Cuts and Feynman amplitudes beyond polylogarithms

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Loops and Legs in Quantum Field Theory
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Based on collaborations with *J. Brödel*, *C. Duhr*, *F. Dulat*, *B. Penante*, *A. von Manteuffel*, *A. Primo*, *E. Remiddi*

Functions **beyond polylogs** appear everywhere in particle physics

- Already in **QED** at two/three loops (form factors, electron self-energy, ...)
- **QCD** Two-loop corrections to H+jet, (similarly V+jet production)
- **QCD** Two-loop corrections to HH (similarly VV production)
- **QCD** Two-loop corrections to $t\bar{t}$
- **NNNLO** QCD corrections to Higgs production
- **NNLO** QCD corrections to heavy quarkonium production and decays
- More or less "any two-loop amplitude" in the **EW theory** ....

Precision physics at the **LHC** seems to require "elliptic functions"[1]
[See **C. Duhr**, **A. De Freitas**, **S. Weinzierl** talks]

[1] Or, at least, reliable numerical results for scattering amplitudes that would evaluate to elliptic generalizations of MPLs...

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One possible point of view – **Differential equations method**

[Kotikov '90, Remiddi '97, **Gehrmann-Remiddi '00**,..., **J. Henn '13**; C. Papadopoulos '14]

⇓

Direct consequence of **Integration-by-parts (IBPs)** identities in d-dimensions!

$$\int \prod_{j=1}^{l} \frac{d^d\, k_j}{(2\pi)^d} \left(\frac{\partial}{\partial k_j^{\mu}}\, v_{\mu}\, \frac{S_1^{\sigma_1} \dots S_s^{\sigma_s}}{D_1^{\alpha_1} \dots D_n^{\alpha_n}} \right) = 0\,, \qquad v^{\mu} = k_j^{\mu}, p_k^{\mu}$$

Reduced to N master integrals, $I_i(d; x_k)$ with $i = 1, ..., N$.

⇓

Differentiating the masters and using the **IBPs** we get a system of N coupled differential equations

$$\frac{\partial}{\partial\, x_k} I_i(d; x_k) = \sum_{j=1}^{N} c_{ij}(d; x_k)\, I_j(d; x_k)\,.$$

Let's look more in detail - *we should recall* that equations are in block form

$$I_j(d;x_k) = (m_j(d;x_k)\,,\ \ sub_j(d;x_k))$$

$$\Downarrow$$

$$\frac{\partial}{\partial x_k} m_i(d;x_k) = \sum_{j=1}^{N} h_{ij}(d;x_k)\, m_j(d;x_k) + \sum_{j=1}^{M} nh_{ij}(d;x_k)\, sub_j(d;x_k)\,.$$

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homogeneous piece is MAIN source of complexity

$$\Downarrow$$

Loosely speaking:

- if decoupled as $d \to 4$, **MPLs**
- if coupled as $d \to 4$, **Elliptic...?** [L.Tancredi '15; L.Adams, S.Weinzierl '17]

How do we solve them? In general:

1- Solve the **homogeneous equations** in the limit $d \to 4$

$$\frac{d}{dx}\vec{I}(d;x) = A(x)\vec{I}(d;x) + (d-4)B(x)\vec{I}(d;x) + \mathcal{O}(d-4)^2\,,$$

with $A(x)$ $n \times n$, depending on how many eqs are coupled

Find $n \times n$ matrix homogeneous solutions $G(x)$, with

$$\frac{d}{dx}G(x) = A(x)G(x)\,, \qquad \to \qquad \vec{I}(d;x) = G(x)\vec{m}(d;x)$$

then

$$\frac{d}{dx}\vec{m}(d;x) = (d-4)G^{-1}(x)\,B(x)G(x)\,\vec{m}(d;x) + \mathcal{O}(d-4)^2\,,$$

2- Solution given by iterative integrals over kernels that contain **homogeneous solutions**, and previous orders

By expanding in $(d-4)$:

$$\vec{m}^{\,[n]}(x) = \int^x dy\, G^{-1}(y)\, B(y) G(y)\, \vec{m}^{\,[n-1]}(y) \;\; + \;\; \text{simpler terms}\,,$$

Or equivalently for the original functions

$$\vec{I}^{\,[n]}(x) = G(x) \int^x dy\, G^{-1}(y)\, B(y)\, \vec{I}^{\,[n-1]}(y) \;\; + \;\; \text{simpler terms}\,,$$

Two Questions:

a) How do I get the matrix $G(x)$ if the system is coupled?
b) What are the functions defined by the integrals above?

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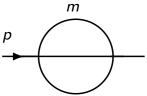
Solution:

- Take an older idea by [S.Laporta, E.Remiddi '04]

- Generalize it to all cases [A.Primo, L.Tancredi '16, '17]

$$\left(\frac{d^2}{d\, s^2} + A(d;s)\, \frac{d}{d\, s} + B(d;s) \right)$$



$$+ G(d;s)\, \mathrm{Tad}(d;m^2) = 0$$

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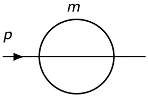
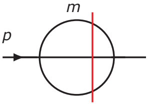


$$+\, G(d;s)\,\mathrm{Tad}(d;m^2) = 0$$

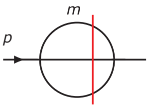
Cut $\rightarrow$ $$\left(\frac{d^2}{d\,s^2} + A(d;s)\,\frac{d}{d\,s} + B(d;s) \right)$$



$$= 0$$

Maximal cut solves homogeneous differential equations

[A.Primo, L.Tancredi '16, '17]



$$= \frac{1}{\sqrt{(3m-\sqrt{s})(\sqrt{s}+m)^3}}\,\mathrm{K}\left(\frac{16m^3\sqrt{s}}{(3m-\sqrt{s})(\sqrt{s}+m)^3}\right)$$

where $\mathrm{K}(x)$ is the complete elliptic integral of the first kind.

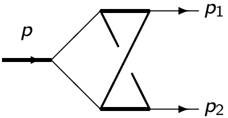
$$\mathrm{K}(x) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)\,(1-x\,t^2)}}$$

Computation of the maximal cut can be simplified in **Baikov representation**
[Papadopoulos, Frellesvig '17; Bosma, Sogaard, Zhang '17; Harley, Moriello, Schabinger '17]

What are the functions defined by these **iterated integrals**?

[See C. Duhr, S. Weinzierl, ...]

An explicit example: [M. Czakon, A. Mitov '08; A. von manteuffel, L. Tancredi '17]

$T(d;x) =$ 

- $p_1^2 = p_2^2 = 0$, four massive lines
- $x = -p^2/m^2$
- 2 master integrals, $T_1(x)$, $T_2(x)$
- Satisfy 2 coupled diff. eqs
- Needed for NNLO $\gamma\gamma$, $t\bar{t}$, ...

All subtopologies can be written in terms of (*not trivial!*)

$$\ln\left(f(l_i)\right), \;\; \mathrm{Li}_n(f(l_i)), \;\; \mathrm{Li}_{2,2}(f(l_i), g(l_j)),$$

with

$$l_i = \{\sqrt{x}, \tfrac{1}{2}(\sqrt{x}+\sqrt{x+4}), \sqrt{x+4}, \tfrac{1}{2}(\sqrt{x}+\sqrt{x-4}), \sqrt{x-4}\}$$

Maximal cut gives homogeneous solutions as **Elliptic Integrals**:

$$I_1(t) = \sqrt{x}\,\mathrm{K}\left(\frac{x}{16}\right), \qquad J_1(x) = \sqrt{x}\,\mathrm{K}\left(1-\frac{x}{16}\right).$$

$$I_2(x) = -\sqrt{x}\,\mathrm{E}\left(\frac{x}{16}\right), \qquad J_2(x) = \sqrt{x}\left[\mathrm{E}\left(1-\frac{x}{16}\right) - \mathrm{K}\left(1-\frac{x}{16}\right)\right],$$

Solving 2nd order diff eq. we find for the **finite part** in $\epsilon = 0$:
[A. von Manteuffel, L.Tancredi '17]

$$T_1^{(0)}(x) = \frac{2}{\pi}\left[J_1(x)\int_0^x dt\, I_1(t)\,Q(t) - I_1(x)\int_0^x dt\, J_1(t)\,Q(t)\right],$$
$$T_2^{(0)}(x) = \frac{2}{\pi}\left[J_2(x)\int_0^x dt\, I_1(t)\,Q(t) - I_2(x)\int_0^x dt\, J_1(t)\,Q(t)\right].$$

with

$$Q(x) = 5\ln^2(l_2) - l_1\,\frac{3/2\,\zeta_2 + 3\ln^2(l_4) + 3\mathrm{Li}_2(-1/l_4^2)}{l_5}$$

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Can we write these functions in terms of **Elliptic polylogarithms**?

[See. C. Duhr's Talk]

Let us start from an **elliptic curve** given by

$$P_4(x) = (x - a_1)(x - a_2)(x - a_3)(x - a_4) = y^2$$

I will use the definition of **elliptic polylogarithms** as iterated integrals over the following kernels [See C. Duhr's talk]:

$$\mathrm{E}_4\left(\begin{smallmatrix} n_1 & \cdots & n_k \\ c_1 & \cdots & c_k \end{smallmatrix}; x\right) = \int_0^x dt\, \psi_{n_1}(c_1, t)\, \mathrm{E}_4\left(\begin{smallmatrix} n_2 & \cdots & n_k \\ c_2 & \cdots & c_k \end{smallmatrix}; t\right),$$

$$\psi_0(0,x) = \frac{c_4}{y}, \qquad \psi_1(c,x) = \frac{1}{x-c},$$

$$\psi_{-1}(c,x) = \frac{y_c}{y(x-c)}, \qquad \psi_{-1}(\infty,x) = \frac{x}{y},$$

$$c_4 = \frac{\sqrt{(a_1-a_3)(a_2-a_4)}}{2},$$

$$y_c = \sqrt{P_4(c)}.$$

And **(important!)**

$$G(c_1, ..., c_n, x) = \mathrm{E}_4\left(\begin{smallmatrix} 1 & \cdots & 1 \\ c_1 & \cdots & c_n \end{smallmatrix}; x\right)$$

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N.B.

There is actually an **infinite tower** of kernels, but they are mostly irrelevant for physical results and their definition is immaterial for what follows.

It turns out to be easier to go through **Feynman parameters** (in $d = 4$!)

$$T(4;x) = \int \frac{d^4k\, d^4l}{(k^2-m^2)(l^2-m^2)((k-p_1)^2-m^2)((l-p_2)^2-m^2)(k-l-p_1)^2(l-k-p_2)^2}$$
$$= \int_0^1 \prod_{i=1}^{6} dx_i\ \delta\left(1-\sum_{i=1}^{6} x_i\right) \frac{1}{[F(x_1,x_2,x_3,x_4,x_5,x_6,p^2,m^2)]^2}$$

with

$$F(x_1,x_2,x_3,x_4,x_5,x_6,p^2,m^2) =$$
$$\Big(p^2\left[\left(\left(x_3+x_4\right)x_5+x_2\left(x_3+x_5\right)\right)x_6+x_1x_5\left(x_4+x_6\right)\right]+m^2\left(x_1+x_2+x_3+x_4\right)$$
$$\times\left(x_3x_4+x_5x_4+x_6x_4+x_3x_5+x_3x_6+x_2\left(x_3+x_5+x_6\right)+x_1\left(x_2+x_4+x_5+x_6\right)\right)\Big)$$

Cheng-Wu theorem, re-arrange Feynman parameters, given any subset S:

$$\int_0^1 \prod_{i=1}^{N} dx_i \, \delta\left(1-\sum_{i=1}^{N} x_i\right) f(x_1,...,x_N)$$
$$= \int_0^\infty \prod_{i\notin S} dx_i \, \int_0^1 \prod_{i\in S} dx_i \, \delta\left(1-\sum_{i\in S} x_i\right) f(x_1,...,x_N)$$

By using parametrization by [M. Hidding, F. Moriello '17] we have (forget prefactors for simplicity)

$$T(4;a) = \int_0^1 dx_2 \int_0^{1-x_2} dx_4 \int_0^{-1-x_2-x_4} dx_3$$
$$\times \int_0^\infty dx_5 \int_0^\infty dx_6 \, \frac{1}{\left[F(x_1,x_2,x_3,x_4,x_5,x_6,p^2,m^2)\right]^2}$$

where $a = 1/x = -m^2/p^2$.

After the first two integrations in x_5 and x_6 we are left with:

$$
\begin{aligned}
T(4;p^2) = a^2 \int_0^1 dx_2 \int_0^{1-x_2} dx_4 \int_0^{1-x_2-x_4} dx_3 \\
\times \frac{1}{(a - x_3 x_4)\left(a - x_2\left(1 - x_2 - x_3 - x_4\right)\right)} \\
\times \Big[\log\left(a + x_2 x_3\right) + \log\left(a + x_4\left(1 - x_2 - x_3 - x_4\right)\right) \\
- \log(a) - \log\left(1 - x_2 - x_4\right) - \log\left(x_2 + x_4\right) \Big]
\end{aligned}
$$

It is then trivial to rewrite the log-s as $G(\ldots, x_3)$ and perform in this way the integration in x_3.

Finally (with a small variable re-definition $x_2 \to \bar{x}_2$)

$$
\begin{aligned}
T(4;a) = {} & \frac{1}{3}\int_0^1 d\bar{x}_2 \frac{a^2}{\sqrt{(\bar{x}_2-1)\,\bar{x}_2}\sqrt{4a+(\bar{x}_2-1)\,\bar{x}_2}} \\
& \times \Bigg[6\Bigg(G\left((\bar{x}_2-1)\,\bar{x}_2,a\right)\left(G_m\left(-\frac{a}{\bar{x}_2-1},\bar{x}_2\right)+2G_m\left(\bar{x}_2,\bar{x}_2\right)\right) \\
& + G\left(0,\bar{x}_2\right)\left(2G_m\left(\bar{x}_2,\bar{x}_2\right)-G_m\left(\frac{a}{\bar{x}_2-1}+\bar{x}_2,\bar{x}_2\right)\right)\Bigg)-G\left(1,\bar{x}_2\right)G_m\left(\frac{a}{\bar{x}_2-1}+\bar{x}_2,\bar{x}_2\right) \\
& + 2G_m\left(0,\frac{a}{\bar{x}_2-1},\bar{x}_2\right)+G_m\left(-\frac{a}{\bar{x}_2-1},\frac{a-\bar{x}_2^2+\bar{x}_2}{1-\bar{x}_2},\bar{x}_2\right) \\
& + 2G_m\left(\bar{x}_2,\frac{a-\bar{x}_2^2+\bar{x}_2}{1-\bar{x}_2},\bar{x}_2\right)+G_m\left(\frac{a}{\bar{x}_2-1}+\bar{x}_2,\frac{a}{\bar{x}_2-1},\bar{x}_2\right) \\
& - 2\log(a)G_m\left(\bar{x}_2,\bar{x}_2\right)+\log(a)G_m\left(\frac{a}{\bar{x}_2-1}+\bar{x}_2,\bar{x}_2\right)+2G\left(1,\bar{x}_2\right)G_m\left(\bar{x}_2,\bar{x}_2\right)\Bigg) \\
& - G_m\left(\bar{x}_2\right)\Bigg(6G\left(0,\bar{x}_2\right)G\left((1-\bar{x}_2)\,\bar{x}_2,a\right)+6G\left(1,\bar{x}_2\right)G\left((1-\bar{x}_2)\,\bar{x}_2,a\right) \\
& + 6G\left(0,(1-\bar{x}_2)\,\bar{x}_2,a\right)+6G\left(0,(\bar{x}_2-1)\,\bar{x}_2,a\right)-6\log(a)G\left((1-\bar{x}_2)\,\bar{x}_2,a\right)+\pi^2\Bigg)\Bigg]
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& + 2G_m\left(\bar{x}_2, \frac{a-\bar{x}_2^2+\bar{x}_2}{1-\bar{x}_2}, \bar{x}_2\right) + G_m\left(\frac{a}{\bar{x}_2-1}+\bar{x}_2, \frac{a}{\bar{x}_2-1}, \bar{x}_2\right) \\
& - 2\log(a) G_m\left(\bar{x}_2,\bar{x}_2\right) + \log(a) G_m\left(\frac{a}{\bar{x}_2-1}+\bar{x}_2, \bar{x}_2\right) + 2G\left(1,\bar{x}_2\right) G_m\left(\bar{x}_2,\bar{x}_2\right)\Bigg) \\
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$$

Finally (with a small variable re-definition $x_2 \to \bar{x}_2$)

$$T(4;a) = \frac{1}{3}\int_0^1 d\bar{x}_2 \frac{a^2}{\sqrt{(\bar{x}_2-1)\,\bar{x}_2}\sqrt{4a+(\bar{x}_2-1)\,\bar{x}_2}}$$

$$\times\Bigg[6\Bigg(G\left((\bar{x}_2-1)\,\bar{x}_2,a\right)\left(G_m\left(-\frac{a}{\bar{x}_2-1},\bar{x}_2\right)+2G_m\left(\bar{x}_2,\bar{x}_2\right)\right)$$

$$+G\left(0,\bar{x}_2\right)\left(2G_m\left(\bar{x}_2,\bar{x}_2\right)-G_m\left(\frac{a}{\bar{x}_2-1}+\bar{x}_2,\bar{x}_2\right)\right)\Bigg)-G\left(1,\bar{x}_2\right)G_m\left(\frac{a}{\bar{x}_2-1}+\bar{x}_2,\bar{x}_2\right)$$

$$+2G_m\left(0,\frac{a}{\bar{x}_2-1},\bar{x}_2\right)+G_m\left(-\frac{a}{\bar{x}_2-1},\frac{a-\bar{x}_2^2+\bar{x}_2}{1-\bar{x}_2},\bar{x}_2\right)$$

$$+2G_m\left(\bar{x}_2,\frac{a-\bar{x}_2^2+\bar{x}_2}{1-\bar{x}_2},\bar{x}_2\right)+G_m\left(\frac{a}{\bar{x}_2-1}+\bar{x}_2,\frac{a}{\bar{x}_2-1},\bar{x}_2\right)$$

$$-2\log(a)G_m\left(\bar{x}_2,\bar{x}_2\right)+\log(a)G_m\left(\frac{a}{\bar{x}_2-1}+\bar{x}_2,\bar{x}_2\right)+2G\left(1,\bar{x}_2\right)G_m\left(\bar{x}_2,\bar{x}_2\right)\Bigg)$$

$$-G_m\left(\bar{x}_2\right)\Bigg(6G\left(0,\bar{x}_2\right)G\left((1-\bar{x}_2)\,\bar{x}_2,a\right)+6G\left(1,\bar{x}_2\right)G\left((1-\bar{x}_2)\,\bar{x}_2,a\right)$$

$$+6G\left(0,(1-\bar{x}_2)\,\bar{x}_2,a\right)+6G\left(0,(\bar{x}_2-1)\,\bar{x}_2,a\right)-6\log(a)G\left((1-\bar{x}_2)\,\bar{x}_2,a\right)+\pi^2\Bigg)\Bigg]$$

The **quartic root** defines an **elliptic curve**!

$$P_4(x) = x(x-1)(4a + x(x-1))$$

and we defined

$$G_m(\vec{n}, x) = G(r_p, \vec{n}, x) - G(r_m, \vec{n}, x)$$

with

$$r_{p/m} = \frac{-\bar{x}_2(1-\bar{x}_2) \pm \sqrt{(\bar{x}_2 - 1)\,\bar{x}_2}\sqrt{4a - \bar{x}_2(1-\bar{x}_2)}}{2\,(\bar{x}_2 - 1)}$$

$$= \frac{-\bar{x}_2(1-\bar{x}_2) \pm \sqrt{P_4(\bar{x}_2)}}{2\,(\bar{x}_2 - 1)}$$

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Now, it turns out that Gs with a quartic root in the indices can be written in terms elliptic polylogs

[E. Remiddi, L. Tancredi '17; J. Broedel, F. Dulat, C. Duhr, B. Penante, L. Tancredi]

$$G_m(\bar{x}_2) = G(r_p, \bar{x}_2) - G(r_m, \bar{x}_2) = \log\left(\frac{\bar{x}_2(1-\bar{x}_2) + \sqrt{P_4(\bar{x}_2)}}{\bar{x}_2(1-\bar{x}_2) - \sqrt{P_4(\bar{x}_2)}}\right)$$

such that

$$\frac{\partial}{\partial \bar{x}_2} G_m(\bar{x}_2) = \frac{1-2\bar{x}_2}{\sqrt{(\bar{x}_2-1)\,\bar{x}_2}\sqrt{4a-\bar{x}_2(1-\bar{x}_2)}}$$

and integrating back in $\bar{x}_2$

$$G_m(\bar{x}_2) = \int_0^{\bar{x}_2} \frac{dt}{\sqrt{P_4(t)}} - 2\int_0^{x_2} \frac{dt\ t}{\sqrt{P_4(t)}} + c$$
$$= \frac{4}{1+\sqrt{1-16a}} \mathrm{E}_4\left({0 \atop 0}; \bar{x}_2\right) - 2\mathrm{E}_4\left({-1 \atop \infty}; \bar{x}_2\right) \pm i\pi$$

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We can rewrite all Gs in terms of E_4 functions, and then perform trivially the last integration.

We obtain [PRELIMINARY!]:

$$
\begin{aligned}
T(4,a) = \frac{2a^2}{c_4^2}\Bigg[& 5\mathrm{E}_4\left(\begin{smallmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & r_{mp} \end{smallmatrix};1\right) + 5\mathrm{E}_4\left(\begin{smallmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & r_{pp} \end{smallmatrix};1\right) + 5\mathrm{E}_4\left(\begin{smallmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & r_{mp} \end{smallmatrix};1\right) + 5\mathrm{E}_4\left(\begin{smallmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & r_{pp} \end{smallmatrix};1\right) \\
& - 3\mathrm{E}_4\left(\begin{smallmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & r_{mm} & 0 \end{smallmatrix};1\right) - 3\mathrm{E}_4\left(\begin{smallmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & r_{mm} & 1 \end{smallmatrix};1\right) - 3\mathrm{E}_4\left(\begin{smallmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & r_{pm} & 0 \end{smallmatrix};1\right) - 3\mathrm{E}_4\left(\begin{smallmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & r_{pm} & 1 \end{smallmatrix};1\right) \\
& + 3\log a\left(\mathrm{E}_4\left(\begin{smallmatrix} 0 & 0 & 1 \\ 0 & 0 & r_{mm} \end{smallmatrix};1\right) + \mathrm{E}_4\left(\begin{smallmatrix} 0 & 0 & 1 \\ 0 & 0 & r_{pm} \end{smallmatrix};1\right)\right)\Bigg] \\
- \frac{4a^2}{c_4}\Bigg[& 5\mathrm{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 & 1 \\ 0 & \infty & 0 & r_{mp} \end{smallmatrix};1\right) + 5\mathrm{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 & 1 \\ 0 & \infty & 0 & r_{pp} \end{smallmatrix};1\right) + 5\mathrm{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 & 1 \\ 0 & \infty & 1 & r_{mp} \end{smallmatrix};1\right) + 5\mathrm{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 & 1 \\ 0 & \infty & 1 & r_{pp} \end{smallmatrix};1\right) \\
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\end{aligned}
$$

with

$$c_4 = \frac{1+\sqrt{1-16a}}{4}$$

$$r_{mm} = \frac{1-\sqrt{1-4a}}{2} \quad r_{mp} = \frac{1-\sqrt{1+4a}}{2} \quad r_{pm} = \frac{1+\sqrt{1-4a}}{2} \quad r_{pp} = \frac{1+\sqrt{1+4a}}{2}$$

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SUMMARY and OUTLOOK

What have we learnt:

- Remeber [C. Duhr's talk], E_4 functions are equivalent to **Brown-Levin's Elliptic polylogarithms**
- Until now these functions used only for Sunrise graph (2-point function with trivial subtopologies)
- First example of a **3-point function** (very much relevant for pheno!) that can be expressed in terms of these functions

What remains to be done:

- Result expressed in terms of $E_4(\ldots, 1)$, **non-unique representation** (there might be **hidden zeros**!)
- Current representation has **fake singularities on the integration contour**. Find representation where these can be removed explicitly.
- Relation with **iterated integrals over modular forms**.

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THANKS!