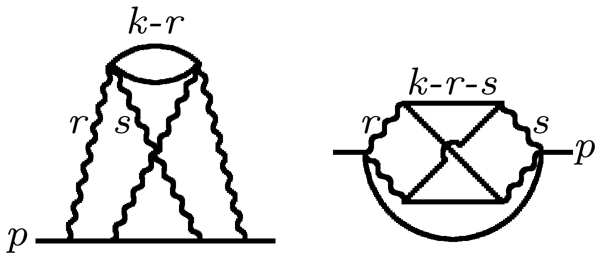
The elements of this family suffices just to fit the combinations of the coefficients of the $\epsilon-$expansions of the elliptic master integrals occurring in the contributions of diagrams. To fit each coefficient one needs to enlarge the basis including also factors like $\ln(s+3)$, $\mathrm{Li}_2(\mathrm{s}/9)$, $\mathrm{Li}_2((\mathrm{s}-1)/8, \ldots$, and/or integrating on different intervals ($[0,1]$, $[9,\infty]$)

For example the fit of M_{702} (ϵ^3 coefficient of $I(a)$) requires 187 elliptic elements, most of them outside the basis $f_1(i,j,k)$, $f_2(i,j,k)$.

$$C_4 = T + \sqrt{3}V_a + V_b + W_b + \sqrt{3}E_a + E_b + U$$

The term containing the ϵ^0 coefficients of the ϵ−expansion of six master integrals (see f, f', f'', g, g', g''):

$$U = -\frac{541}{300}C_{81a} - \frac{629}{60}C_{81b} + \frac{49}{3}C_{81c} - \frac{327}{160}C_{83a} + \frac{49}{36}C_{83b} + \frac{37}{6}C_{83c} \ .$$



(f, f', f'') (g, g', g'')

(f, f', f'') and (g, g', g'') have numerators respectively equal to $(1, p.k, (p.k)^2)$

These master integrals appear in topologies 81 and 83 (gauge-invariant sets 24 and 25, vacuum polarization diagrams containing a light-light scattering).

Last minute add: new results

- 4-loop renormalization constants Z_2^{OS} and Z_m^{OS} in QED
- Part of them $\rightarrow$ coefficients of some color structure in QCD Z_2^{OS} and Z_m^{OS}

$$m_0 = Z_m m, \quad \psi_0 = \sqrt{Z_2}\psi$$

Z_m, Z_2 estracted from the electron self-energy

Does the elliptic basis used for $g-2$ suffice for Z_2 and Z_m ?

Last minute add: new results

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Z_m, Z_2 estracted from the electron self-energy

Does the elliptic basis used for $g-2$ suffice for Z_2 and Z_m ?

Answer: yes

Last minute add: new results: Z_2 and Z_m in QED

Analytical fits of the coefficient of α^4 (α_0^4 for Z_2 and Z_m)

- $F_2^{(4)}(0) \to$ 121 term + divergent parts
- $Z_2^{(4)}(0) \to$ 118 term + divergent parts (missing three HPL(3-5,$e^{i\pi/2}$)
- $Z_m^{(4)}(0) \to$ 73 term + divergent parts (missing many terms at weight 6 and 7)

Last minute add: new results: Z_2 and Z_m in QED

Numerical values ($e^{\epsilon\gamma}$ normalization)

$$Z_2^{(4)} = +\,0.20502387152777\ldots\epsilon^{-4} + 0.59774667245370\ldots\epsilon^{-3} - 0.89328249574801\ldots\epsilon^{-2} - 6.18821133900575\ldots\epsilon^{-1} - 17.2691387464077\ldots + O(\epsilon)$$

to be compared with *Marquard, Smirnov, Smirnov, Steinhauser* (2018)

$$Z_2^{(4)} = +\,0.20500(37)\epsilon^{-4} + 0.5980(27)\epsilon^{-3} - 0.895(21)\epsilon^{-2} - 6.18(17)\ldots\epsilon^{-1} - 17.4(1.6)\ldots + O(\epsilon)$$

Conclusions

- fitting analytically families of master integrals is complicated
- it needs very high precision numerical values
- it needs a deep analysis of some key master integrals
- in the case of g-2 only a few percent of the total number of elements of the basis with polylogarithms of complex argument are needed
- current elliptical basis suffices to fit $F_2(0)$, Z_2, Z_m
- some guide to promptly identify only the needed elements of the basis would be useful, especially at 5-loop level

The End