Multiloop Euler-Heisenberg Lagrangians, Schwinger pair creation, and the QED N - photon amplitudes

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based on collaboration with

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U M S N H

Loops and Legs in Quantum Field Theory
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The Euler-Heisenberg Lagrangian at one loop

1936 W. Heisenberg and H. Euler: One-loop QED effective Lagrangian in a constant field ("Euler-Heisenberg Lagrangian")

$$\mathcal{L}^{(1)}(a,b) = -\frac{1}{8\pi^2}\int_0^\infty \frac{dT}{T^3}\,\mathrm{e}^{-m^2T}\left[\frac{(eaT)(ebT)}{\tanh(eaT)\tan(ebT)} - \frac{e^2}{3}(a^2-b^2)T^2 - 1\right]$$

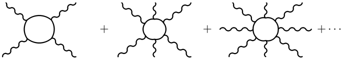
Here a, b are the two invariants of the Maxwell field, related to **E**, **B** by $a^2 - b^2 = B^2 - E^2, \quad ab = \mathbf{E}\cdot\mathbf{B}$.

1936 V. Weisskopf: Analogously for Scalar QED

$$\mathcal{L}^{(1)}_{\rm scal}(a,b) = \frac{1}{16\pi^2}\int_0^\infty \frac{dT}{T^3}\,\mathrm{e}^{-m^2T}\left[\frac{(eaT)(ebT)}{\sinh(eaT)\sin(ebT)} + \frac{e^2}{6}(a^2-b^2)T^2 - 1\right]$$

Low-energy N-photon amplitudes

The Euler-Heisenberg Lagrangian has the information on the N - photon amplitudes in the low energy limit (where all photon energies are small compared to the electron mass, $\omega_i \ll m$).



L.C. Martin, C. S., V. M. Villanueva, NPB 668 (2003) 335:
Explicit construction of the amplitudes from the weak field expansion coefficients c_{kl}, defined by

$$\mathcal{L}(a,b) = \sum_{k,l} c_{kl}\, a^{2k} b^{2l}$$

For each N and each helicity assignment, the dependence on the momentum and polarization vectors is absorbed by a single invariant χ_N.

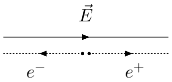
Imaginary part of the effective action

If the field has an electric component ($b \neq 0$) there are poles on the integration contour at $ebT = k\pi$ which create an imaginary part. For the purely electric case one gets (J. Schwinger 1951)

$$
\begin{aligned}
\mathrm{Im}\mathcal{L}^{(1)}(E) &= \frac{m^4}{8\pi^3}\beta^2 \sum_{k=1}^{\infty} \frac{1}{k^2} \exp\left[-\frac{\pi k}{\beta}\right] \\
\mathrm{Im}\mathcal{L}^{(1)}_{\mathrm{scal}}(E) &= -\frac{m^4}{16\pi^3}\beta^2 \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} \exp\left[-\frac{\pi k}{\beta}\right]
\end{aligned}
$$

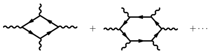
($\beta = eE/m^2$).

- The kth term relates to coherent creation of k pairs in one Compton volume.
- Weak field limit $\beta \ll 1 \Rightarrow$ only $k = 1$ relevant.
- $\mathrm{Im}\mathcal{L}(E)$ depends on E non-perturbatively (nonanalytically), which is consistent with the interpretation of pair creation as vacuum tunneling (F. Sauter 1931):



Relation to pair creation

The relation to pair creation is based on the Optical Theorem, which relates



to the "cut diagrams"

However, the latter individually all vanish for a constant field, which can emit only zero-energy photons.
Thus what counts is the asymptotic behaviour for a large number of photons.

Borel dispersion relation

Thus for a constant field we cannot use dispersion relations for individual diagrams; the appropriate generalization is a Borel dispersion relation: define the weak field expansion by

$$\mathcal{L}(E) = \sum_{n=2}^{\infty} c(n) \left(\frac{eE}{m^2} \right)^{2n}$$

$$c(n) \overset{n \to \infty}{\sim} c_\infty \Gamma[2n-2]$$

then

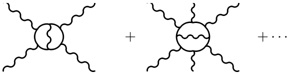
$$\mathrm{Im}\mathcal{L}(E) \sim) \overset{\beta \to 0}{\sim} c_\infty \, \mathrm{e}^{-\frac{\pi m^2}{eE}}$$

G.V. Dunne & C.S. 1999 NPB 564 (2000) 591

Beyond one loop

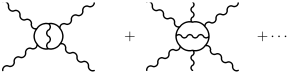
Two loop (one-photon exchange) corrections:

Euler-Heisenberg Lagrangian:



Schwinger pair creation:

2-Loop Euler-Heisenberg Lagrangian



V. I. Ritus 1975, S.L. Lebedev & V.I. Ritus 1984, W. Dittrich & M. Reuter 1985, M. Reuter, M.G. Schmidt & C.S. 1997: The two-loop correction $\mathcal{L}^{(2)}(E)$ to the Euler-Heisenberg Lagrangian leads to rather intractable integrals. However, the imaginary part $\mathrm{Im}\mathcal{L}^{(2)}(E)$ becomes extremely simple in the weak-field limit:

$$\mathrm{Im}\mathcal{L}^{(1)}(E) + \mathrm{Im}\mathcal{L}^{(2)}(E) \overset{\beta\to 0}{\sim} \frac{m^4\beta^2}{8\pi^3}\left(1+\alpha\pi\right) \mathrm{e}^{-\frac{\pi}{\beta}}$$

The exponentiation conjecture

S.L. Lebedev & V.I. Ritus 1984: Assuming that higher orders will lead to exponentiation

$$\mathrm{Im}\mathcal{L}^{(1)}(E) + \mathrm{Im}\mathcal{L}^{(2)}(E) + \mathrm{Im}\mathcal{L}^{(3)}(E) + \dots \overset{\beta\to 0}{\sim} \frac{m^4\beta^2}{8\pi^3}\exp\Bigl[-\frac{\pi}{\beta} + \alpha\pi\Bigr] = \mathrm{Im}\mathcal{L}^{(1)}(E)\ \mathrm{e}^{\alpha\pi}$$

then the result can be interpreted in the tunneling picture as the corrections to the Schwinger pair creation rate due to the pair being created with a negative Coulomb interaction energy

$$m(E) \approx m + \delta m(E), \quad \delta m(E) = -\frac{\alpha}{2}\frac{eE}{m}$$

where $\delta m(E)$ is the Ritus mass shift, originally derived from the crossed process of one-loop electron propagation in the field:

Pair creation by the field

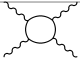
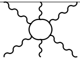
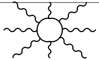
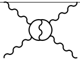
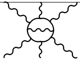
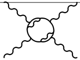
Electron propagation in the field

Exponentiation in Scalar QED

For Scalar QED, the corresponding conjecture was established already two years earlier by (I.K. Affleck, O. Alvarez, N.S. Manton 1982), using Feynman's worldline path integral formalism and a semi-classical approximation ("worldline instanton").

$$\begin{aligned} \sum_{l=1}^{\infty} \mathrm{Im}\mathcal{L}^{(l)}_{\mathrm{scal}}(E) \quad &\overset{\beta\to 0}{\sim} \quad -\frac{m^4\beta^2}{16\pi^3}\exp\Big[-\frac{\pi}{\beta}+\alpha\pi\Big] \\ &= \quad \mathrm{Im}\mathcal{L}^{(1)}_{\mathrm{scal}}(E)\, \mathrm{e}^{\alpha\pi} \end{aligned}$$

Diagrams contributing to the exponentiation formula

	Number of external legs			
Number of loops	4	6	8	$\cdots$
1				$\cdots$
2			$\cdots$	$\cdots$
3		$\cdots$	$\ddots$	$\vdots$
$\vdots$	$\vdots$	$\ddots$	$\ddots$	$\vdots$

- Involves diagrams with any number of loops and legs.
- Includes also all counterdiagrams from mass renormalization.
- Not included: diagrams with more than one fermion loop (get suppressed in the weak-field limit).
- Summation over the external legs $\rightarrow$ Schwinger factor $e^{-\frac{\pi}{\beta}}$.
- Summation over the internal photons insertions $\rightarrow$ Ritus-Lebedev/Affleck-Alvarez-Manton factor $e^{\alpha\pi}$.

Large N limit of the QED N-photon amplitudes

Curious: a summation over photon insertions into an electron loop has produced the analytic factor $\mathrm{e}^{\alpha\pi}$!
By Borel analysis, this factor can also be transferred to the large - N limit of the N - photon amplitudes:

$$\lim_{N\to\infty}\frac{\Gamma^{(\mathrm{all-loop})}[k_1,\varepsilon_1^+;\dots;k_N,\varepsilon_N^+]}{\Gamma^{(1)}[k_1,\varepsilon_1^+;\dots;k_N,\varepsilon_N^+]} = \mathrm{e}^{\alpha\pi}\,.$$

(G.V. Dunne and C.S. 2005).

QED in 1+1 dimensions

The exponentiation conjecture has been verified at two loops in Scalar and Spinor QED. A three-loop check is in order, but calculating the three-loop EHL in $D = 4$ is too difficult.

M. Krasnansky 2005: Studied the Scalar QED EHL in $D = 2, 4, 6$.
For $D = 2$:

$$\begin{aligned} \mathcal{L}^{(2)(2D)}_{\rm scal}(\kappa) &= -\frac{e^2}{32\pi^2}\left(\xi_{2D}^2 - 4\kappa\xi'_{2D}\right), \\ \xi_{2D} &= -\Big(\psi(\kappa+\frac{1}{2}) - \ln(\kappa)\Big) \end{aligned}$$

($\psi(x) = \Gamma'(x)/\Gamma(x)$, $\kappa = m^2/(2ef)$, $f^2 = \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$).

Simpler, but nontrivial $\rightarrow$ Suggests to establish and verify the above predictions for 2D QED.

The EHL in 2S Spinor QED

I. Huet, D.G.C. McKeon and C.S., JHEP 12 (2010) 036:

- Used the method of Affleck at al. to generalize the exponentiation conjecture to the 2D case:

$$\mathrm{Im}\mathcal{L}_{2D}^{(all-loop)} \sim \mathrm{e}^{-\frac{m^2\pi}{eE}+\tilde{\alpha}\pi^2\kappa^2}$$

where $\tilde{\alpha} = \frac{2e^2}{\pi m^2}\kappa = m^2/(2ef)$, $f^2 = \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$.

- Calculated the one- and two-loop EHL in 2D spinor QED:

$$\begin{aligned}
\mathcal{L}^{(1)}(\kappa) &= -\frac{m^2}{4\pi}\frac{1}{\kappa}\Big[\ln\Gamma(\kappa) - \kappa(\ln\kappa - 1) + \frac{1}{2}\ln\big(\frac{\kappa}{2\pi}\big)\Big] \\
\mathcal{L}^{(2)}(f) &= \frac{m^2}{4\pi}\frac{\tilde{\alpha}}{4}\Big[\tilde{\psi}(\kappa) + \kappa\tilde{\psi}'(\kappa) + \ln(\lambda_0 m^2) + \gamma + 2\Big]
\end{aligned}$$

where

$$\tilde{\psi}(x) \equiv \psi(x) - \ln x + \frac{1}{2x}$$

Weak-field expansion coefficients

Explicit formulas not only for $c^{(1)}(n)$ but $c^{(2)}(n)$:

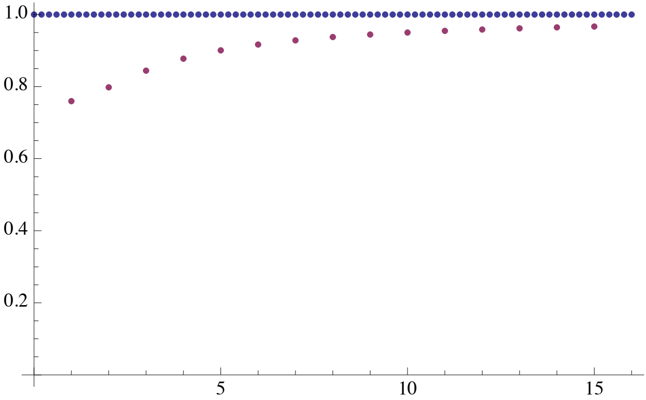
$$
\begin{aligned}
c^{(1)}(n) &= (-1)^{n+1}\frac{B_{2n}}{4n(2n-1)} \\
c^{(2)}(n) &= (-1)^{n+1}\frac{\tilde{\alpha}}{8}\frac{2n-1}{2n}B_{2n}
\end{aligned}
$$

Using properties of the Bernoulli numbers B_n, we can easily verify that

$$
\lim_{n\to\infty}\frac{c^{(2)}(n)}{c^{(1)}(n+1)} = \tilde{\alpha}\pi^2
$$

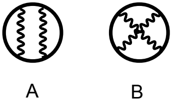
Asymptotic prediction for the 2-loop EHL in 2D QED

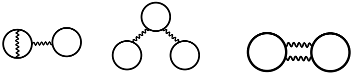
Rapid convergence of $c^{(2)}(n)$ to the asymptotic prediction:



Three-loop EHL in 2D Spinor QED: diagrams

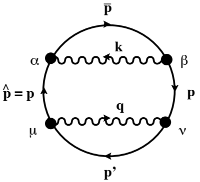
Diagrams contributing to the 3-loop EHL:





- Only the one fermion-loop diagrams A and B are relevant for the exponentiation conjecture.
- At three-loop, the 2D EHL is already UV finite.
- There are spurious IR - divergences, but they can be removed by going to the traceless gauge $\xi = -2$.

Diagram A



$$\mathcal{L}^{3A}(f) = \frac{\tilde{\alpha}^2 m^2}{32\pi} \int_0^\infty dw dw' d\hat{w} d\bar{w} \; I_A \; \mathrm{e}^{-a}$$

$$I_A = \frac{\rho^3}{A^2 \cosh \rho w \cosh \rho \hat{w} (\cosh \rho \bar{w} \cosh \rho w')^2} \left[\frac{\cosh \rho (w - \hat{w})}{2\rho} - \frac{1}{A \cosh \rho w \cosh \rho \hat{w}} \right]$$

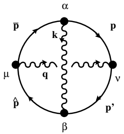
where $\rho = \frac{ef}{m^2}$, $a = w + w' + \hat{w} + \bar{w}$, $A = \tanh\rho w + \tanh\rho w' + \tanh\rho\hat{w} + \tanh\rho\bar{w}$.

(Relatively) easy to compute the weak-field expansion coefficients $c^{(3)A}(n) = \frac{\tilde{\alpha}^2}{64} \Gamma_n^A$, which are rational numbers:

$$\Gamma_0^A = -\frac{1}{3}, \Gamma_1^A = -\frac{1}{30}, \Gamma_2^A = \frac{17}{63}, \Gamma_3^A = \frac{251}{99}, \ldots$$

(13 coefficients so far)

Diagram B



$$\mathcal{L}^{3B}(f) = \frac{\tilde{\alpha}^2 m^2}{128\pi} \int_0^\infty dw dw' d\hat{w} d\bar{w} \; I_B \; \mathrm{e}^{-a} ,$$

$$I_B = \frac{\rho^3}{\cosh^2 \rho w \cosh^2 \rho w' \cosh^2 \rho \hat{w} \cosh^2 \rho \bar{w}} \frac{B}{A^3 C} - \rho \frac{\cosh(\rho \tilde{w})}{\cosh \rho w \cosh \rho w' \cosh \rho \hat{w} \cosh \rho \bar{w}} \Big[\frac{1}{A} - \frac{C}{G^2} \ln\Big(1 + \frac{G^2}{AC}\Big) \Big]$$

$$B = (\tanh^2 \mathrm{z} + \tanh^2 \hat{\mathrm{z}})(\tanh \mathrm{z}' + \tanh \bar{\mathrm{z}}) + (\tanh^2 \mathrm{z}' + \tanh^2 \bar{\mathrm{z}})(\tanh \mathrm{z} + \tanh \hat{\mathrm{z}})$$

$$C = \tanh \mathrm{z} \tanh \mathrm{z}' \tanh \hat{\mathrm{z}} + \tanh \mathrm{z} \tanh \mathrm{z}' \tanh \bar{\mathrm{z}} + \tanh \mathrm{z} \tanh \hat{\mathrm{z}} \tanh \bar{\mathrm{z}} + \tanh \mathrm{z}' \tanh \hat{\mathrm{z}} \tanh \bar{\mathrm{z}}$$

$$G = \tanh \mathrm{z} \tanh \hat{\mathrm{z}} - \tanh \mathrm{z}' \tanh \bar{\mathrm{z}}$$

($z = \rho w$ etc.).

Weak-field expansion of diagram B

For diagram B, the calculation of the weak-field expansion coefficients turned out to be much more difficult than for A:

- More difficult integrals.
- Expansion in the external field creates huge numerator polynomials in the Feynman parameters.

In a first attempt using numerical integration we obtained only six coefficients - too few for our purposes!

I. Huet, C. S. and M. Rausch de Traubenberg (in preparation):
Solution for both problems: Use the high symmetry of the diagram!

Integration-by-parts algorithm

Introduce the operator

$$\tilde{d} \equiv \frac{\partial}{\partial w} - \frac{\partial}{\partial w'} + \frac{\partial}{\partial \hat{w}} - \frac{\partial}{\partial \bar{w}}$$

which acts simply on he trigonometric building blocks of the integrand.
Integrating by parts with this operator, it is possible to write the integrand of the n-th coefficient β_n as a total derivative $\beta_n = \tilde{d}\theta_n$. Then

$$\int_0^\infty dwdw'd\hat{w}d\bar{w}\ \mathrm{e}^{-a}\beta_n = \int_0^\infty dwd\bar{w}d\hat{w}dw'\tilde{d}\ \mathrm{e}^{-(w+w'+\hat{w}+\bar{w})}\theta_n = 4\int_0^\infty dwdw'd\hat{w}\ \mathrm{e}^{-(w+w'+\hat{w})}\ \theta_n|_{\bar{w}=0}$$

The remaining integrals are already of a standard type. In this way we obtained the first two coefficients:

$$\Gamma_0^B = -\frac{3}{2} + \frac{7}{4}\zeta_3$$
$$\Gamma_1^B = -\frac{251}{120} + \frac{35}{16}\zeta_3$$

All coefficients will be of the form $r_1 + r_2\zeta_3$ with rational numbers r_1, r_2.

Using the polynomial invariants of D_4

Diagram B has the symmetries

$$\begin{aligned} w &\leftrightarrow \hat{w} \\ w' &\leftrightarrow \bar{w} \\ (w, \hat{w}) &\leftrightarrow (w', \bar{w}) \end{aligned}$$

Those generate the dihedral group D_4. This allows one to rewrite the numerator polynomials as polynomials in the variable $\tilde{w} = w - w' + \hat{w} - \bar{w}$ with coefficients that are polynomials in the four D_4 - invariants a, v, j, h,

$$\begin{aligned} a &= w + w' + \hat{w} + \bar{w} \\ v &= 2(w\hat{w} + w'\bar{w}) + (w + \hat{w})(w' + \bar{w}) \\ j &= a\tilde{w} - 4(w\hat{w} - w'\bar{w}) \\ h &= a(ww'\hat{w} + ww'\bar{w} + w\hat{w}\bar{w} + w'\hat{w}\bar{w}) + (w\hat{w} - w'\bar{w})^2 \end{aligned}$$

These invariants are moreover chosen such that they are annihiliated by $\tilde{d}$. Thus they are well-adapted to the integration-by-parts algorithm.
This significantly reduces the size of the expressions generated by the expansion in the field.

Outlook

- We are confident to get enough expansion coefficients to settle the question of the validity of the exponentiation conjecture in $1+1$ dimensional QED by the time of Loops and Legs 2020.
- The techniques that we have developed for the calculation of the 3-loop EHL in two dimension should in part also become useful in an eventual calulation of this Lagrangian in four dimensions.

Thank you for your attention!