Differential equations for loop integrals without squared propagators

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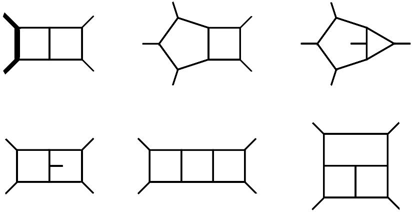

Loops and Legs 2018
St. Goar, 3rd of May 2018

Based on 1712.03760
with Jorrit Bosma and Yang Zhang

Motivation

Differential equations: powerful tool for computing loop integrals.

Given a basis $\boldsymbol{\mathcal{I}} = (\mathcal{I}_1, \ldots, \mathcal{I}_N)$, differentiate wrt. an external invariant x_m

[T. Gehrmann and E. Remiddi, Nucl. Phys., **B580**, 485 (2000)]

$$\frac{\partial}{\partial x_m} \boldsymbol{\mathcal{I}}(\boldsymbol{x}, \epsilon) = A_m(\boldsymbol{x}, \epsilon) \boldsymbol{\mathcal{I}}(\boldsymbol{x}, \epsilon)$$

To determine A_m, use integration-by-parts reductions.

The differential operators typically produce doubled propagators

$$\frac{\partial}{\partial p^\mu} \frac{1}{(\ell - p)^2} = \frac{2(\ell - p)_\mu}{\left((\ell - p)^2\right)^2}$$

Doubled propagators $\Longrightarrow$ larger linear systems to row reduce.

Question: **can we avoid integrals with squared propagators?**

$\longrightarrow$ *Talks by* **Ben Page**
Fernando Febres-Cordero
Mao Zeng

Baikov representation

Consider a generic loop integral,

$$I(N;\alpha_1,\ldots,\alpha_m;D)=\int\prod_{j=1}^{L}\frac{\mathrm{d}^D\ell_j}{i\pi^{D/2}}\frac{\overbrace{D_{k+1}^{\alpha_{k+1}}\cdots D_m^{\alpha_m}}^{N}}{D_1^{\alpha_1}\cdots D_k^{\alpha_k}}$$

Let $\{v_1,\ldots,v_{E+L}\}\equiv\{p_1,\ldots,p_E,\ell_1,\ldots,\ell_L\}$ and, with $x_{i,j}=v_i\cdot v_j$

$$U=\begin{vmatrix}x_{1,1}&\cdots&x_{1,E}\\ \vdots&\ddots&\vdots\\ x_{E,1}&\cdots&x_{E,E}\end{vmatrix}\quad\text{and}\quad F=\begin{vmatrix}x_{1,1}&\cdots&x_{1,E+L}\\ \vdots&\ddots&\vdots\\ x_{E+L,1}&\cdots&x_{E+L,E+L}\end{vmatrix}.$$

The Baikov variables are the inverse propagators $\left(m=LE+\frac{L(L+1)}{2}\right)$

$$z_\alpha=D_\alpha=\sum_{\beta=1}^{m}A_{\alpha\beta}x_\beta+\sum_{1\le i\le j\le E}(B_\alpha)_{ij}\lambda_{ij}\quad\text{with}\quad A_{\alpha\beta},(B_\alpha)_{ij}\in\mathbb{Z}$$

The Baikov representation is

$$I(N;\alpha;D)\propto U^{\frac{E-D+1}{2}}\int\mathrm{d}z_1\cdots\mathrm{d}z_m\frac{z_{k+1}^{\alpha_{k+1}}\cdots z_m^{\alpha_m}}{z_1^{\alpha_1}\cdots z_k^{\alpha_k}}F^{\frac{D-L-E-1}{2}}$$

Differential equations in Baikov representation

Differentiating the Baikov representation

$$I(N;\alpha;D) \propto U^{\frac{E-D+1}{2}} \int \mathrm{d}z_1 \cdots \mathrm{d}z_m \frac{z_{k+1}^{\alpha_{k+1}} \cdots z_m^{\alpha_m}}{z_1^{\alpha_1} \cdots z_k^{\alpha_k}} F^{\frac{D-L-E-1}{2}}$$

wrt. any external invariant χ produces

$$\frac{\partial}{\partial \chi} I(N;\alpha;D) = \frac{E-D+1}{2U} \frac{\partial U}{\partial \chi} I(N;\alpha;D) + \frac{D-L-E-1}{2} \overbrace{I\left(\frac{1}{F}\frac{\partial F}{\partial \chi} N;\alpha;D\right)}^{(D-2)-\text{dimensional}}$$

The dimension shift can be avoided if $\exists$ polynomial (a_i, b) s.t.

[H. Frellesvig and C. G. Papadopoulos, JHEP 04, 083 (2017)]

$$\frac{\partial F}{\partial \chi} = \sum_{i=1}^{m} a_i \frac{\partial F}{\partial z_i} + bF$$

Namely, integration by parts in z_i then yields

$$\frac{\partial}{\partial \chi} I(N;\alpha;D) = \frac{E-D+1}{2U} \frac{\partial U}{\partial \chi} I(N;\alpha;D) - \sum_{i=1}^{m} I\left(z_i^{\alpha_i} \frac{\partial}{\partial z_i} \frac{a_i N}{z_i^{\alpha_i}};\alpha;D\right) + \frac{D-L-E-1}{2} I(bN;\alpha;D)$$

Differential equations in Baikov representation

Differentiating the Baikov representation

$$I(N;\alpha;D) \propto U^{\frac{E-D+1}{2}} \int \mathrm{d}z_1 \cdots \mathrm{d}z_m \frac{z_{k+1}^{\alpha_{k+1}} \cdots z_m^{\alpha_m}}{z_1^{\alpha_1} \cdots z_k^{\alpha_k}} F^{\frac{D-L-E-1}{2}}$$

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Fundamental ideal membership

We refer to the property

$$\frac{\partial F}{\partial \chi} = \sum_{i=1}^{m} a_i \frac{\partial F}{\partial z_i} + bF$$

as the *fundamental ideal membership*.

General solution for the cofactors (a_i, b):

$$a_{ik,\alpha} = -\frac{\partial z_\alpha}{\partial \lambda_{ik}} - \sum_{j=1}^{E} \sum_{q=E+1}^{E+L} (1+\delta_{iq}) \frac{\partial z_\alpha}{\partial x_{iq}} \left(G_i^{-1}\right)_{kj} x_{jq}$$

$$b_{ik} = 2\left(G_i^{-1}\right)_{ki}$$

where G_i is $\mathrm{Gram}(p_1, \ldots, p_E)$ with the ith column multiplied by 2.

Thus, differential equations can always be expressed in Baikov representation without involving dimension shifts.

Differential equations without squared propagators

We have established the fundamental ideal membership

$$\frac{\partial F}{\partial \chi} \in \left\langle \frac{\partial F}{\partial z_1}, \ldots, \frac{\partial F}{\partial z_m}, F \right\rangle .$$

If the following *enhanced* membership ideal property holds,

$$\frac{\partial F}{\partial \chi} \in \left\langle z_1 \frac{\partial F}{\partial z_1}, \ldots, z_k \frac{\partial F}{\partial z_k}, \frac{\partial F}{\partial z_{k+1}}, \ldots, \frac{\partial F}{\partial z_m}, F \right\rangle$$

then the corresponding decomposition of $\frac{\partial F}{\partial \chi}$ leads to diff. eqs. without squared propagators,

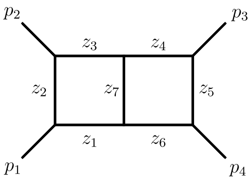
$$\frac{\partial}{\partial \chi} I(N; \alpha; D) = \tfrac{E-D+1}{2U} \tfrac{\partial U}{\partial \chi} I(N; \alpha; D) - \sum_{i=1}^{k} I\Big(z_i \tfrac{\partial}{\partial z_i}(b_i N) + (1-\alpha_i) b_i N; \alpha; D \Big)$$

$$- \sum_{i=k+1}^{m} I\Big(\tfrac{\partial}{\partial z_i}(a_i N) - \tfrac{\alpha_i a_i N}{z_i}; \alpha; D \Big) + \tfrac{D-L-E-1}{2} I(bN; \alpha; D)$$

"linking formula"

as only non-negative powers $z_i^{\alpha_i \geq 0}$ are affected by the red term.

Example 1: massless planar double box (1 of 2)



Set $P_{12} = p_1 + p_2$ and

$$z_1 = \ell_1^2\,, \qquad z_2 = (\ell_1 - p_1)^2\,, \qquad z_3 = (\ell_1 - P_{12})^2$$
$$z_4 = (\ell_2 + P_{12})^2\,, \qquad z_5 = (\ell_2 - p_4)^2\,, \qquad z_6 = \ell_2^2$$
$$z_7 = (\ell_1 + \ell_2)^2\,, \qquad z_8 = (\ell_1 + p_4)^2\,, \qquad z_9 = (\ell_2 + p_1)^2$$

The Baikov polynomial satisfies the enhanced ideal membership,

$$F = \begin{vmatrix} 0 & \frac{s}{2} & -\frac{s+t}{2} & \frac{z_1-z_2}{2} & \frac{z_9-z_6}{2} \\ \frac{s}{2} & 0 & \frac{t}{2} & \frac{z_2-z_3+s}{2} & \frac{z_4-z_9-s}{2} \\ -\frac{s+t}{2} & \frac{t}{2} & 0 & \frac{z_3-z_8-s}{2} & \frac{z_5-z_4+s}{2} \\ \frac{z_1-z_2}{2} & \frac{z_2-z_3+s}{2} & \frac{z_3-z_8-s}{2} & z_1 & \frac{z_7-z_1-z_6}{2} \\ \frac{z_9-z_6}{2} & \frac{z_4-z_9-s}{2} & \frac{z_5-z_4+s}{2} & \frac{z_7-z_1-z_6}{2} & z_6 \end{vmatrix}\,, \qquad \frac{\partial F}{\partial \chi} = \sum_{i=1}^{9} b_i z_i \frac{\partial F}{\partial z_i} + bF$$

The cofactors are, setting $\mathbf{b} = (b_1, \ldots, b_9)$,

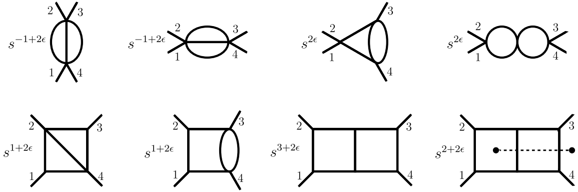
$$\mathbf{b} = \left(\frac{z_3-z_8}{\chi(\chi+1)s}, \frac{z_3-z_8-\chi s-s}{\chi(\chi+1)s}, \frac{z_3-z_8-s}{\chi(\chi+1)s}, \frac{z_4-z_5-s}{\chi(\chi+1)s}, \frac{z_4-z_5-s}{\chi(\chi+1)s}, \frac{z_4-z_5}{\chi(\chi+1)s}, \frac{z_3+z_4-z_5-z_8-s}{\chi(\chi+1)s}, \frac{z_3-z_8-s}{\chi(\chi+1)s}, \frac{z_4-z_5-\chi s-s}{\chi(\chi+1)s} \right)$$

$$b = -\frac{2z_3+2z_4-2z_5-2z_8-2\chi s-3s}{\chi(\chi+1)s}$$

Example 1: massless planar double box (2 of 2)

We insert these **b** into the linking formula and reduce to the basis

[A. Georgoudis, KJL, Y. Zhang, Comput. Phys. Commun. **221** (2017) 203]



and use `Fuchsia` to find a transformation to a canonical basis, yielding

[J. Henn, Phys. Rev. Lett. **110** (2013) 251601]

[O. Gituliar, V. Magerya, Comput. Phys. Commun. **219** (2017) 329]

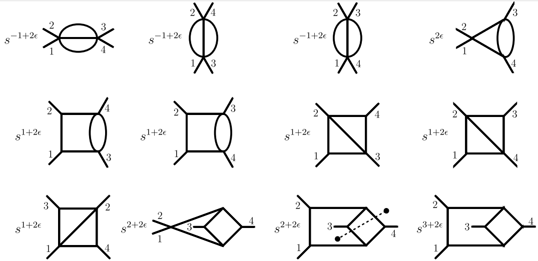
$$\frac{\partial}{\partial \chi}\mathcal{J}(\chi,\epsilon) = \widehat{A}(\chi,\epsilon)\mathcal{J}(\chi,\epsilon)\,, \qquad \widehat{A} = \epsilon\left(\frac{a_0}{\chi} + \frac{a_{-1}}{\chi+1}\right)$$

where a_0 and a_{-1} are matrices with integer entries,

$$a_0 = \begin{pmatrix} -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -60 & -60 & 0 & 0 & -2 & 0 & 0 & 0 \\ 20 & 0 & -4 & 0 & 0 & -2 & 0 & 0 \\ -360 & 360 & 72 & 0 & 12 & 36 & -2 & 0 \\ 540 & -360 & -90 & -9 & -18 & -36 & 1 & 1 \end{pmatrix}, \quad a_{-1} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ -20 & 0 & 4 & 0 & 0 & 1 & 0 & 0 \\ 360 & -720 & -36 & 18 & -12 & -36 & 2 & 2 \\ -540 & 360 & 90 & -9 & 18 & 36 & -1 & -1 \end{pmatrix}$$

Example 2: massless non-planar double box

From the basis



we find

$$\frac{\partial}{\partial \chi}\boldsymbol{\mathcal{J}}(\chi,\epsilon) = \widehat{A}(\chi,\epsilon)\boldsymbol{\mathcal{J}}(\chi,\epsilon)\,, \qquad \widehat{A} = \epsilon\left(\frac{a_0}{\chi} + \frac{a_{-1}}{\chi+1}\right)$$

with

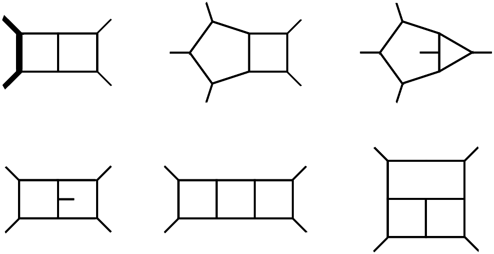
$$a_0 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 20 & 0 & 4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 20 & -4 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ -60 & 0 & -60 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & -60 & -60 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -360 & -60 & 660 & -36 & 0 & -18 & -4 & -16 & -6 & 1 & 1 & -1 \\ -120 & 420 & 420 & -36 & 0 & -18 & -16 & -12 & 6 & 0 & 0 & -2 \end{pmatrix}, \quad a_{-1} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -20 & 0 & -4 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -20 & 4 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 60 & -60 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 60 & 60 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -360 & -180 & -180 & -36 & -18 & 0 & 12 & 0 & 6 & 0 & -2 & 0 \\ -600 & -660 & 60 & -36 & -18 & 0 & 24 & -4 & -6 & 1 & -1 & 1 \end{pmatrix}$$

Further examples of enhanced ideal membership

The enhanced ideal membership

$$\frac{\partial F}{\partial \chi} \in \left\langle z_1 \frac{\partial F}{\partial z_1}, \ldots, z_k \frac{\partial F}{\partial z_k}, \frac{\partial F}{\partial z_{k+1}}, \ldots, \frac{\partial F}{\partial z_m}, F \right\rangle$$

holds true for a large set of non-trivial diagrams.



[S. Badger, C. Brønnum-Hansen, H. Hartanto, T. Peraro, 1712.02229]

[S. Abreu, F. Cordero, H. Ita, B. Page, M. Zeng, 1712.03946]

[D. Chicherin, J. Henn, V. Mitev, 1712.09610]

In all of these examples it is possible to set up differential equations without involving doubled propagators.

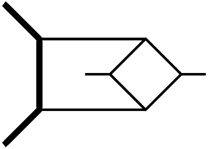
Counterexample of enhanced ideal membership

However, the enhanced ideal membership

$$\frac{\partial F}{\partial \chi} \in \left\langle z_1 \frac{\partial F}{\partial z_1}, \ldots, z_k \frac{\partial F}{\partial z_k}, \frac{\partial F}{\partial z_{k+1}}, \ldots, \frac{\partial F}{\partial z_m}, F \right\rangle$$

is not a general property of the Baikov polynomial.

This diagram provides a counterexample.



Letting G denote a Gröbner basis of the ideal, polynomial division of $\frac{\partial F}{\partial \chi}$ wrt. G leaves a non-vanishing remainder.

Conclusions and Outlook

- Differential equations for loop integrals can be studied in Baikov representation without dimension shifts.
- For a large class of non-trivial diagrams it is possible to avoid integrals with doubled propagators in intermediate stages.

  This limits the set of integrals for which IBP reductions must be determined.
- For the future: classify diagrams for which the enhanced ideal membership holds.