Elliptic polylogarithms and two-loop Feynman integrals

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in collaboration with
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Loops and Legs in Quantum Field Theory

St. Goar, 30 April 2018

Polylogarithms

- Large classes of loop integrals can be expressed in terms of polylogarithms.

$$G(\underbrace{a_1,\ldots,a_n}_{\text{weight } n};z) = \int_0^z \frac{dt}{t-a_1}\, G(a_2,\ldots,a_n;t)$$

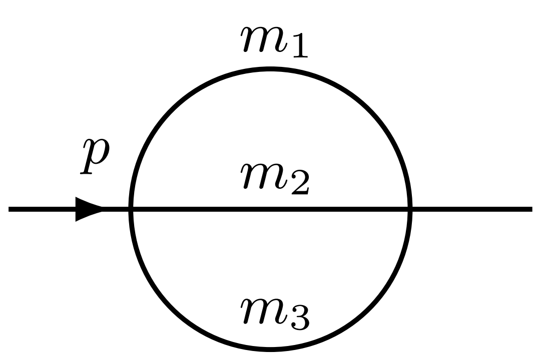
$$G(0;z) = \log z$$

$$G(a_1;z) = \log\left(1-\frac{z}{a_1}\right)$$

$$G(0,1;z) = -\mathrm{Li}_2(z)$$

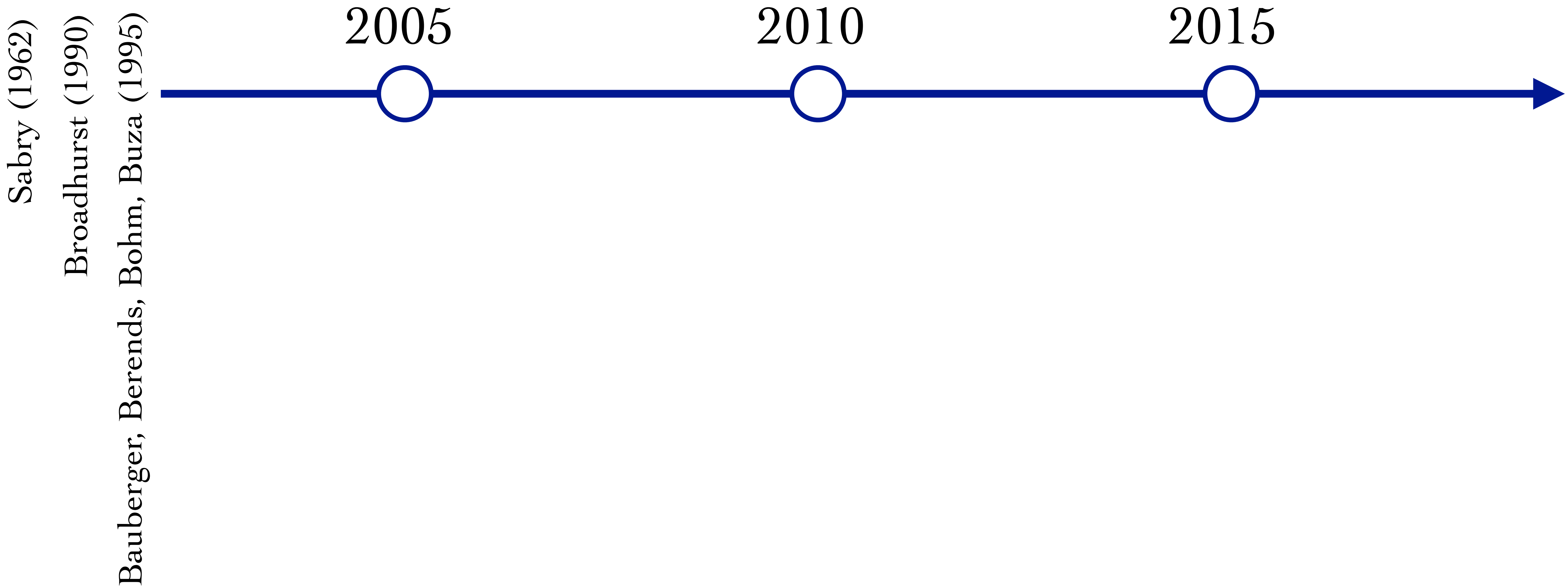
[Goncharov; Brown; Poincaré; Euler; Nielsenl Lappo-Danilevski; …]

- Not all Feynman integrals can be expressed in terms of MPLs!
- Most prominent example: Two-loop sunset integral.



[Broadhurst; Bauberger, Berends, Bohm, Buza; Laporta, Remiddi; Bloch, Vanhove; Remiddi, Tancredi; Adams, Bogner, Schweitzer, Weinzierl; Hidding, Moriello; Brödel, CD, Dulat, Penante, Tancredi]

Elliptic Feynman integrals



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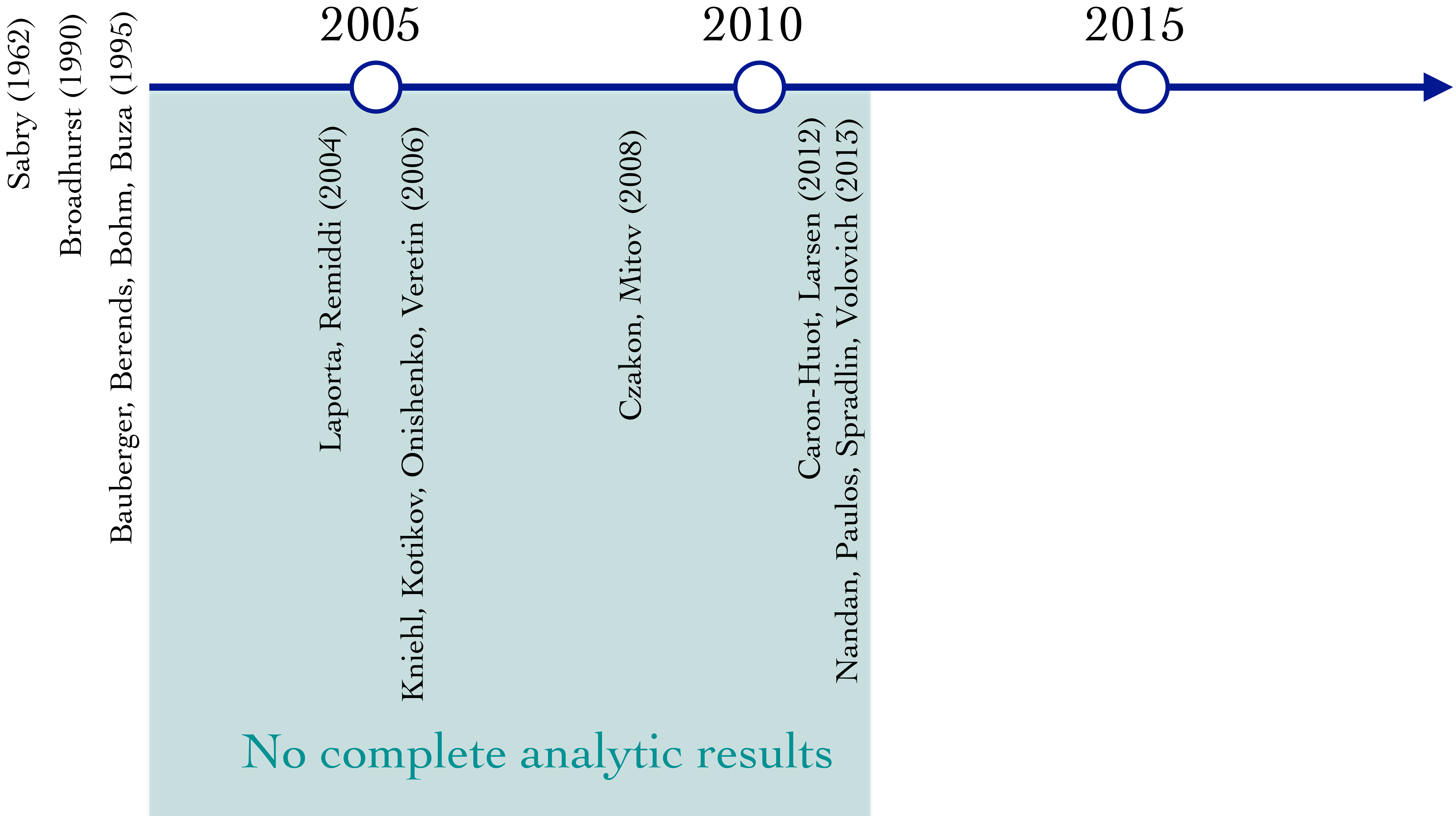


Elliptic Feynman integrals

Sabry (1962)

Broadhurst (1990)

Bauberger, Berends, Bohm, Buza (1995)

2005

Laporta, Remiddi (2004)

Kniehl, Kotikov, Onishenko, Veretin (2006)

2010

Czakon, Mitov (2008)

Caron-Huot, Larsen (2012)

Nandan, Paulos, Spradlin, Volovich (2013)

No complete analytic results

2015

Bloch, Vanhove (2013)

Adams, Bogner, Weinzierl (2013-16)

Bloch, Kerr, Vanhove (2014)

Adams, Bogner, Weinzierl (2015,16)

Bloch, Kerr, Vanhove (2016) Passarino, (2016)

Adams, Bogner, Schweitzer, Weinzierl (2016)

Bonciani, Del Duca, Frellesvig et al., (2016)

Remiddi, Tancredi (2016,17)

Adams, Weinzierl (2017) Chen, Jiang, Qiao, (2017)

Bonciani, Becchetti (2017)

Ablinger, Blümlein, de Freitas, et al. (2017)

Broedel, Dulat, CD, Tancredi (2017)

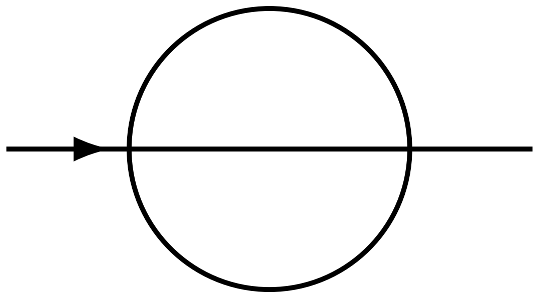
Bourjaily, MacLeod, Spradlin, et al. (2017)

Primo, Tancredi (2017) Hidding, Moriello (2017)

Broedel, Dulat, CD, Penante, Tancredi (2018)

The elliptic sunset

- Integral has the form (dispersion integral):



$$= 2\int_{(m_2+m_3)^2}^{\infty} \frac{dx}{\sqrt{R_2(x,m_2^2,m_3^2)R_2(s,x,m_1^2)}}$$

$$\times \log\left(\frac{x+m_1^2-s+\sqrt{R_2(s,x,m_1^2)}}{x+m_1^2-s-\sqrt{R_2(s,x,m_1^2)}}\right)$$

➡ Elliptic integral of the 1st kind: $\mathrm{K}(\lambda) = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-\lambda x^2)}}$

- Bloch & Vanhove: Sunset integral can be expressed in terms of an elliptic dilogarithm:

$$\mathcal{L}_2(z,\tau) \sim \sum_{n=1}^{\infty} \mathrm{Li}_2(z\,q^n) \qquad q = e^{2\pi i\tau}$$

- Also iterated integrals of modular forms. [Adams, Weinzierl]

Outline

- Aim of this talk:
  - ➡ Define a class of elliptic generalisations of poylogarithms.
  - ➡ Show how integrals like $\int_0^1 \frac{dx}{\sqrt{x(x-1)(x-a)}} \log\left(1-\frac{x}{a}\right)$ are related to eMPLs.
- Outline:
  - ➡ Elliptic polylogarithms … or how to integrate on an elliptic curve.
  - ➡ Further applications … or what happens after the sunset.

Elliptic polylogarithms

… or how to integrate on an elliptic curve

Elliptic polylogarithms

- First goal: generalise notion of polylogarithms to elliptic curves.
  - ➡ MPLs = iterated integrals with logarithmic singularities.
- Definition:

Genus 0: $$G(a_1, \ldots, a_n; z) = \int_0^z \frac{dt}{t - a_1} G(a_2, \ldots, a_n; t) \qquad a_i \in \mathbb{C}$$

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Genus 1: $$\widetilde{\Gamma}\left(\begin{smallmatrix} n_1 & \dots & n_k \\ z_1 & \dots & z_k \end{smallmatrix}; z,\tau\right) = \int_0^z dz'\, g^{(n_1)}(z'-z_1,\tau)\,\widetilde{\Gamma}\left(\begin{smallmatrix} n_2 & \dots & n_k \\ z_2 & \dots & z_k \end{smallmatrix}; z',\tau\right) \qquad n_i \in \mathbb{N},\ z_i \in \mathbb{C}$$

[~ Brown, Levin; Brödel, Mafra, Matthes, Schlotterer]

- Eisenstein-Kronecker series:

$$F(z,\alpha,\tau) = \frac{1}{\alpha}\sum_{n\geq 0} g^{(n)}(z,\tau)\,\alpha^n = \frac{\theta_1'(0,\tau)\,\theta_1(z+\alpha,\tau)}{\theta_1(z,\tau)\,\theta_1(\alpha,\tau)}$$

➡ Each $g^{(n)}$ has (at most) simple poles at $z = m + n\tau,\ m,n \in \mathbb{Z}$.

Elliptic polylogarithms

$$\int_0^1 \frac{dx}{\sqrt{x(x-1)(x-a)}} \log\left(1-\frac{x}{a}\right)$$

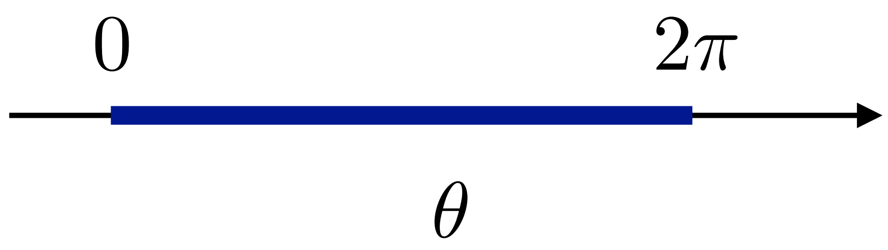
vs.

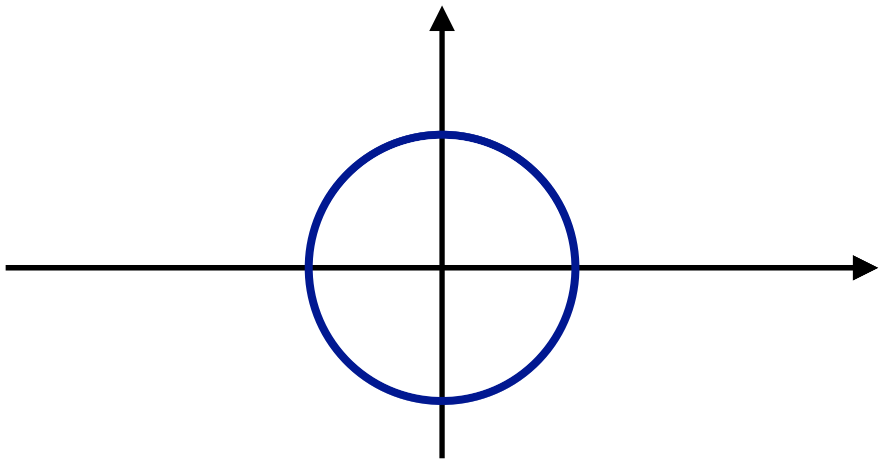
$$\widetilde{\Gamma}\left(\begin{smallmatrix} n_1 & \dots & n_k \\ z_1 & \dots & z_k \end{smallmatrix}; z, \tau\right) = \int_0^z dz'\, g^{(n_1)}(z'-z_1,\tau)\, \widetilde{\Gamma}\left(\begin{smallmatrix} n_2 & \dots & n_k \\ z_2 & \dots & z_k \end{smallmatrix}; z', \tau\right)$$

What is the relation..?

The circle

- How to describe a circle?





(x, y) with $y^2 = 1 - x^2$

The circle

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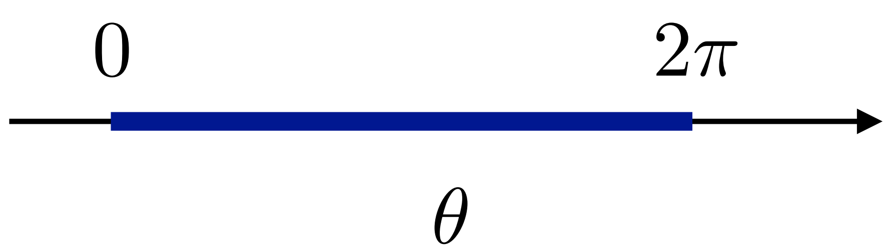
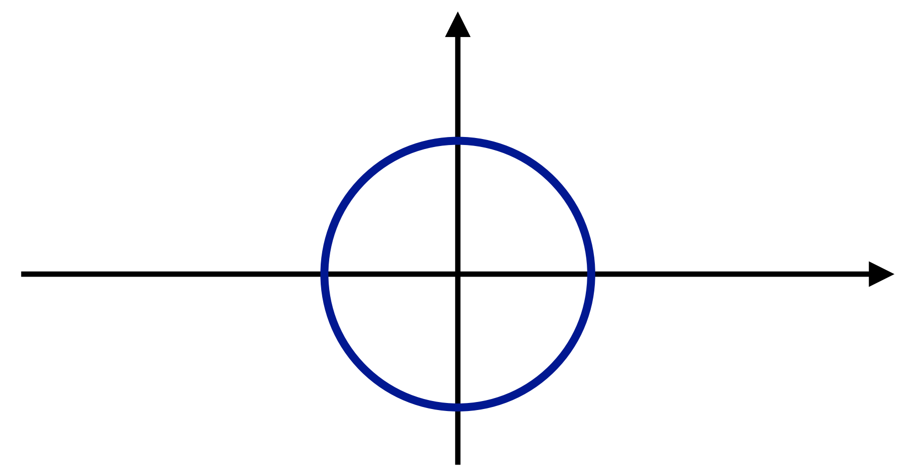




(x, y) with $y^2 = 1 - x^2$

➡ Can rescale 'circumference' to 1.

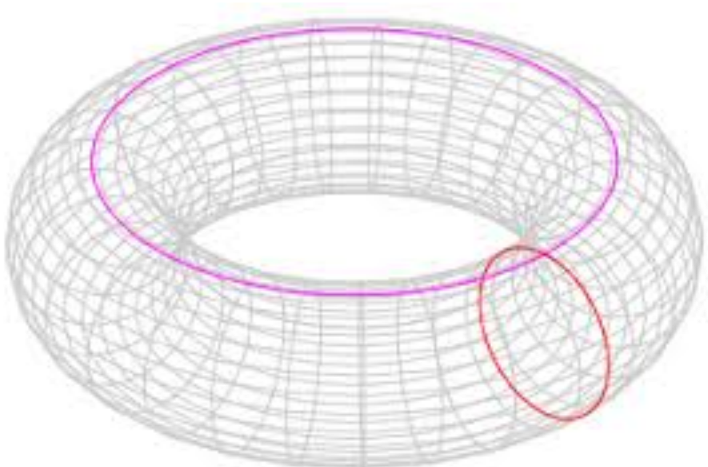
➡ Trigonometric function: $\cos\theta$

$$(\cos'\theta)^2 = 1 - (\cos\theta)^2 \qquad\qquad \cos(\theta + 2\pi) = \cos\theta$$

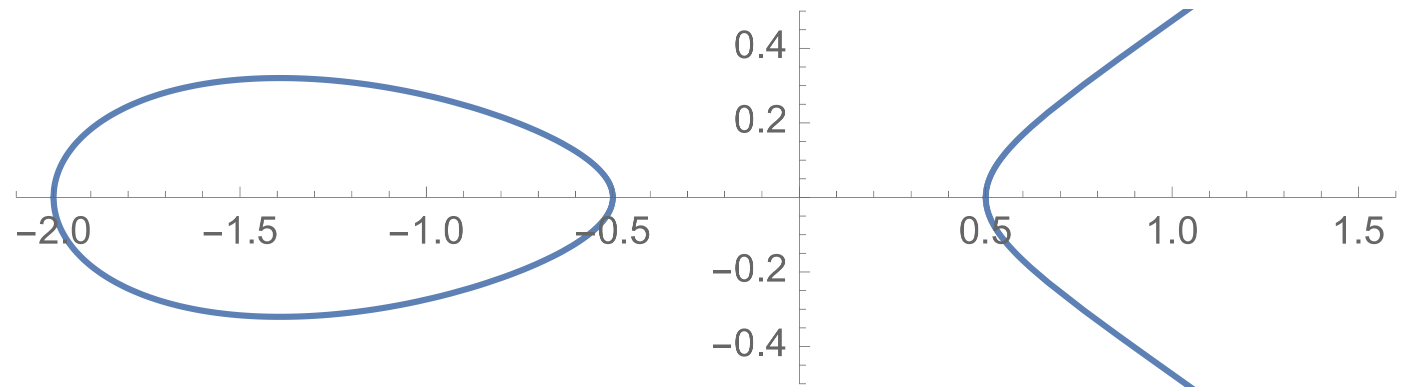
➡ Inverse map: $\theta = -\int_0^x \frac{dx'}{\sqrt{1 - x'^2}}$

Elliptic curves

- Elliptic curves are the same as tori!



z



$[x, y, 1]$ with $y^2 = 4x^3 - g_2 x - g_2$

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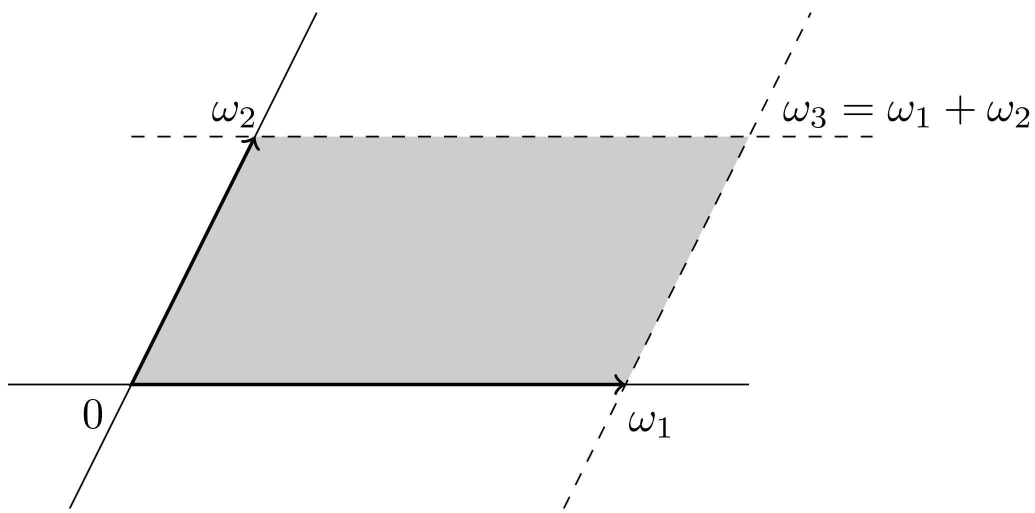

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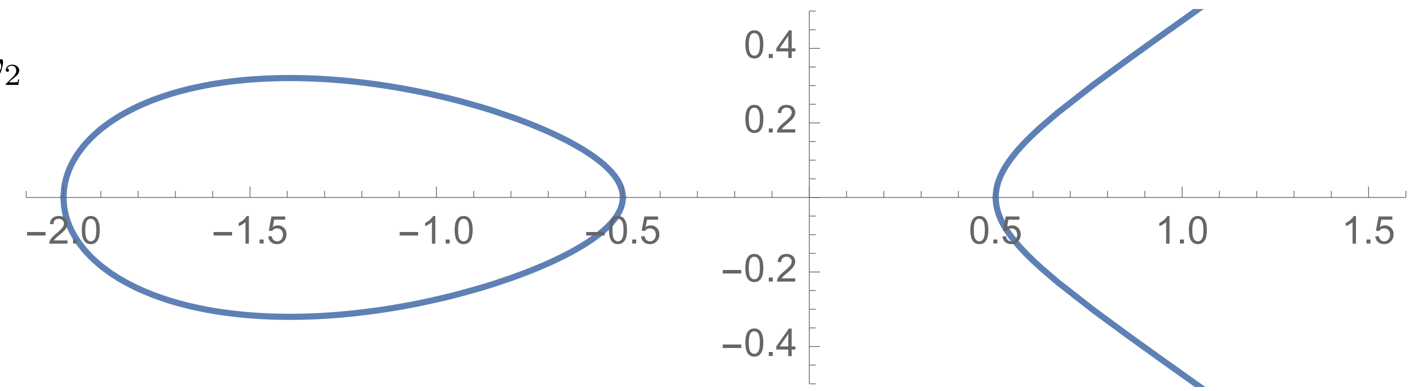

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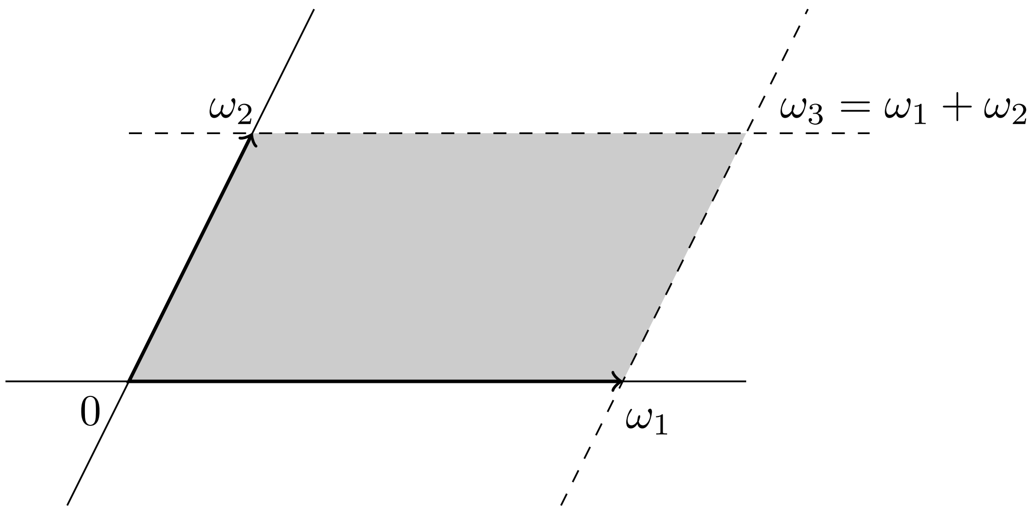
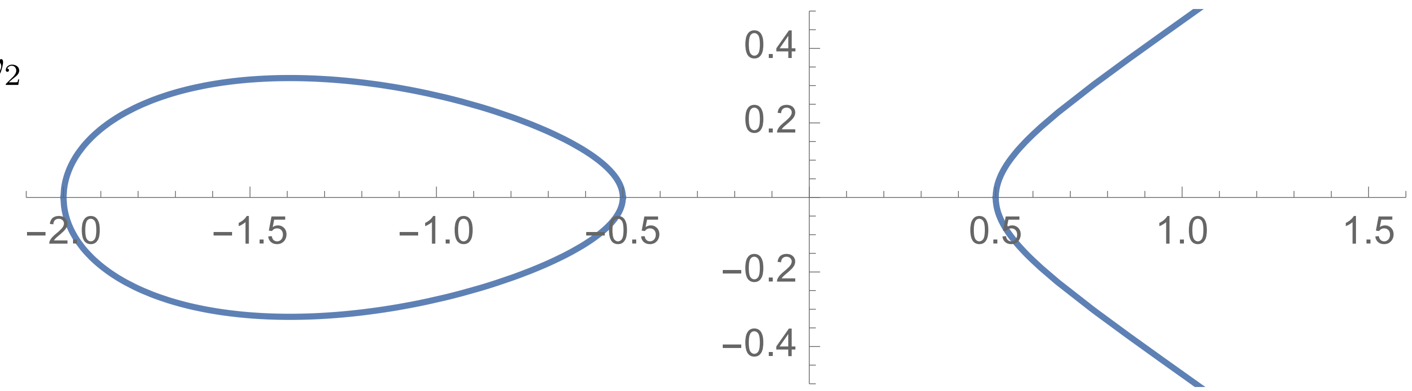




z $\qquad$ $[x, y, 1]$ with $\quad y^2 = 4x^3 - g_2 x - g_2$

➡ Can always rescale one ‘radius’ to 1: $\tau = \omega_2/\omega_1 \quad \text{Im}\,\tau > 0$

➡ Weierstrass $\wp$-function:

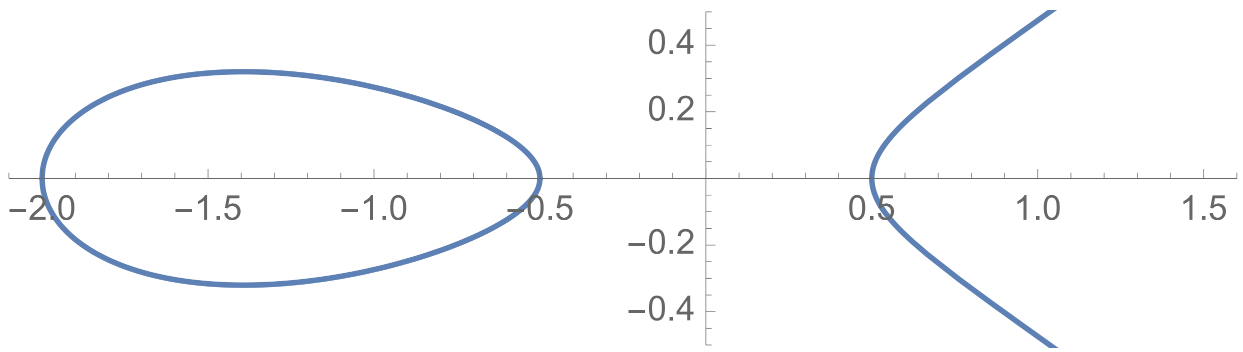
$$\wp(z;\omega_1,\omega_2) = \frac{1}{z^2} + \sum_{(m,n)\neq(0,0)} \left(\frac{1}{(z+m\,\omega_1+n\,\omega_2)^2} - \frac{1}{(m\,\omega_1+n\,\omega_2)^2} \right)$$

$$\wp'^2 = 4\wp^3 - g_2\,\wp - g_3 \qquad\qquad \wp(z+\omega_i;\omega_1,\omega_2) = \wp(z;\omega_1,\omega_2)$$

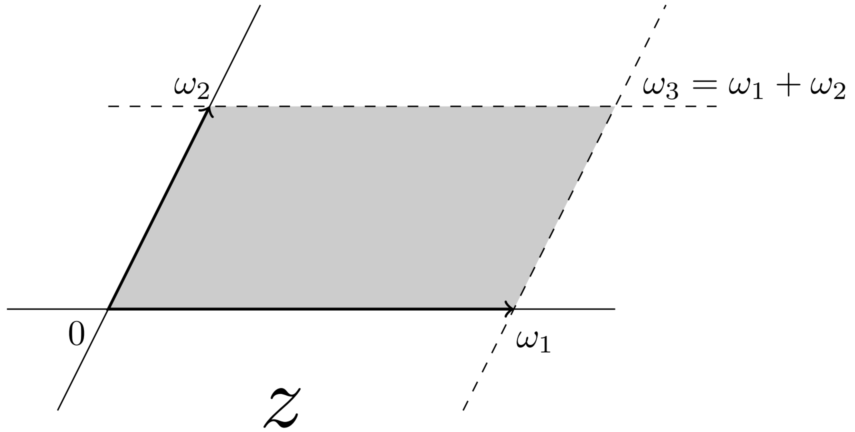
➡ Inverse map: $\quad z = \int_\infty^x \frac{dx'}{\sqrt{4x'^3 - g_2 x' - g_3}}$

Elliptic polylogarithms

- Relation to integrals like $\int_0^1 \frac{dx}{\sqrt{x(x-1)(x-a)}} \log\left(1-\frac{x}{a}\right)$?
  - ➡ They are the same thing! [Brödel, CD, Dulat, Tancredi]



$[x, y, 1]$



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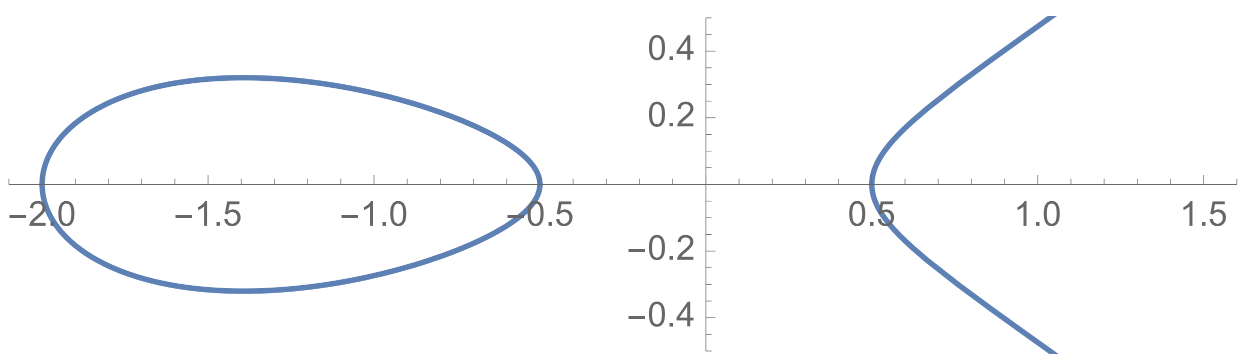

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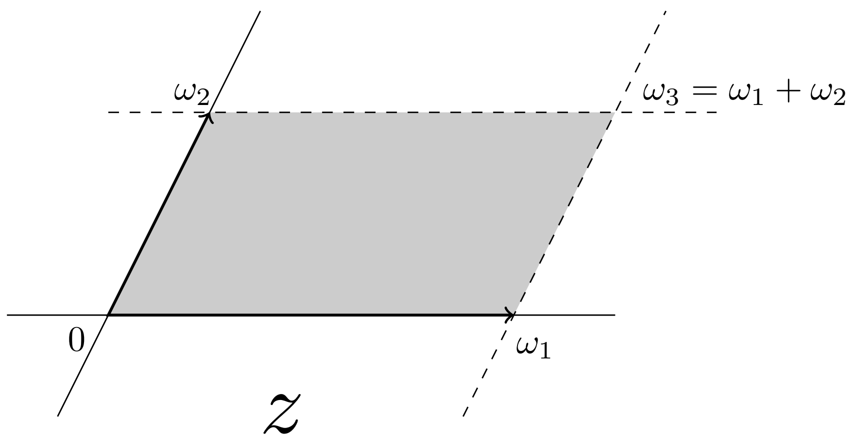

z

$$\frac{dx}{x-c} = dz\left[g^{(1)}(z-z_c) + g^{(1)}(z+z_c) - 2g^{(1)}(z)\right]$$

$$y_c = \sqrt{4c^3 - g_2 c - g_3}$$

$$\frac{y_c\, dx}{y\,(x-c)} = dz\left[g^{(1)}(z-z_c) - g^{(1)}(z+z_c) + 2g^{(1)}(z_c)\right]$$

➡ There is a 1-to-1 map between linearly independent 1-forms in (x, y)-space and z-space.

Various guises of eMPLs

In general (for quartic polynomial with zeroes a_i)

$$E_4\left(\begin{smallmatrix} m_1 & \dots & m_k \\ c_1 & & c_k \end{smallmatrix}; x\right) \qquad m_i \in \mathbb{Z}$$

$$= \int_0^x dx' \, \psi_{m_1}(x, c_1) \, E_4\left(\begin{smallmatrix} m_2 & \dots & m_k \\ c_2 & & c_k \end{smallmatrix}; x\right)$$

$$\widetilde{\Gamma}\left(\begin{smallmatrix} n_1 & \dots & n_k \\ z_1 & \dots & z_k \end{smallmatrix}; z, \tau\right) \qquad n_i \in \mathbb{N} \qquad \tau = \omega_2/\omega_1$$

$$= \int_0^z dz' \, g^{(n_1)}(z' - z_1, \tau) \, \widetilde{\Gamma}\left(\begin{smallmatrix} n_2 & \dots & n_k \\ z_2 & \dots & z_k \end{smallmatrix}; z', \tau\right)$$

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$$a_{ij} = a_i - a_j \qquad c_4 = \frac{1}{2}\sqrt{a_{13}a_{24}}$$

$$dx\, \psi_0(x, c) = \frac{c_4\, dx}{y} \qquad dz\, g^{(0)}(z, \tau) = dz$$

$$dx\, \psi_1(x, c) = \frac{dx}{x - c} \qquad dz\, \Big[g^{(1)}(z - z_c, \tau) + g^{(1)}(z + z_c, \tau) - g^{(1)}(z - z_\infty, \tau) - g^{(1)}(z - z_\infty, \tau)\Big]$$

$$dx\, \psi_{-1}(x, c) = \frac{y_c\, dx}{y(x - c)} \qquad dz\, \Big[g^{(1)}(z - z_c, \tau) - g^{(1)}(z + z_c, \tau) + g^{(1)}(z_c - z_\infty, \tau) + g^{(1)}(z_c - z_\infty, \tau)\Big]$$

etc.

Some properties

- Direct connection between E_4 and eMPLs of Brown and Levin.
- Integration kernels are linearly independent (w.r.t. IBP).
- Space of eMPLs and rational functions on an elliptic curve is closed under taking primitives.
- Ordinary MPLs can be written in terms of eMPLs.

$$G(c_1,\ldots,c_k;x) = E_4\left(\begin{smallmatrix}1 & \cdots & 1\\ c_1 & & c_k\end{smallmatrix};x\right)$$

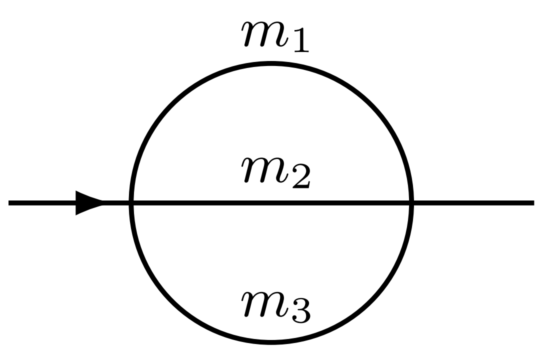
- All classical elliptic integrals are part of eMPLs.

$$\mathrm{K}(\lambda) = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-\lambda x^2)}} = \frac{2}{1+\sqrt{\lambda}}\, E_4\left(\begin{smallmatrix}0\\0\end{smallmatrix};1\right)$$

$$(a_1\ldots a_4) = (-1/\sqrt{\lambda},-1,1,1/\sqrt{\lambda})$$

The sunset

- Sunset integral can be computed in a similar way:



$$= 2\int_{(m_2+m_3)^2}^{\infty} \frac{dx}{\sqrt{R_2(x,m_2^2,m_3^2)R_2(s,x,m_1^2)}}$$

$$\times \log\left(\frac{x+m_1^2-s+\sqrt{R_2(s,x,m_1^2)}}{x+m_1^2-s-\sqrt{R_2(s,x,m_1^2)}}\right)$$

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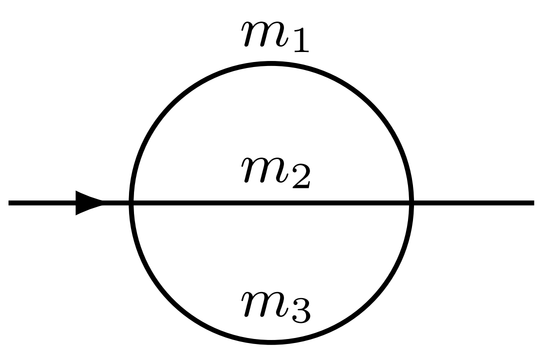

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$$\times \log\left(\frac{x+m_1^2-s+\sqrt{R_2(s,x,m_1^2)}}{x+m_1^2-s-\sqrt{R_2(s,x,m_1^2)}}\right)$$

$$= \frac{4\sqrt{2}}{\sqrt{P_0}}\,F_1^{(1)}(s,m_1^2,m_2^2,m_3^2)$$

$$F_1^{(1)}(s,m_1^2,m_2^2,m_3^2)$$

$$= \mathrm{E}_4\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix};Q_2\right) + \mathrm{E}_4\left(\begin{smallmatrix}0&1\\0&-1/q\end{smallmatrix};Q_2\right) - \mathrm{E}_4\left(\begin{smallmatrix}0&1\\0&-q\end{smallmatrix};Q_2\right) - \ln q\,\mathrm{E}_4\left(\begin{smallmatrix}0\\0\end{smallmatrix};Q_2\right)$$

➡ P_0 and Q_2 are complicated algebraic function of the kinematics.

[Brödel, CD, Dulat, Tancredi]

The sunset

- Can write sunset in terms of $\widetilde{\Gamma}$.
  - ➡ Interesting feature: all arguments have the form $\dfrac{r}{12} + \dfrac{s}{12}\,\tau$.

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- How do iterated integrals of Eisenstein series fit into this picture?

[Adams, Weinzierl]

The sunset

- Can write sunset in terms of $\widetilde{\Gamma}$.
  - ➡ Interesting feature: all arguments have the form $\frac{r}{12}+\frac{s}{12}\tau$.
- How do iterated integrals of Eisenstein series fit into this picture? [Adams, Weinzierl]
- At rational points $\frac{r}{N}+\frac{s}{N}\tau$ eMPLs reduce to iterated integrals of Eisenstein series. [Brödel, CD, Dulat, Penante, Tancredi]
  - ➡ Sunset can be written in terms of iterated integrals of Eisenstein series with $N=12$.
  - ➡ Explains the findings of [Adams, Weinzierl].
  - ➡ Also show that this is very special, and will not be true in general!

Further applications

… or what happens after the sunset?

Some applications

- Some hypergeometric functions give rise to eMPLs. [Brödel, CD, Dulat, Penante, Tancredi]

- Remiddi & Tancredi have introduced a class of 'elliptic generalisations of polylogarithms'.

$$R_4(b,u,m) = b(b-4^2)(b-(\sqrt{u}-m)^2)(b-(\sqrt{u}+m)^2)$$

$$\int_0^{4m^2} \frac{db}{\sqrt{R_4(b,u,m)}}\, G(4m^2;b)$$

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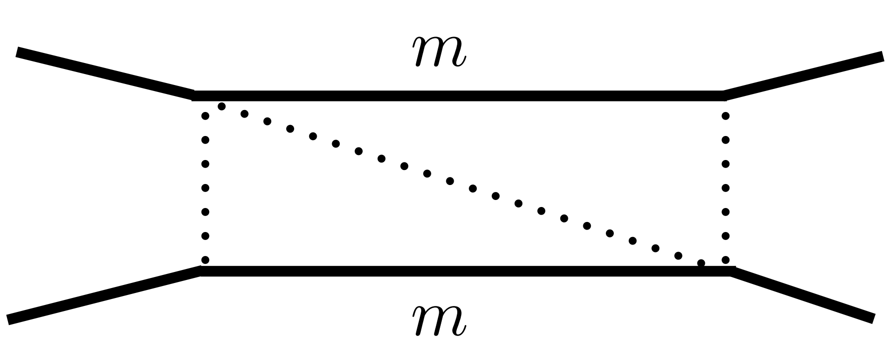
$$\int_0^{4m^2} \frac{db}{\sqrt{R_4(b,u,m)}} G(4m^2;b) = \frac{4\mathrm{K}\left(\frac{(3m-\sqrt{u})(m+\sqrt{u})^3}{(3m+\sqrt{u})(m-\sqrt{u})^3}\right)}{\sqrt{(3m+\sqrt{u})(\sqrt{u}-m)^3}} \left[\widetilde{\Gamma}\left(\begin{smallmatrix}0 & 1\\ 0 & \tau/2\end{smallmatrix};\frac{\tau}{2},\tau\right) + \widetilde{\Gamma}\left(\begin{smallmatrix}0 & 1\\ 0 & -\tau/2\end{smallmatrix};\frac{\tau}{2},\tau\right) - \widetilde{\Gamma}\left(\begin{smallmatrix}0 & 1\\ 0 & \tau/2\end{smallmatrix};\frac{1}{3},\tau\right) - \widetilde{\Gamma}\left(\begin{smallmatrix}0 & 1\\ 0 & -1/3\end{smallmatrix};\frac{\tau}{2},\tau\right)\right]$$

- ➡ Can always be written in terms of eMPLs with arguments of the form $\frac{r}{6} + \tau\frac{s}{6}$.
- ➡ Can always be written in terms of iterated integrals of Eisenstein series.

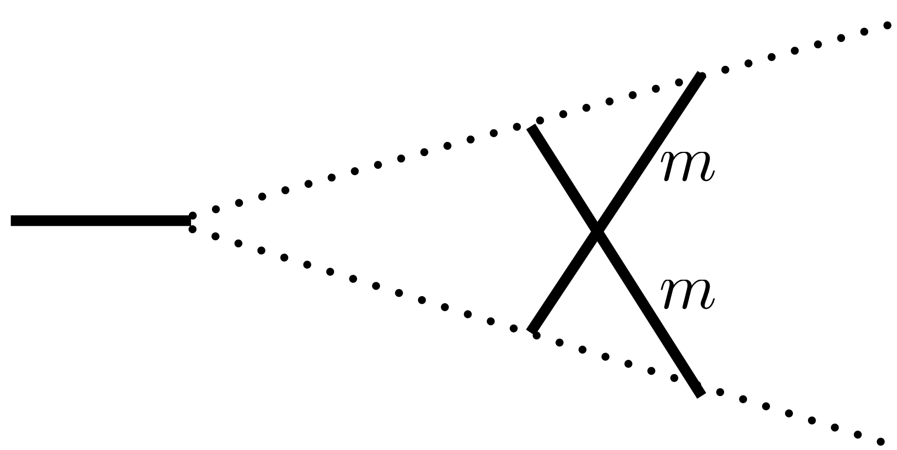
Beyond the sunset

- We are currently looking for applications beyond the sunset graph (and the kite).
  - ➡ A 4-point master integral for Bhabha scattering

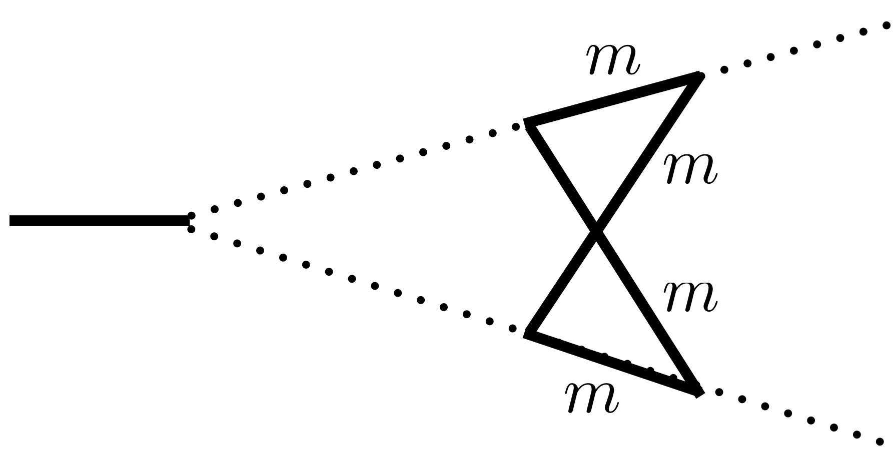


[Henn, Smirnov, Smirnov; Brödel, CD, Dulat, Penante, Tancredi to appear]

  - ➡ Some three-points functions for the electroweak form factor and t-tbar:



[Aglietti, Bonciani, Grassi, Remiddi; Brödel, CD, Dulat, Penante, Tancredi, in progress]



[Tancredi, von Manteuffel; Brödel, CD, Dulat, Penante, Tancredi, in progress]

Bhabha scattering

- The planar two-loop integrals considered by Henn, Smirnov.[2]
  - ➡ Kinematics parametrised by Landau variables:

$$\frac{-s}{m^2} = \frac{(1-x)^2}{x} \qquad \frac{-t}{m^2} = \frac{(1-y)^2}{y}$$

- All integrals but one are expressed in terms of polylogarithms.
  - ➡ Differential equation for remaining integral:

$$d\,f_{11}^{(4)} = g_1\,d\,\log\left(\frac{1-Q}{1+Q}\right) + g_2\,d\,\log\left(\frac{(1+x)+(1-x)Q}{(1+x)-(1-x)Q}\right) + g_3\,d\,\log\left(\frac{(1+y)+(1-y)Q}{(1+y)-(1-y)Q}\right)$$

$$g_1, g_2, g_2 = \text{'simple' polylogarithms}$$

$$g_2 = -\frac{8}{3}G(0,x)^3 - \frac{8\pi^2}{3}G(0,x)$$

$$Q = \sqrt{\frac{(x+y)(1+xy)}{x+y-4xy+x^2y+xy^2}}$$

  - ➡ Solution only known as 'Chen iterated integrals' (equivalent to symbol plus boundary condition).

Bhabha scattering

$$d\log\left(\frac{1-Q}{1+Q}\right) = \frac{dx}{\xi}\left(\frac{1}{x}-x\right)+\frac{dy}{\xi}\left(\frac{1}{y}-y\right)\frac{x}{y} \qquad \xi^2=(x+y)(x+1/y)\left(x^2+xy-4x+x/y+1\right)$$

Rational function on elliptic curve — Defines an elliptic curve

➡ Differential equation can be solved in terms of eMPLs!

➡ N.B.: This does NOT mean that there is no way to write the solution in terms of ordinary polylogarithms!

➡ Solution is a pure function of eMPLs of uniform weight.

$$32\,\widetilde{\Gamma}\left(\begin{smallmatrix}1 & 1 & 1 & 1\\ 1/2 & 1/4 & 3/8 & 1/8\end{smallmatrix};z-1/4,\tau\right)+32\,\widetilde{\Gamma}\left(\begin{smallmatrix}1 & 1 & 1 & 1\\ 1/2 & 1/4 & 3/8 & 3/8\end{smallmatrix};z-1/4,\tau\right)$$

$$+64\,\widetilde{\Gamma}\left(\begin{smallmatrix}1 & 1 & 1 & 1\\ 1/2 & 1/4 & \tau/2 & 1/8\end{smallmatrix};z-1/4,\tau\right)-64\,\widetilde{\Gamma}\left(\begin{smallmatrix}1 & 1 & 1 & 1\\ 1/2 & 1/4 & \tau/2 & 3/8\end{smallmatrix};z-1/4,\tau\right)$$

$$+[\mathcal{O}(700)\text{ more terms}]$$

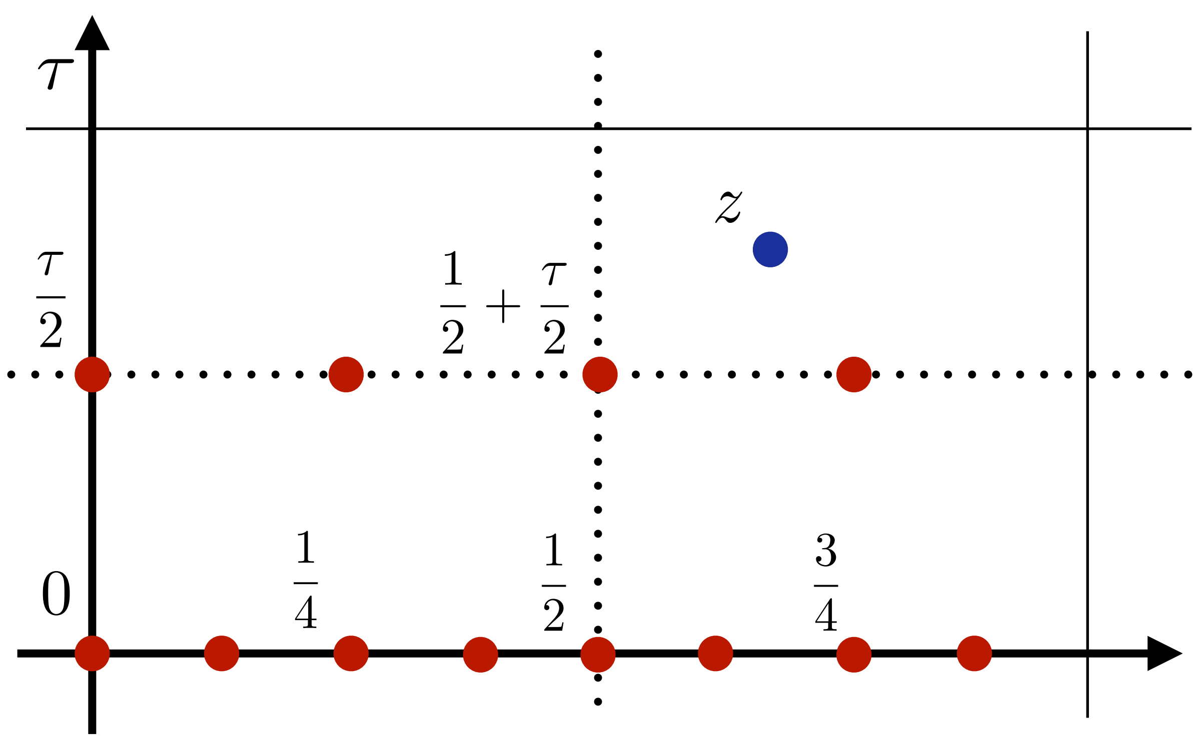
Bhabha scattering

$$32\,\widetilde{\Gamma}\left(\begin{smallmatrix}1 & 1 & 1 & 1\\ 1/2 & 1/4 & 3/8 & 1/8\end{smallmatrix}; z-1/4,\tau\right) + 32\,\widetilde{\Gamma}\left(\begin{smallmatrix}1 & 1 & 1 & 1\\ 1/2 & 1/4 & 3/8 & 3/8\end{smallmatrix}; z-1/4,\tau\right) + \dots$$

- Change of variables: $(x,y) \leftrightarrow (z,\tau)$

$$z = \frac{\sqrt{a_{13}a_{24}}}{4\,\mathrm{K}(\lambda)}\int_{a_1}^{1-x}\frac{d\chi}{\xi} \qquad \tau = i\,\frac{\mathrm{K}(1-\lambda)}{\mathrm{K}(\lambda)} \qquad \lambda \equiv \frac{a_{14}a_{23}}{a_{13}a_{24}}$$

$$\xi^2 = (\chi+y)(\chi+1/y)\left(\chi^2+\chi y-4\chi+\chi/y+1\right) = (\chi-a_1)\dots(\chi-a_4)$$



➡ Singularities all located at

$$z = \frac{r}{8} + \frac{s}{8}\tau$$

Conclusions

- Elliptic polylogarithms are a natural generalisation of ordinary polylogarithms, and share many of their properties:
  - ➡ Shuffle algebra
  - ➡ Closure under integration
  - ➡ Admit a definition of symbols, etc.
- Closely related to other objects, like modular forms.
- Seem to be applicable to a wide range of problems beyond the sunset integral.
- We are just at the beginning of uncovering the mathematics of these objects!

Stay tuned!