INFRARED SINGULARITIES OF QCD SCATTERING AMPLITUDES IN THE REGGE LIMIT TO ALL ORDERS

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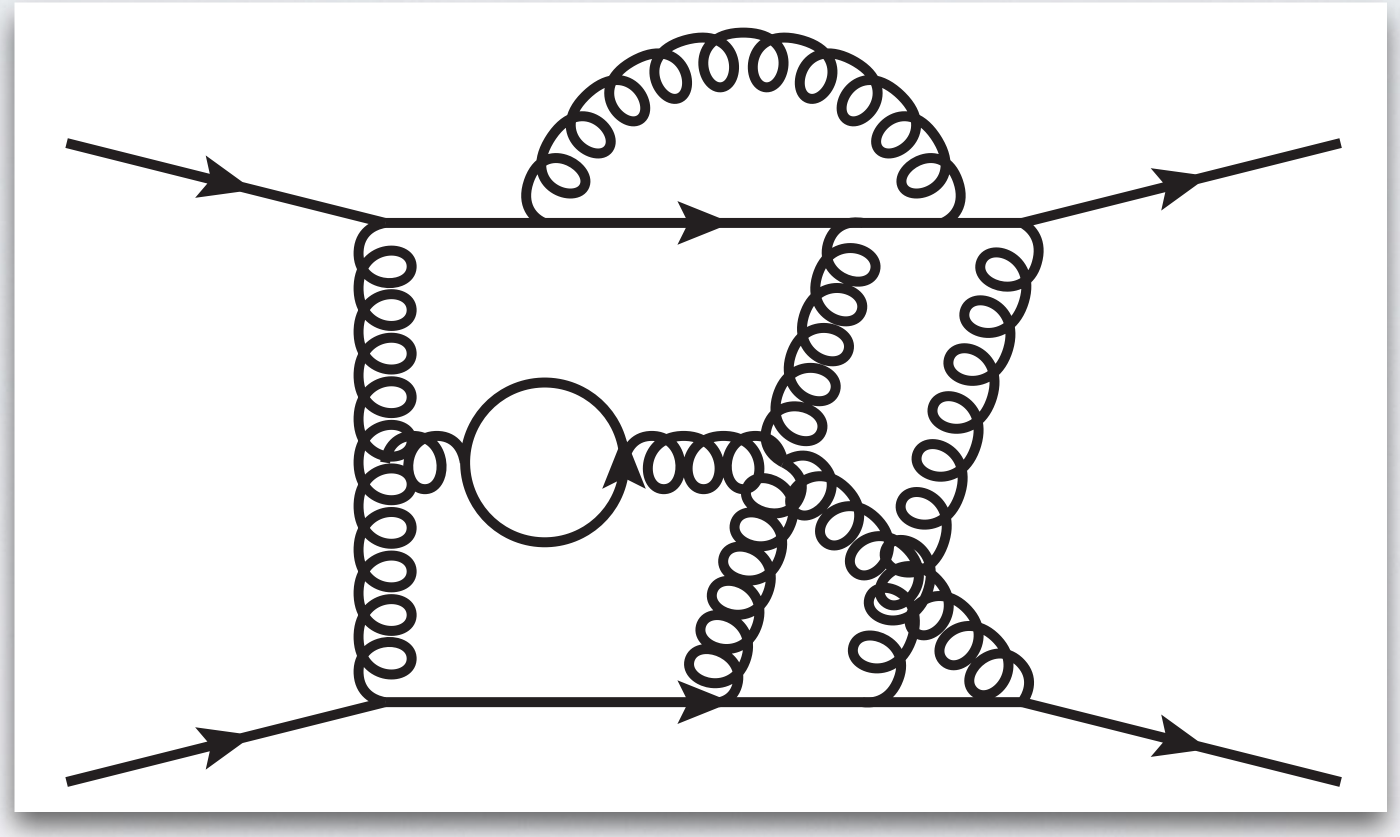
OUTLINE

- **Factorisation of amplitudes in the high-energy and in the infrared limit**
- **The two-Reggeon cut: scattering amplitudes by iterated solution of the BFKL equation**
- **Infrared singularities to all orders**

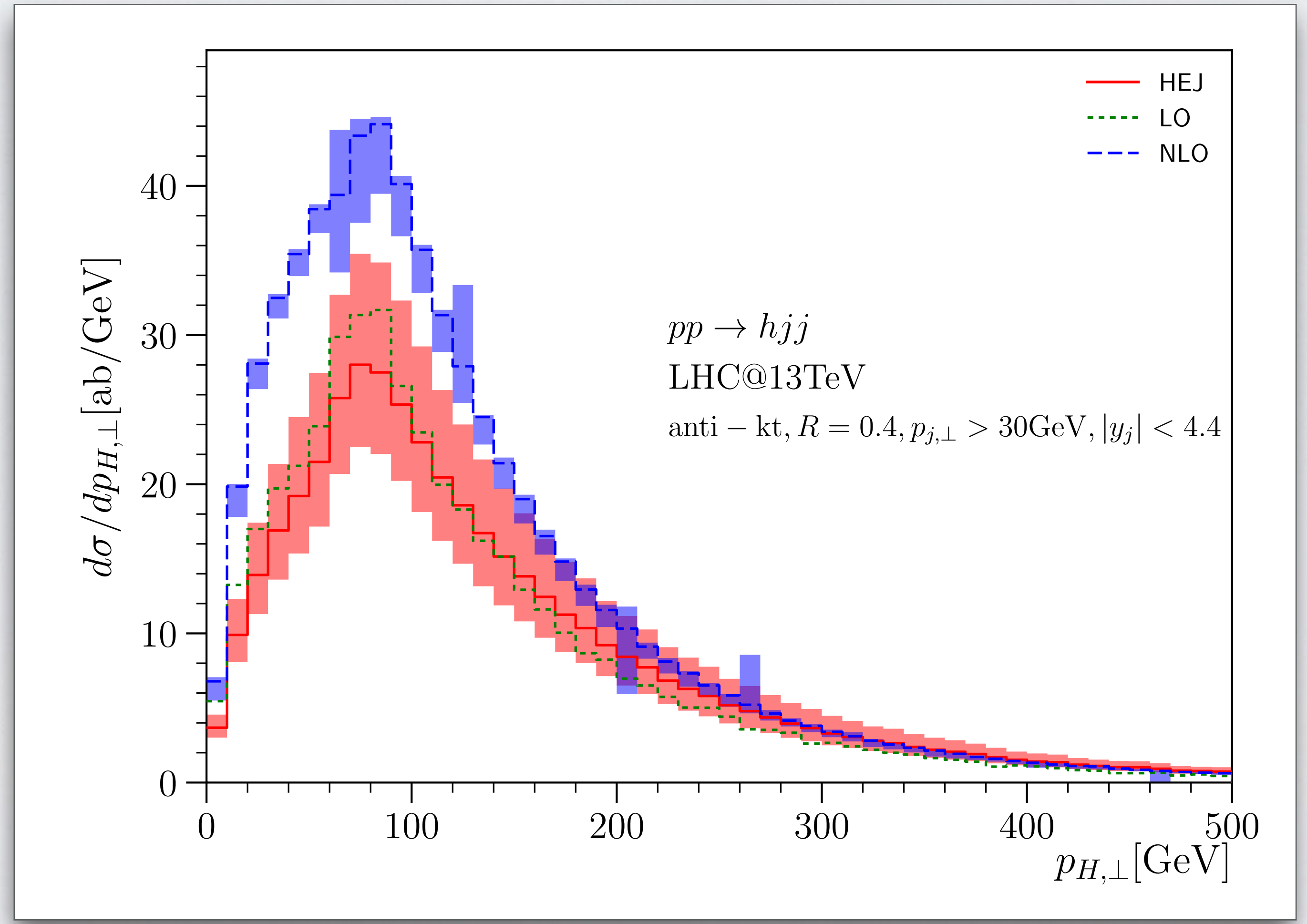
  - ***In collaboration with***
    Simon Caron-Huot, Einan Gardi and Joscha Reichel,
  - ***Based on***
    arXiv:1701.05241 (JHEP 1706 (2017) 016),
    arXiv:1711.04850 (JHEP 1803 (2018) 098)

FACTORISATION OF AMPLITUDES IN THE HIGH-ENERGY AND IN THE INFRARED LIMIT



PARTICLE SCATTERING IN KINEMATICAL LIMITS

- The motivation to study particle scattering in kinematical limits is twofold:
- From a phenomenological perspective, differential distributions in kinematic limits develop large logarithms, which may spoil the convergence of the perturbative expansion, and need to be resummed.

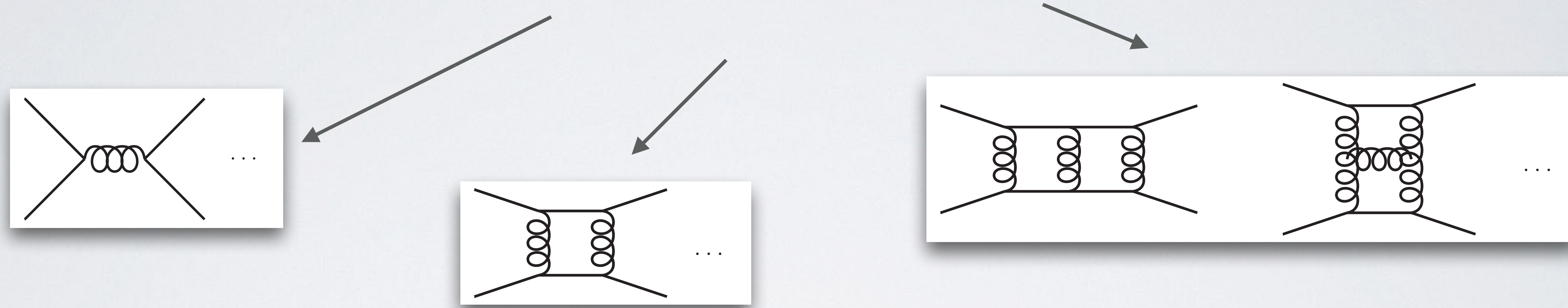


Andersen, Hapola, Maier, Smillie, 2017

PARTICLE SCATTERING IN KINEMATICAL LIMITS

- From a theoretical perspective, scattering amplitudes are complicated functions of the kinematical invariants, their calculation is non-trivial, and it is subject of intense study.

$$\mathcal{M} = 4\pi\alpha_s\left[\mathcal{M}^{(0)} + \frac{\alpha_s}{4\pi}\mathcal{M}^{(1)} + \left(\frac{\alpha_s}{4\pi}\right)^2\mathcal{M}^{(2)} + ...\right].$$

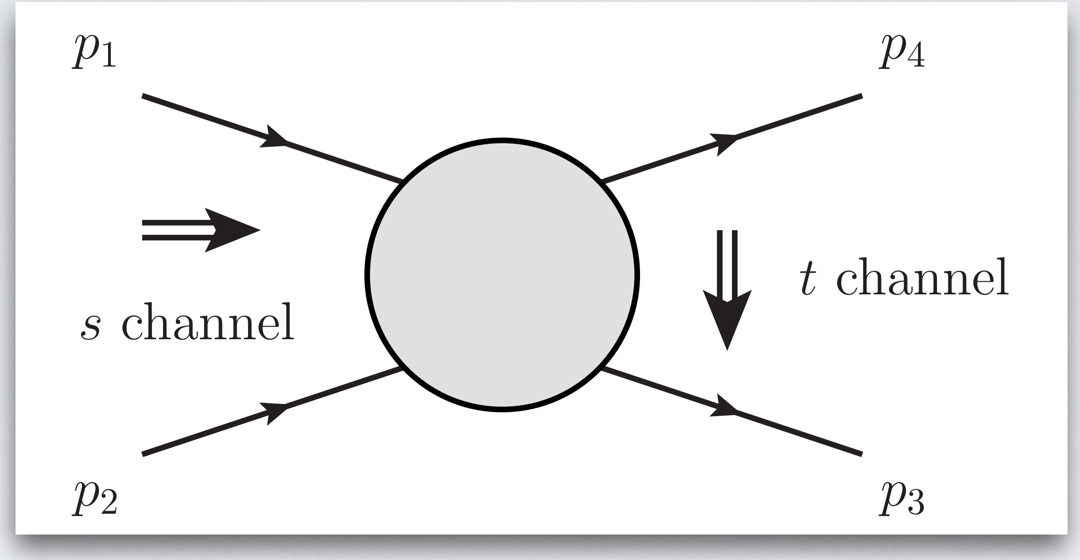


- Express Feynman integrals in terms of known functions (HPLs, elliptic integrals, etc)
- Analytic structure of infrared divergences.

- Information and constraints can be obtained by considering kinematical limits:
  - the number of invariants is reduced;
  - identify factorisation properties and iterative structures of the amplitude.

➔ this talk

2 ➔ 2 SCATTERING IN THE HIGH-ENERGY LIMIT



- Consider 2 → 2 scattering amplitudes in the high-energy limit:

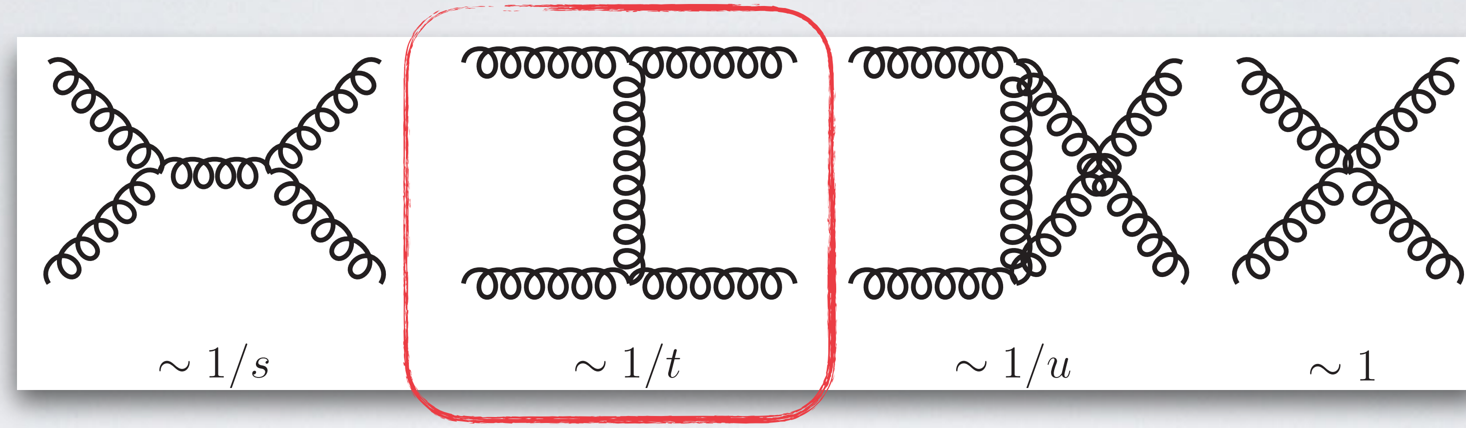
$$s = (p_1 + p_2)^2 \gg -t = -(p_1 - p_4)^2 > 0.$$

- The amplitude is expanded in the small ratio |t/s|; we consider here the leading power term:

$$\mathcal{M}_{ij\to ij}(s,t,\mu^2) = \frac{s}{t}\mathcal{M}^{[-1]}_{ij\to ij}\left(\frac{-t}{\mu^2}\right) + \mathcal{M}^{[0]}_{ij\to ij}\left(\frac{-t}{\mu^2}\right) + \frac{t}{s}\mathcal{M}^{[1]}_{ij\to ij}\left(\frac{-t}{\mu^2}\right) + \ldots$$

2 ➔ 2 SCATTERING IN THE HIGH-ENERGY LIMIT

- Gluon-gluon scattering amplitude at tree level:



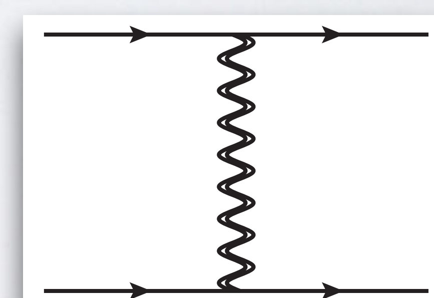
- In the high-energy limit only the second diagram contributes at leading power.

$$\mathcal{M}^{(0)}_{ij\to ij} = \frac{2s}{t}\,(T^b_i)_{a_1 a_4}(T^b_j)_{a_2 a_3}\,\delta_{\lambda_1\lambda_4}\delta_{\lambda_2\lambda_3}\,.$$

- The amplitude at higher orders contains logarithms of the ratio |s/t|. They can be characterised in terms of Regge poles and cuts: at LL

Regge, Gribov

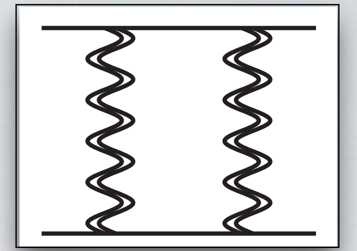
$$\mathcal{M}_{ij\to ij}|_{\rm LL} = \left(\frac{s}{-t}\right)^{\frac{\alpha_s}{\pi}\,C_A\,\alpha_g^{(1)}(t)} 4\pi\alpha_s\,\mathcal{M}^{(0)}_{ij\to ij},$$



- The function $\alpha_g(t)$ is known as the Regge trajectory

$$\alpha_g^{(1)}(t) = \frac{r_\Gamma}{2\epsilon}\left(\frac{-t}{\mu^2}\right)^{-\epsilon} \overset{\mu^2\to -t}{=} \frac{r_\Gamma}{2\epsilon}, \qquad r_\Gamma = e^{\epsilon\gamma_{\rm E}}\,\frac{\Gamma(1-\epsilon)^2\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)} \approx 1 - \frac{1}{2}\zeta_2\,\epsilon^2 - \frac{7}{3}\zeta_3\,\epsilon^3 + \ldots$$

2 ➔ 2 SCATTERING IN THE HIGH-ENERGY LIMIT



- Determining the amplitude beyond LL requires to understand Regge cuts.
- Regge structure becomes evident decomposing the amplitude into even and odd parts (projection onto eigenstates of signature, or crossing symmetry s ↔ u):

$$\mathcal{M}^{(\pm)}(s,t) = \tfrac{1}{2}\Big(\mathcal{M}(s,t) \pm \mathcal{M}(-s-t,t)\Big).$$

- $M^{(+)}$ and $M^{(-)}$ are respectively imaginary and real, when expressed in terms of the signature-even combination of logs

$$L \equiv \log\left|\frac{s}{t}\right| - i\frac{\pi}{2} = \frac{1}{2}\left(\log\frac{-s-i0}{-t} + \log\frac{-u-i0}{-t}\right).$$

- Beyond tree level the amplitude has a non-trivial color structure

$$\mathcal{M}(s,t) = \sum_i c^{[i]}\,\mathcal{M}^{[i]}(s,t).$$

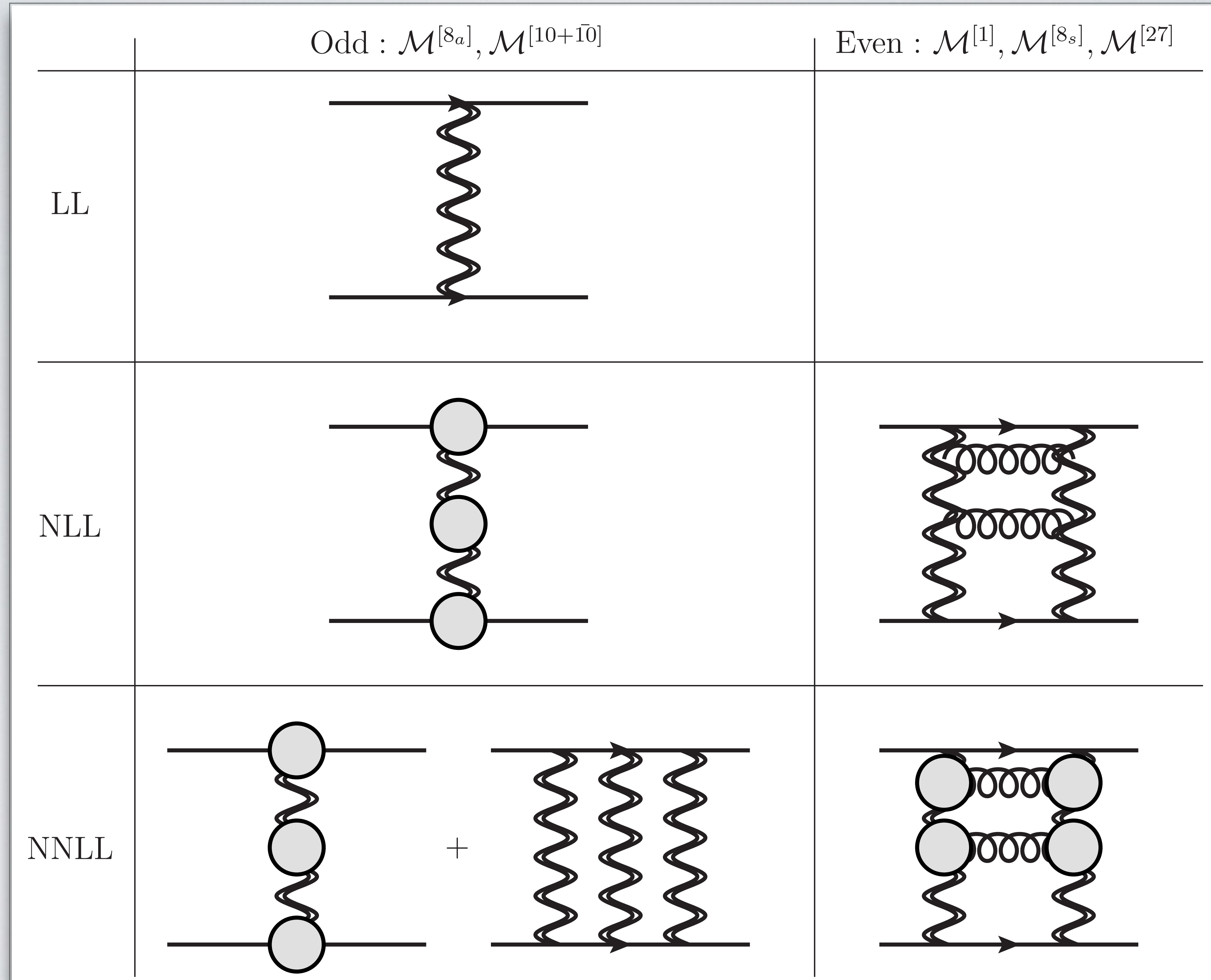
- Decompose the amplitude in a color orthonormal basis in the t-channel

$$8 \otimes 8 = 1 \oplus 8_s \oplus 8_a \oplus 10 \oplus \overline{10} \oplus 27$$

- Invoking Bose symmetry we deduce

$$\text{odd: } \mathcal{M}^{[8_a]}, \mathcal{M}^{[10+\overline{10}]}, \qquad \text{even: } \mathcal{M}^{[1]}, \mathcal{M}^{[8s]}, \mathcal{M}^{[27]} \qquad (gg \text{ scattering}).$$

2 ➔ 2 SCATTERING IN THE HIGH-ENERGY LIMIT



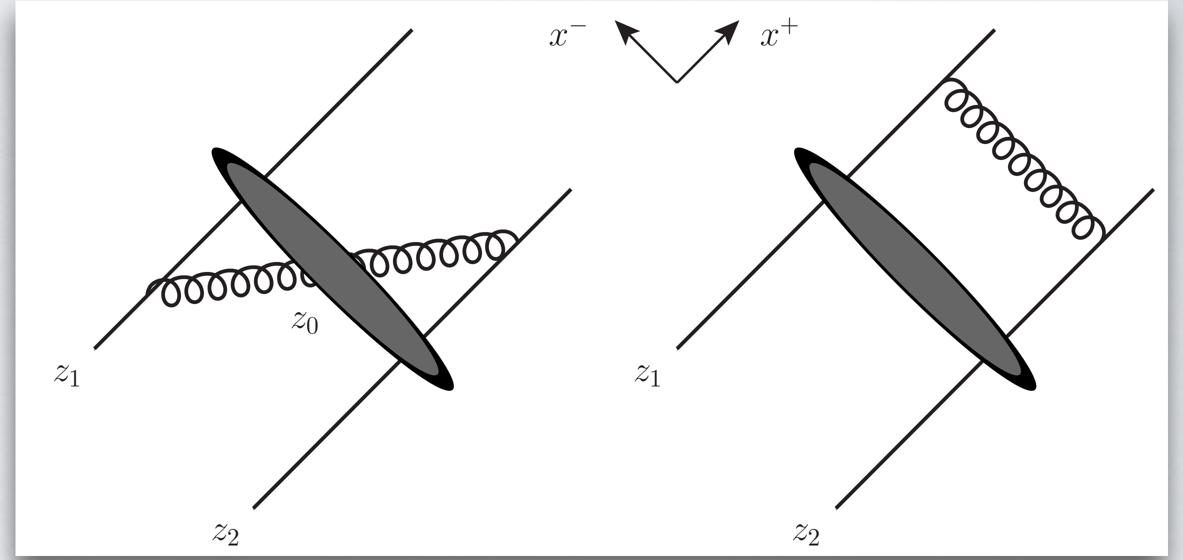
Caron-Huot, 2013; Del Duca, Falcioni, Magnea, LV, 2014, Caron-Huot, Gardi, LV, 2017

2 ➔ 2 SCATTERING IN THE HIGH-ENERGY LIMIT

- High-energy limit = forward scattering: to leading power, the fast projectile and target described in terms of Wilson lines:

$$U(z_\perp) = \mathcal{P}\exp\left[ig_s\int_{-\infty}^{+\infty} A_+^a(x^+,x^-{=}0,z_\perp)dx^+T^a\right].$$

Korchemskaya, Korchemsky, 1994, 1996;
Babansky, Balitsky, 2002, Caron-Huot, 2013



- The Wilson line stretches from $-\infty$ to $+\infty$ and thus develops rapidity divergencies. The regularised Wilson lines obeys the Balitsky-JIMWLK evolution equation:

$$-\frac{d}{d\eta}\left[U(z_1)\ldots U(z_n)\right] = \sum_{i,j=1}^{n} H_{ij}\cdot\left[U(z_1)\ldots U(z_n)\right],$$

with

$$H_{ij} = \frac{\alpha_s}{2\pi^2}\int[dz_i][dz_j][dz_0]K_{ij;0}\left[T^a_{i,L}T^a_{j,L} + T^a_{i,R}T^a_{j,R} - U^{ab}_{\rm ad}(z_0)\left(T^a_{i,L}T^b_{j,R} + T^a_{j,L}T^b_{i,R}\right)\right] + \mathcal{O}(\alpha_s^2).$$

- Evolution in rapidity resums the high-energy log:

$$\eta = L \equiv \log\left|\frac{s}{t}\right| - i\frac{\pi}{2}.$$

2 ➔ 2 SCATTERING IN THE HIGH-ENERGY LIMIT

- In perturbation theory the unitary matrices $U(z)$ will be close to identity and so can be usefully parametrised by a field W

$$U(z) = e^{i g_s \, T^a W^a(z)} \, .$$

Caron-Huot, 2013

- The color-adjoint field W sources a BFKL Reggeised gluon. A generic projectile, created with four-momentum p_1 and absorbed with p_4, can thus be expanded at weak coupling as

$$\begin{aligned} |\psi_i\rangle &\equiv g_s \, D_{i,1}(t) \, |W\rangle + g_s^2 \, D_{i,2}(t) \, |WW\rangle + g_s^3 \, D_{i,3}(t) \, |WWW\rangle + \dots \\ &\equiv |\psi_{i,1}\rangle \;\; + \;\; |\psi_{i,2}\rangle \;\; + \;\; |\psi_{i,3}\rangle + \dots . \end{aligned}$$

- Focus on the Regge-cut contributions: define a "reduced" amplitude by removing the Reggeized gluon and collinear divergences

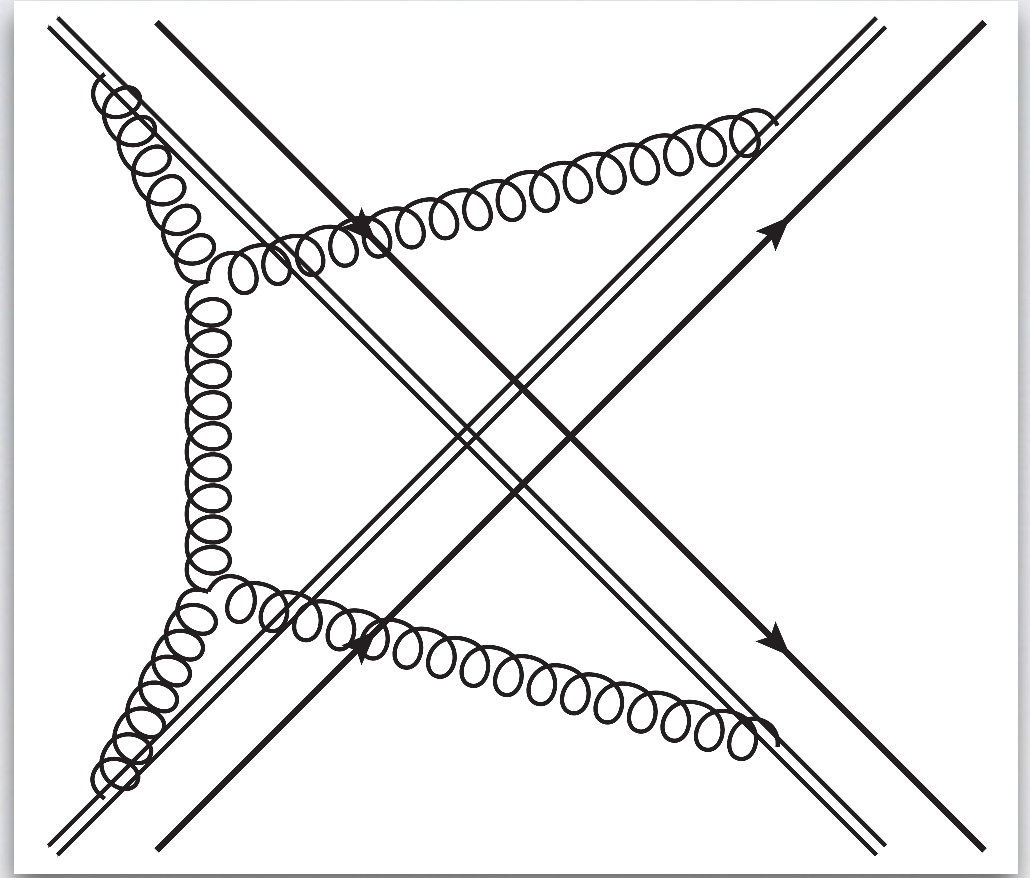
$$\hat{\mathcal{M}}_{ij\to ij} \equiv (Z_i Z_j)^{-1} \, e^{-\, \mathbf{T}_t^2 \, \alpha_g(t) \, L} \, \mathcal{M}_{ij\to ij} \, ,$$

- The scattering amplitude is obtained by taking the expectation value of Wilson lines evolved to equal rapidity:

$$\frac{i}{2s} \hat{\mathcal{M}}_{ij\to ij} \xrightarrow{\text{Regge}} \frac{i}{2s} \left(\hat{\mathcal{M}}^{(+)}_{ij\to ij} + \hat{\mathcal{M}}^{(-)}_{ij\to ij} \right) \equiv \langle \psi_j^{(+)} | e^{-\hat{H} L} | \psi_i^{(+)} \rangle + \langle \psi_j^{(-)} | e^{-\hat{H} L} | \psi_i^{(-)} \rangle .$$

Caron-Huot, 2013, Caron-Huot, Gardi, LV, 2017

INTERMEZZO: REGGE VS INFRARED FACTORIZATION



REGGE VS INFRARED FACTORISATION

- We have a tool to calculate scattering amplitudes to high orders in perturbation theory (in the high-energy limit).
- Application: test (and predict) the analytic structure of infrared divergences in gauge theories.

- The infrared divergences of amplitudes are controlled by a renormalization group equation:

$$\mathcal{M}_n\left(\{p_i\},\mu,\alpha_s(\mu^2)\right) = \mathbf{Z}_n\left(\{p_i\},\mu,\alpha_s(\mu^2)\right)\mathcal{H}_n\left(\{p_i\},\mu,\alpha_s(\mu^2)\right),$$

Becher, Neubert, 2009; Gardi, Magnea, 2009

where $\mathbf{Z}_n$ is given as a path-ordered exponential of the soft-anomalous dimension:

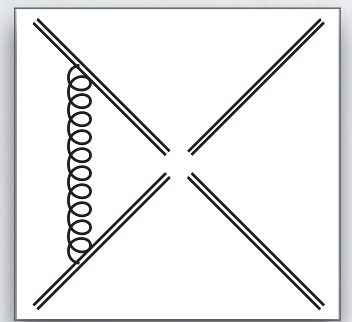
$$\mathbf{Z}_n\left(\{p_i\},\mu,\alpha_s(\mu^2)\right) = \mathcal{P}\exp\left\{-\frac{1}{2}\int_0^{\mu^2}\frac{d\lambda^2}{\lambda^2}\mathbf{\Gamma}_n\left(\{p_i\},\lambda,\alpha_s(\lambda^2)\right)\right\},$$

- The soft anomalous dimension for scattering of massless partons ($p_i^2 = 0$) is an operators in color space given, to three loops, by

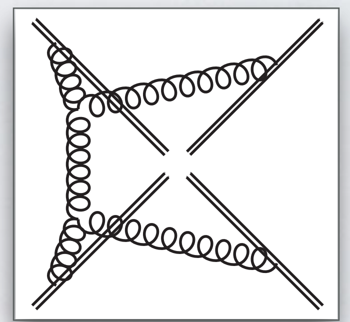
$$\mathbf{\Gamma}_n\left(\{p_i\},\lambda,\alpha_s(\lambda^2)\right) = \mathbf{\Gamma}_n^{\rm dip.}\left(\{p_i\},\lambda,\alpha_s(\lambda^2)\right) + \mathbf{\Delta}_n\left(\{\rho_{ijkl}\}\right).$$

REGGE VS INFRARED FACTORISATION

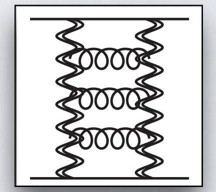
$$\mathbf{\Gamma}_n\left(\{p_i\},\lambda,\alpha_s(\lambda^2)\right) = \mathbf{\Gamma}_n^{\text{dip.}}\left(\{p_i\},\lambda,\alpha_s(\lambda^2)\right) + \mathbf{\Delta}_n\left(\{\rho_{ijkl}\}\right).$$

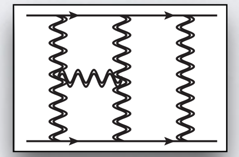


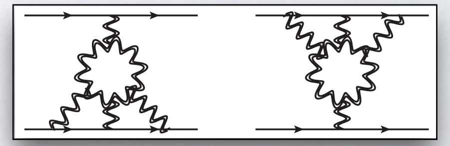
"dipole formula"



"quadrupole correction"

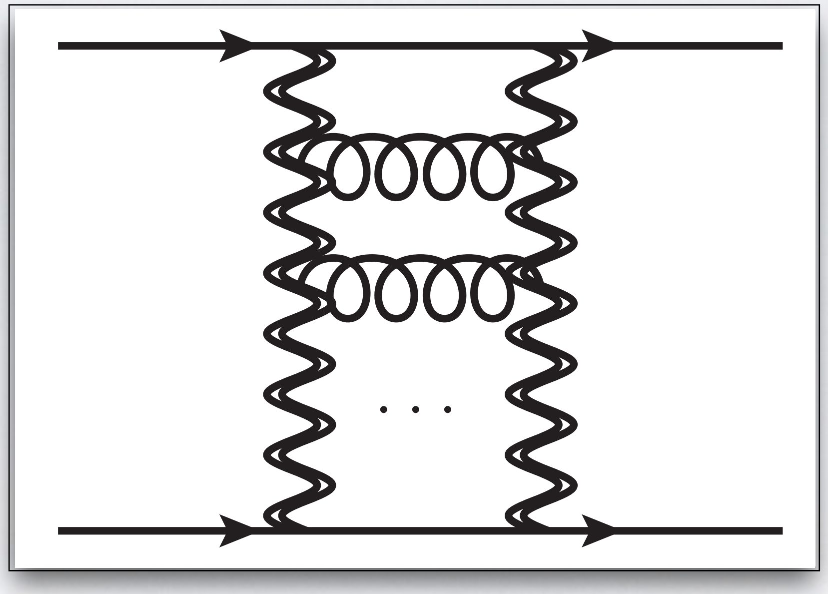




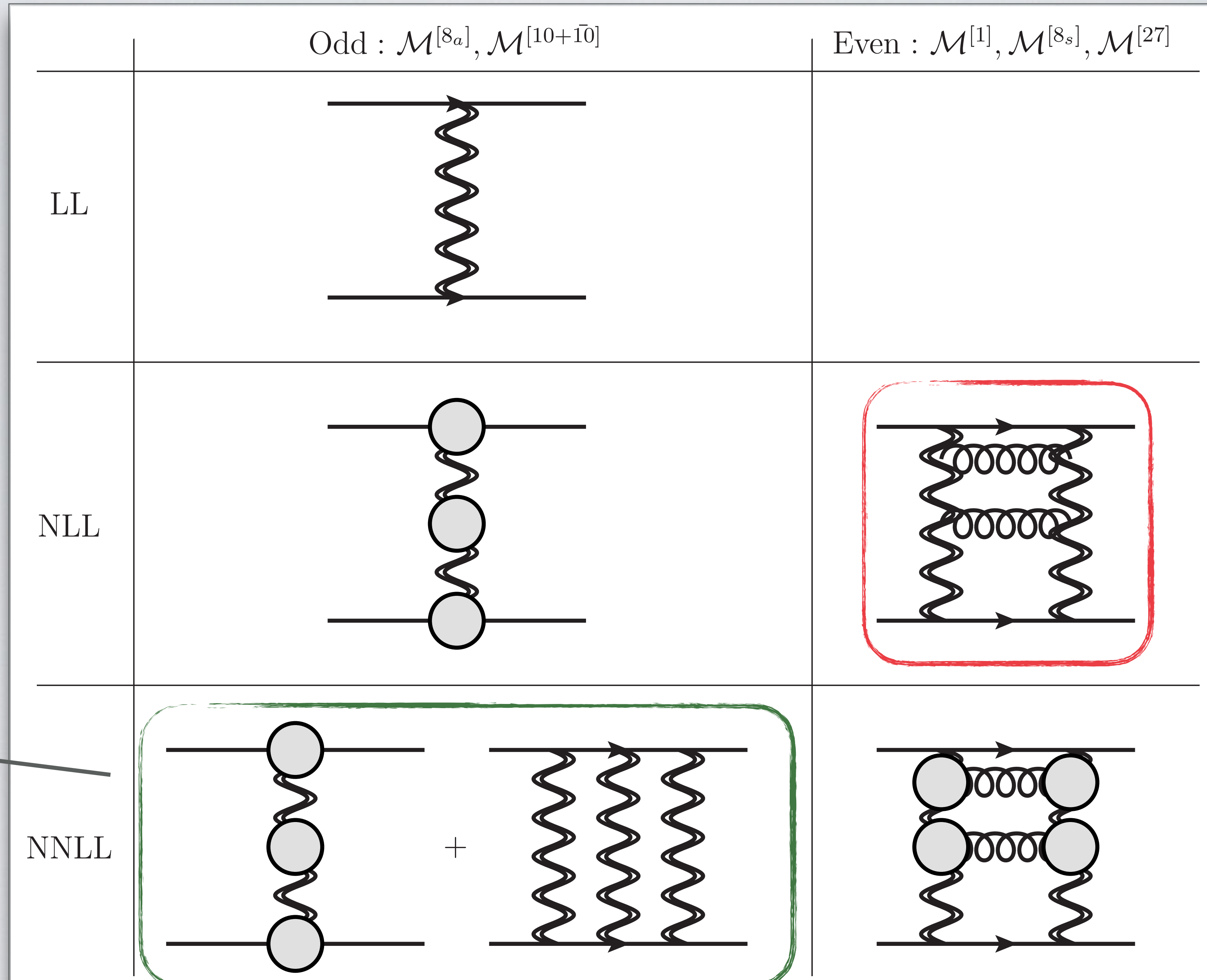


- Early studies of constraints from soft-collinear factorisation, collinear limits, and the high-energy limit in Becher, Neubert, 2009; Dixon, Gardi, Magnea, 2009; Del Duca, Duhr, Gardi, Magnea, White, 2011; Neubert, LV, 2012;
- first evidence of "beyond dipole" contribution at four loops in Caron-Huot, 2013;
- finally calculated exactly in Almelid, Duhr, Gardi, 2015, 2016;
- confirmed, in 2 ➔ 2 scattering in N=4 SYM in Henn, Mistlberger, 2016;
- confirmed, in the high energy limit, in Caron-Huot, Gardi, LV, 2017;
- re-derived based on a bootstrap approach in Almelid, Duhr, Gardi, McLeod, White, 2017.

THE TWO REGGEON CUT



THE 2-REGGEON CUT



See Caron-Huot, Gardi, LV, 2017

- Here we calculate

$$\frac{i}{2s}\hat{\mathcal{M}}^{(+),\mathrm{NLL}}_{ij\to ij} \equiv \langle\psi^{(+)}_{j,2}|e^{-\hat{H}L}|\psi^{(+)}_{i,2}\rangle.$$

THE 2-REGGEON CUT

- The amplitude takes the form of an iterated integral over the BFKL kernel:

$$\hat{\mathcal{M}}^{(+,\ell)}_{\text{NLL}} = -i\pi \frac{(B_0)^\ell}{(\ell-1)!} \int [\text{D}k] \, \frac{p^2}{k^2(k-p)^2} \, \Omega^{(\ell-1)}(p,k) \, \mathbf{T}^2_{s-u} \, \mathcal{M}^{(0)},$$

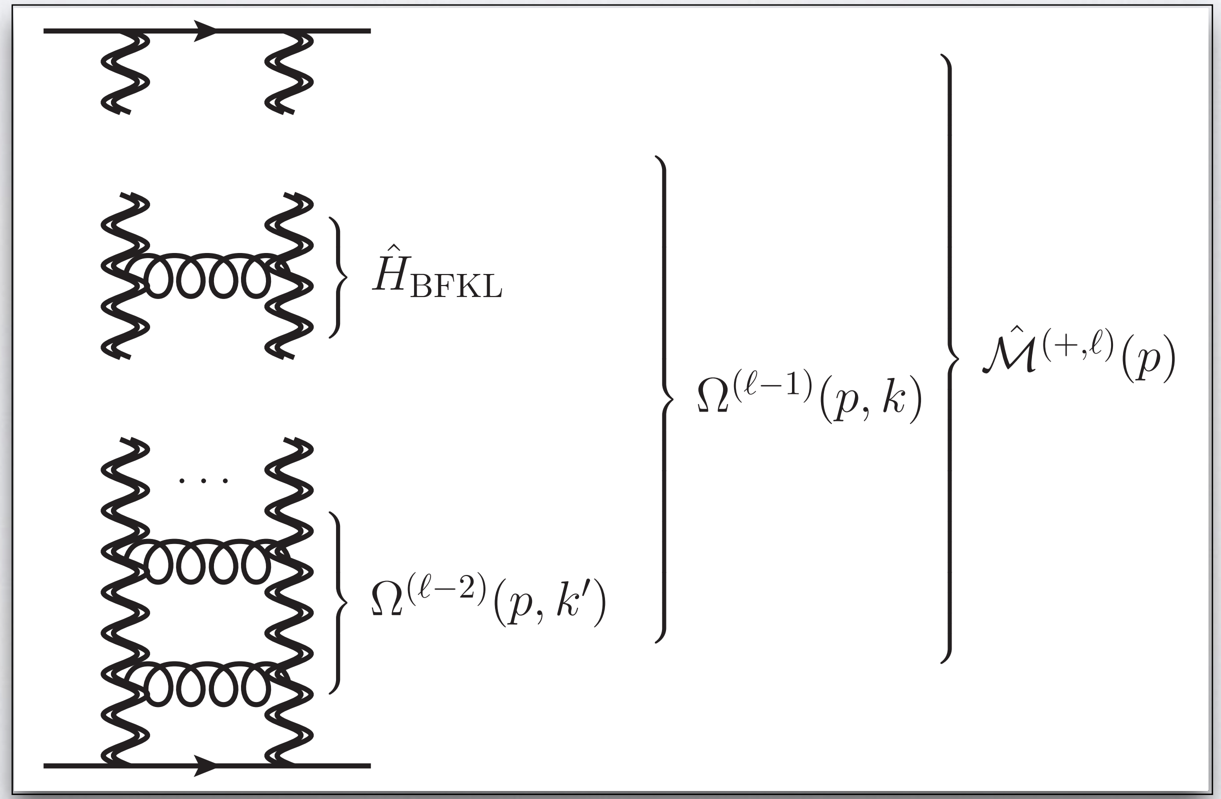
with

$$B_0 = e^{\epsilon\gamma_{\text{E}}} \frac{\Gamma^2(1-\epsilon)\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)}.$$

- The "target averaged wave function" reads:

$$\Omega^{(\ell-1)}(p,k) = \hat{H} \, \Omega^{(\ell-2)}(p,k), \qquad \hat{H} = (2C_A - \mathbf{T}^2_t) \, \hat{H}_{\text{i}} + (C_A - \mathbf{T}^2_t) \, \hat{H}_{\text{m}}$$

- Graphically:



THE 2-REGGEON CUT

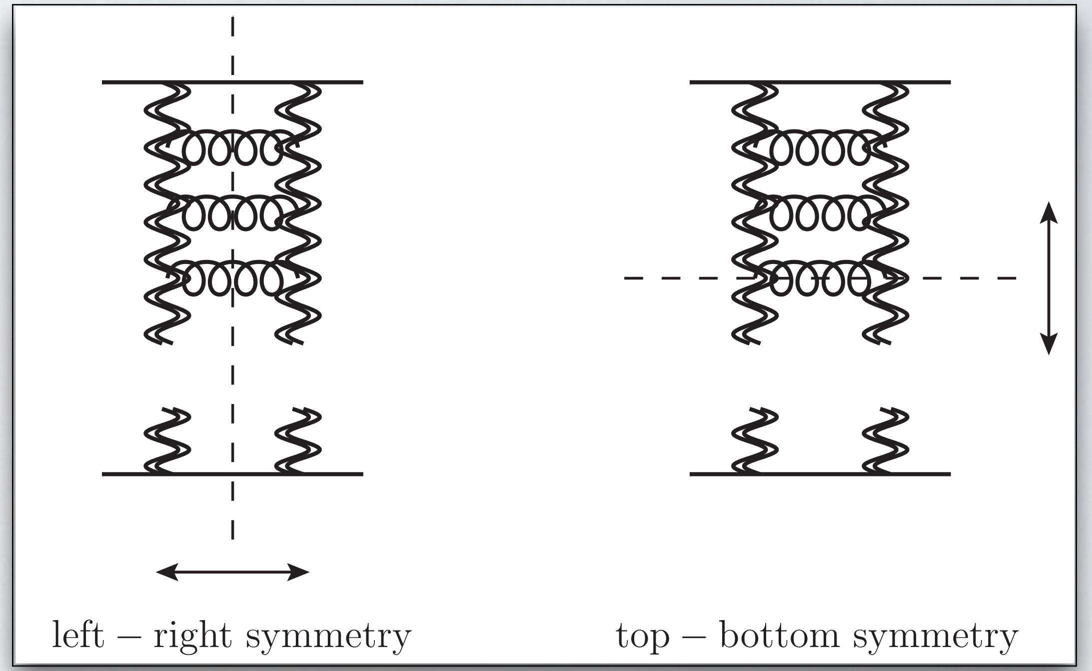
- Wavefunction evolution: two color structures

$$\Omega^{(\ell-1)}(p,k) = \hat{H}\,\Omega^{(\ell-2)}(p,k), \qquad \hat{H} = (2C_A - \mathbf{T}_t^2)\,\hat{H}_\mathrm{i} + (C_A - \mathbf{T}_t^2)\,\hat{H}_\mathrm{m}$$

with

$$\hat{H}_\mathrm{i}\,\Psi(p,k) = \int [\mathrm{D}k']\, f(p,k,k') \Big[\Psi(p,k') - \Psi(p,k)\Big],$$

$$\hat{H}_\mathrm{m}\,\Psi(p,k) = J(p,k)\,\Psi(p,k),$$



Initial condition:

$$\Omega^{(0)}(p,k) = 1.$$

- The function f is the BFKL kernel

$$f(p,k',k) = \frac{k'^2}{k^2(k-k')^2} + \frac{(p-k')^2}{(p-k)^2(k-k')^2} - \frac{p^2}{k^2(p-k)^2},$$

$$J(p,k) = -2\epsilon \int [\mathrm{D}k']\, f(p,k,k').$$

- Wavefunction can be expressed in terms of single-valued HPLs (Dixon, Pennington, Duhr, 2012; Del Duca, Dixon, Pennington, Duhr, 2013; Del Duca, Druc, Drummond, Duhr, Dulat, Marzucca, Papathanasiou, Verbeek 2016, …).

THE 2-REGGEON CUT

- Up to four loops one gets

$$\hat{\mathcal{M}}^{(+,1)}_{\rm NLL} = -i\pi \frac{B_0}{2\epsilon} \mathbf{T}^2_{s-u} \mathcal{M}^{(0)},$$

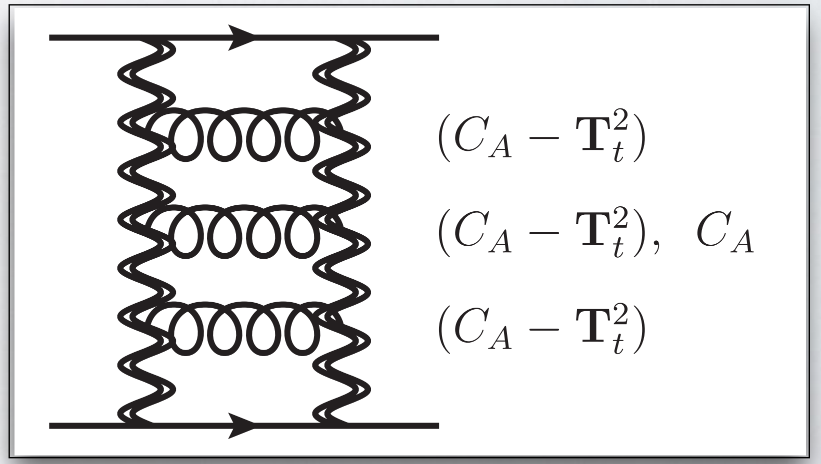
$$\hat{\mathcal{M}}^{(+,2)}_{\rm NLL} = i\pi \frac{(B_0)^2}{2} \left[\frac{1}{(2\epsilon)^2} + \frac{9\zeta_3}{2}\epsilon + \frac{27\zeta_4}{4}\epsilon^2 + \frac{63\zeta_5}{2}\epsilon^3 + \mathcal{O}(\epsilon^4) \right] (C_A - \mathbf{T}^2_t)\, \mathbf{T}^2_{s-u} \mathcal{M}^{(0)},$$

$$\hat{\mathcal{M}}^{(+,3)}_{\rm NLL} = i\pi \frac{(B_0)^3}{3!} \left[\frac{1}{(2\epsilon)^3} - \frac{11\zeta_3}{4} - \frac{33\zeta_4}{8}\epsilon - \frac{357\zeta_5}{4}\epsilon^2 + \mathcal{O}(\epsilon^3) \right] (C_A - \mathbf{T}^2_t)^2\, \mathbf{T}^2_{s-u} \mathcal{M}^{(0)},$$

$$\hat{\mathcal{M}}^{(+,4)}_{\rm NLL} = i\pi \frac{(B_0)^4}{4!} \left\{ (C_A - \mathbf{T}^2_t)^3 \left(\frac{1}{(2\epsilon)^4} + \frac{175\zeta_5}{2}\epsilon + \mathcal{O}(\epsilon^2) \right) + C_A (C_A - \mathbf{T}^2_t)^2 \left(-\frac{\zeta_3}{8\epsilon} + \frac{3}{16}\zeta_4 - \frac{167\zeta_5}{8}\epsilon + \mathcal{O}(\epsilon^2) \right) \right\} \mathbf{T}^2_{s-u} \mathcal{M}^{(0)}.$$

Caron-Huot, 2013

- At four loop a new color structure appear, with a single pole not predicted by the dipole formula of infrared divergences!
- The fact that it arises only at four loops is a consequence of the "top-bottom" symmetry of the ladder.



2-REGGEON CUT: SOFT APPROXIMATION

- Can calculate the amplitude to higher orders - the calculation becomes rapidly involved.
- However, here we are interested in the infrared singularities only.
- Shortcut: closer inspection of the Hamiltonian reveals that the wavefunction is finite!

$$\hat{H}_{\rm i}\,\Psi(p,k) = \int [{\rm D}k']\, f(p,k,k') \Big[\Psi(p,k') - \Psi(p,k)\Big],$$

$$\hat{H}_{\rm m}\,\Psi(p,k) = J(p,k)\,\Psi(p,k),$$

finite!

- All divergences must arise from the last integration!

$$\hat{\mathcal{M}}^{(+,\ell)}_{\rm NLL} = -i\pi \frac{(B_0)^\ell}{(\ell-1)!} \int [{\rm D}k]\, \frac{p^2}{k^2(k-p)^2}\, \Omega^{(\ell-1)}(p,k)\, \mathbf{T}^2_{s-u}\, \mathcal{M}^{(0)},$$

- Divergences arises only from the limit $k \to p$ or $k \to 0$ limit. Consider one of the two regions, and multiply the result by two.
- This is consistent, because evolution in the soft region ($k \to 0$) stays within the soft region.

2-REGGEON CUT: SOFT APPROXIMATION

- In the soft limit the integrations becomes trivial. We obtain an all-order solution

$$\Omega_s^{(\ell-1)}(p,k) = \frac{(C_A - \mathbf{T}_t^2)^{\ell-1}}{(2\epsilon)^{\ell-1}} \sum_{n=0}^{\ell-1} (-1)^n \binom{\ell-1}{n} \left(\frac{p^2}{k^2}\right)^{n\epsilon} \prod_{m=0}^{n-1} \left\{1 + \hat{B}_m(\epsilon) \frac{2C_A - \mathbf{T}_t^2}{C_A - \mathbf{T}_t^2}\right\},$$

where

$$\hat{B}_n(\epsilon) = 1 - \frac{B_n(\epsilon)}{B_0(\epsilon)}, \quad \text{and} \quad B_n(\epsilon) = e^{\epsilon\gamma_E} \frac{\Gamma(1-\epsilon)}{\Gamma(1+n\epsilon)} \frac{\Gamma(1+\epsilon+n\epsilon)\Gamma(1-\epsilon-n\epsilon)}{\Gamma(1-2\epsilon-n\epsilon)}.$$

- It is immediate to get the reduced amplitude

$$\hat{\mathcal{M}}_{\rm NLL}^{(+,\ell)}|_s = i\pi \frac{1}{(2\epsilon)^\ell} \frac{B_0^\ell(\epsilon)}{\ell!} \left(1 + \hat{B}_{-1}\right) (C_A - \mathbf{T}_t^2)^{\ell-1} \sum_{n=1}^{\ell} (-1)^{n+1} \binom{\ell}{n} \times \prod_{m=0}^{n-2} \left[1 + \hat{B}_m(\epsilon) \frac{2C_A - \mathbf{T}_t^2}{C_A - \mathbf{T}_t^2}\right] \mathbf{T}_{s-u}^2 \, \mathcal{M}^{(0)} + \mathcal{O}(\epsilon^0).$$

- The result is valid up to the single poles, which allows one to achieve a tremendous simplification

$$\hat{\mathcal{M}}_{\rm NLL}^{(+,\ell)}|_s = i\pi \frac{1}{(2\epsilon)^\ell} \frac{B_0^\ell(\epsilon)}{\ell!} \left(1 - R(\epsilon) \frac{C_A}{C_A - \mathbf{T}_t^2}\right)^{-1} (C_A - \mathbf{T}_t^2)^{\ell-1} \, \mathbf{T}_{s-u}^2 \, \mathcal{M}^{(0)} + \mathcal{O}(\epsilon^0),$$

where

Caron-Huot, Gardi, Reichel, LV, 2017

$$R(\epsilon) \equiv \frac{B_0(\epsilon)}{B_{-1}(\epsilon)} - 1 = \frac{\Gamma^3(1-\epsilon)\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)} - 1 = -2\zeta_3\,\epsilon^3 - 3\zeta_4\,\epsilon^4 - 6\zeta_5\epsilon^5 - \left(2\zeta_3^2 + 10\zeta_6\right)\epsilon^6 + \mathcal{O}(\epsilon^7).$$

TWO REGGEON CUT: SOFT APPROXIMATION

- Expand for a few orders in the strong coupling constant:

$$\hat{\mathcal{M}}_{\rm NLL}^{(+,\ell=1,2,3)}|_s = i\pi \frac{B_0^\ell(\epsilon)}{\ell!\,(2\epsilon)^\ell} (C_A - \mathbf{T}_t^2)^{\ell-1} \mathbf{T}_{s-u}^2 \mathcal{M}^{(0)} + \mathcal{O}(\epsilon^0),$$

$$\hat{\mathcal{M}}_{\rm NLL}^{(+,\ell=4,5,6)}|_s = i\pi \frac{B_0^\ell(\epsilon)}{\ell!\,(2\epsilon)^\ell} \left\{ (C_A - \mathbf{T}_t^2)^{\ell-1} + R(\epsilon)\, C_A (C_A - \mathbf{T}_t^2)^{\ell-2} \right\} \mathbf{T}_{s-u}^2 \mathcal{M}^{(0)} + \mathcal{O}(\epsilon^0),$$

$$\hat{\mathcal{M}}_{\rm NLL}^{(+,\ell=7,8,9)}|_s = i\pi \frac{B_0^\ell(\epsilon)}{\ell!\,(2\epsilon)^\ell} \left\{ (C_A - \mathbf{T}_t^2)^{\ell-1} + R(\epsilon)\, C_A (C_A - \mathbf{T}_t^2)^{\ell-2} + R^2(\epsilon)\, C_A^2 (C_A - \mathbf{T}_t^2)^{\ell-3} \right\} \mathbf{T}_{s-u}^2 \mathcal{M}^{(0)} + \mathcal{O}(\epsilon^0),$$

$$\hat{\mathcal{M}}_{\rm NLL}^{(+,\ell=10,11,12)}|_s = i\pi \frac{B_0^\ell(\epsilon)}{\ell!\,(2\epsilon)^\ell} \left\{ (C_A - \mathbf{T}_t^2)^{\ell-1} + R(\epsilon)\, C_A (C_A - \mathbf{T}_t^2)^{\ell-2} + R^2(\epsilon)\, C_A^2 (C_A - \mathbf{T}_t^2)^{\ell-3} + R^3(\epsilon)\, C_A^3 (C_A - \mathbf{T}_t^2)^{\ell-4} \right\} \mathbf{T}_{s-u}^2 \mathcal{M}^{(0)} + \mathcal{O}(\epsilon^0).$$

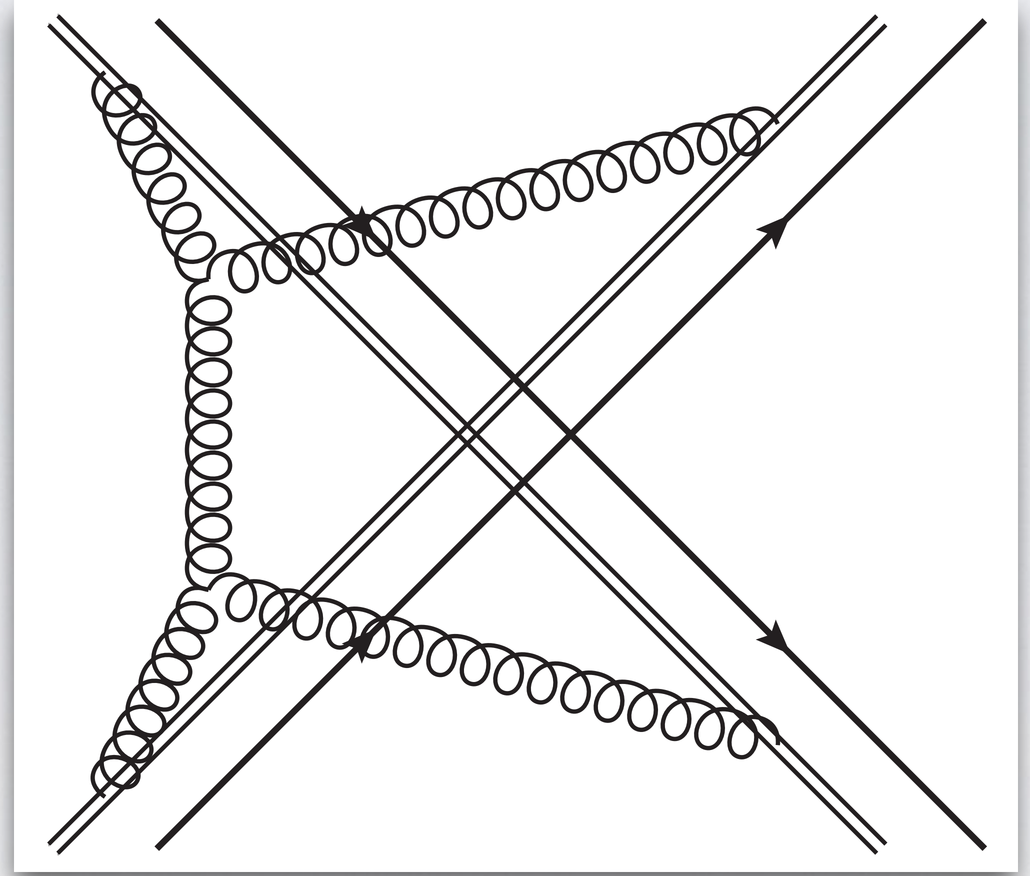
A new color structure appears every three loops!

Caron-Huot, Gardi, Reichel, LV, 2017

- Resumming the amplitude to all loops we get

$$\hat{\mathcal{M}}_{\rm NLL}^{(+)}|_s = 4\pi\alpha_s \frac{i\pi}{L(C_A - \mathbf{T}_t^2)} \left(1 - R(\epsilon) \frac{C_A}{C_A - \mathbf{T}_t^2}\right)^{-1} \left[\exp\left\{ \frac{B_0(\epsilon)}{2\epsilon} \frac{\alpha_s}{\pi} L (C_A - \mathbf{T}_t^2) \right\} - 1 \right] \mathbf{T}_{s-u}^2 \mathcal{M}^{(0)} + \mathcal{O}(\epsilon^0).$$

2-REGGEON CUT: INFRARED SINGULARITIES



2-REGGEON CUT: INFRARED SINGULARITIES

- Recall the infrared factorisation formula

$$\mathcal{M}\left(\{p_i\},\mu,\alpha_s(\mu^2)\right) = \mathbf{Z}\left(\{p_i\},\mu,\alpha_s(\mu^2)\right)\mathcal{H}\left(\{p_i\},\mu,\alpha_s(\mu^2)\right),$$

with

$$\mathbf{Z}\left(\{p_i\},\mu,\alpha_s(\mu^2)\right) = \mathcal{P}\exp\left\{-\frac{1}{2}\int_0^{\mu^2}\frac{d\lambda^2}{\lambda^2}\mathbf{\Gamma}\left(\{p_i\},\lambda,\alpha_s(\lambda^2)\right)\right\},$$

Expand the soft anomalous dimension in the high-energy logarithm:

$$\mathbf{\Gamma}\left(\alpha_s(\lambda)\right) = \mathbf{\Gamma}_{\rm LL}\left(\alpha_s(\lambda),L\right) + \mathbf{\Gamma}_{\rm NLL}\left(\alpha_s(\lambda),L\right) + \mathbf{\Gamma}_{\rm NNLL}\left(\alpha_s(\lambda),L\right) + \ldots .$$

- At LL one has

$$\mathbf{\Gamma}_{\rm LL}\left(\alpha_s(\lambda)\right) = \frac{\alpha_s(\lambda)}{\pi}\frac{\gamma_K^{(1)}}{2}L\,\mathbf{T}_t^2 = \frac{\alpha_s(\lambda)}{\pi}L\,\mathbf{T}_t^2 .$$

- At NLL

$$\mathbf{\Gamma}_{\rm NLL} = \mathbf{\Gamma}_{\rm NLL}^{(+)} + \mathbf{\Gamma}_{\rm NLL}^{(-)},$$

- with

$$\mathbf{\Gamma}_{\rm NLL}^{(+)} = \frac{\alpha_s(\lambda)}{\pi}\sum_{i=1}^{2}\left(\frac{\gamma_K^{(1)}}{2}C_i\log\frac{-t}{\lambda^2} + 2\gamma_i^{(1)}\right) + \left(\frac{\alpha_s(\lambda)}{\pi}\right)^2\frac{\gamma_K^{(2)}}{2}L\,\mathbf{T}_t^2,$$

$$\mathbf{\Gamma}_{\rm NLL}^{(-)} = i\pi\frac{\alpha_s(\lambda)}{\pi}\mathbf{T}_{s-u}^2 + O(\alpha_s^4L^3).$$

2-REGGEON CUT: INFRARED SINGULARITIES

- We get the infrared-factorised representation of the reduced amplitude:

$$\exp\left\{\frac{1-B_0(\epsilon)}{2\epsilon}\frac{\alpha_s}{\pi}L(C_A-\mathbf{T}_t^2)\right\}\hat{\mathcal{M}}_{\text{NLL}}$$
$$= -\int_0^p \frac{d\lambda}{\lambda}\exp\left\{\frac{1}{2\epsilon}\frac{\alpha_s(p)}{\pi}L(C_A-\mathbf{T}_t^2)\left[1-\left(\frac{p}{\lambda}\right)^{\epsilon}\right]\right\}\mathbf{\Gamma}^{(-)}_{\text{NLL}}\left(\alpha_s(\lambda)\right)\mathcal{M}^{(\text{tree})}+\mathcal{O}(\epsilon^0).$$

- Matching with the result from the Regge theory allows us to obtain

$$\mathbf{\Gamma}^{(-,\ell)}_{\text{NLL}} = \frac{i\pi}{(\ell-1)!}\left(1-R\left(\frac{x}{2}(C_A-\mathbf{T}_t^2)\right)\frac{C_A}{C_A-\mathbf{T}_t^2}\right)^{-1}\Bigg|_{x^{\ell-1}}\mathbf{T}^2_{s-u}\,,$$

and recall that

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$$R(\epsilon)=\frac{\Gamma^3(1-\epsilon)\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)}-1=-2\zeta_3\,\epsilon^3-3\zeta_4\,\epsilon^4-6\zeta_5\epsilon^5-\left(2\zeta_3^2+10\zeta_6\right)\epsilon^6+\ldots.$$

2-REGGEON CUT: INFRARED SINGULARITIES

- Explicitly, for the first few orders we have:

$$\mathbf{\Gamma}_{\rm NLL}^{(-,1)} = i\pi\, \mathbf{T}_{s-u}^2, \qquad \mathbf{\Gamma}_{\rm NLL}^{(-,2)} = 0, \qquad \mathbf{\Gamma}_{\rm NLL}^{(-,3)} = 0,$$

$$\mathbf{\Gamma}_{\rm NLL}^{(-,4)} = -i\pi\, \frac{\zeta_3}{24}\, C_A (C_A - \mathbf{T}_t^2)^2\, \mathbf{T}_{s-u}^2,$$

$$\mathbf{\Gamma}_{\rm NLL}^{(-,5)} = -i\pi\, \frac{\zeta_4}{128}\, C_A (C_A - \mathbf{T}_t^2)^3\, \mathbf{T}_{s-u}^2,$$

$$\mathbf{\Gamma}_{\rm NLL}^{(-,6)} = -i\pi\, \frac{\zeta_5}{640}\, C_A (C_A - \mathbf{T}_t^2)^4\, \mathbf{T}_{s-u}^2,$$

$$\mathbf{\Gamma}_{\rm NLL}^{(-,7)} = i\pi \frac{1}{720} \left[\frac{\zeta_3^2}{16}\, C_A^2 (C_A - \mathbf{T}_t^2)^4 + \frac{1}{32} \left(\zeta_3^2 - 5\zeta_6 \right) C_A (C_A - \mathbf{T}_t^2)^5 \right] \mathbf{T}_{s-u}^2,$$

$$\mathbf{\Gamma}_{\rm NLL}^{(-,8)} = i\pi \frac{1}{5040} \left[\frac{3\zeta_3\zeta_4}{32}\, C_A^2 (C_A - \mathbf{T}_t^2)^5 + \frac{3}{64} \left(\zeta_3\zeta_4 - 3\zeta_7 \right) C_A (C_A - \mathbf{T}_t^2)^6 \right] \mathbf{T}_{s-u}^2.$$

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- The result can be used as constraint in a bootstrap approach to the soft anomalous dimension.

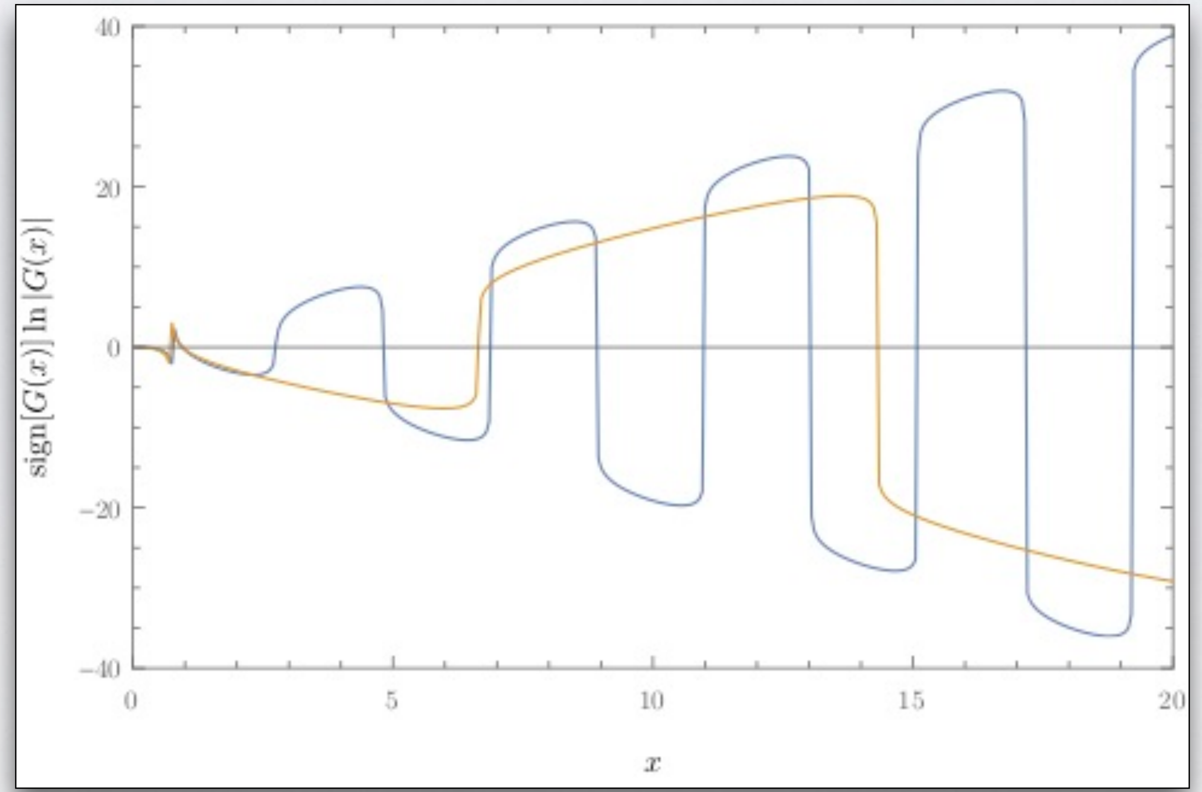
See e.g. **Almelid, Duhr, Gardi, McLeod, White, 2017**

2-REGGEON CUT: INFRARED SINGULARITIES

- The anomalous dimension has an infinite radius of convergence as a function $x = L\,\alpha_s/\pi$, i.e. it is an entire function, free of any singularities for any finite x. Write it as

$$\mathbf{\Gamma}^{(-)}_{\rm NLL} = i\pi\frac{\alpha_s}{\pi}\,G\left(\frac{\alpha_s}{\pi}L\right)\mathbf{T}^2_{s-u}\,, \qquad\qquad G(x) = \sum_{\ell=1}^{\infty} x^{\ell-1}G^{(\ell)}\,.$$

- Plotting $G(x)$ for larger values of x reveals oscillations with a constant period and an exponentially growing amplitude. Here we plot the logarithm of $|G(x)|$ weighted by the sign of $G(x)$:



Caron-Huot, Gardi, Reichel, LV, 2017

- The function is well approximated by

$$G(x) \to c\,e^{ax}\cos\left(bx+d\right)\;,$$

	a	b	c	d
1	1.97	1.52	0.25	0.48
27	1.46	0.41	0.58	2.01

CONCLUSION

- We solved the BFKL evolution of even $2 \rightarrow 2$ scattering amplitudes at NLL in the high-energy logarithms, in the soft limit.
- This allows us to determine the structure of infrared divergences of this amplitude to all orders in perturbation theory, and extract the corresponding soft anomalous dimension.
- From a perturbative point of view, new infrared divergences proportional to a new color structure appear every three loops.
- From an analytic point of view, the soft anomalous dimension is given in terms of an entire function, which can be parameterised asymptotically in terms of a few parameters.
- The information obtained concerning infrared singularities can be used to constrain the structure of the soft anomalous dimension in general kinematics.