Precise Charm and Bottom Quark Masses: an Update

J. H. Kühn

KIT (Karlsruhe)

in collaboration with

K. Chetyrkin, Y. Kiyo, A. Maier, P. Maierhöfer, A. Smirnov,
M. Steinhauser and C. Sturm

NPB 619 (2001) 588
EPJ C48 (2006) 107
NPB 778 (2008) 05413
PLB 669 (2008) 88
NPB 823 (2009) 269
PRD 80 (2009) 074010
NPB 824 (2010) 1
PRD 97 (2018) 056016
⇒ PRD 96 (2017) 116007

I. WHY precise masses?

B-decays:

$$\Gamma\left(B \to X_{\mathrm{u}} l \bar{\nu}\right) \approx G_{\mathrm{F}}^2 m_{\mathrm{b}}^5 \left|V_{\mathrm{ub}}\right|^2$$
$$\Gamma\left(B \to X_{\mathrm{c}} l \bar{\nu}\right) \approx G_{\mathrm{F}}^2 m_{\mathrm{b}}^5 f\left(m_{\mathrm{c}}^2 / m_{\mathrm{b}}^2\right) \left|V_{\mathrm{cb}}\right|^2$$
$$B \to X_{\mathrm{s}} \gamma$$

Υ-spectroscopy:

$$m\left(\Upsilon\left(1\mathrm{S}\right)\right) = 2M_{\mathrm{b}} - \left(\frac{4}{3}\alpha_{\mathrm{S}}\right)^2 \frac{M_{\mathrm{b}}}{4} + \ldots$$

Higgs decay:

$$\Gamma\left(H \to b\bar{b}\right) = \frac{G_{\mathrm{F}} M_{\mathrm{H}}}{4\sqrt{2}\pi} m_{\mathrm{b}}^2 \widetilde{R}$$

$$\widetilde{R} = 1 + 5.6667 a_{\mathrm{S}} + 29.147 a_{\mathrm{S}}^2 + 41.758 a_{\mathrm{S}}^3 - 825.7 a_{\mathrm{S}}^4 \quad \left(a_{\mathrm{S}} \equiv \frac{\alpha_{\mathrm{S}}}{\pi}\right)$$

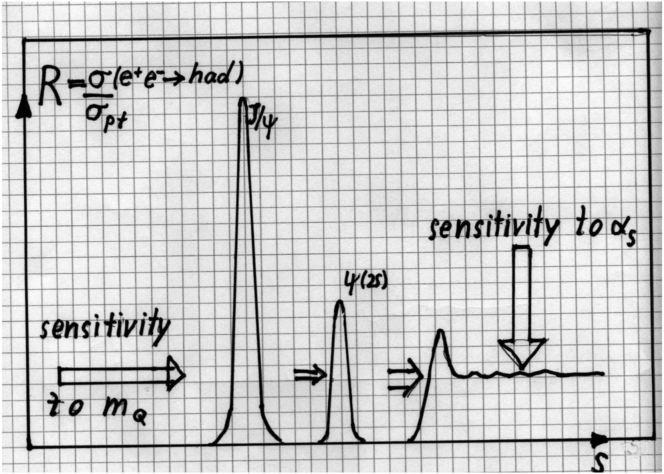
$$= 1 + 0.2062 + 0.0386 + 0.0020 - 0.00145$$

a_{S}^4-term = 5-loop calculation [Baikov, Chetyrkin, JK]
and running of $m_b(M_H)$, as determined from $m_b(10\text{ GeV})$
Theory uncertainty $(M_H/3 < \mu < 3M_H)$ reduced from $5 \cdot 10^{-4}$ to $1.5 \cdot 10^{-4}$
similarly for m_c

Yukawa Unification:

$\lambda_\tau \sim \lambda_{\mathrm{b}}$ or $\lambda_\tau \sim \lambda_{\mathrm{b}} \sim \lambda_{\mathrm{t}}$ at GUT scale
top-bottom $\to m_{\mathrm{t}}/m_{\mathrm{b}} \sim$ ratio of vacuum expectation values
request $\frac{\delta m_{\mathrm{b}}}{m_{\mathrm{b}}} \sim \frac{\delta m_{\mathrm{t}}}{m_{\mathrm{t}}} \Rightarrow \delta m_{\mathrm{t}} \approx 1\ \mathrm{GeV} \Rightarrow \delta m_{\mathrm{b}} \approx 25\ \mathrm{MeV}$
precise quark masses crucial

II. m_Q from SVZ Sum Rules, Moments and Tadpoles



Some definitions:

$$\left(-q^2 g_{\mu\nu} + q_\mu q_\nu\right) \Pi(q^2) \equiv i \int \mathrm{d}x e^{iqx} \left\langle T j_\mu(x) j_\nu(0) \right\rangle$$

with the electromagnetic current j_μ.

$$R(s) = 12\pi \mathrm{Im}\left[\Pi(q^2 = s + i\epsilon)\right]$$

Taylor expansion:

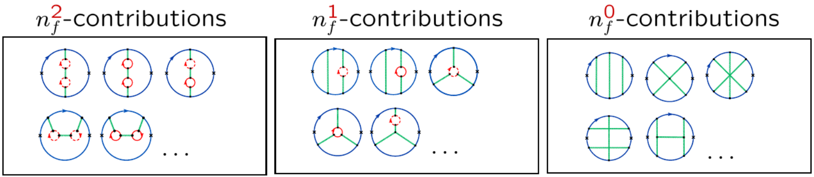
$$\Pi_Q(q^2) = Q_Q^2 \frac{3}{16\pi^2} \sum_{n \geq 0} \overline{C}_n z^n$$

with $z = q^2/(4m_Q^2)$ and $m_Q = m_Q(\mu)$ the $\overline{\mathrm{MS}}$ mass.

$$\overline{C}_n = \overline{C}_n^{(0)} + \frac{\alpha_s}{\pi} \overline{C}_n^{(1)} + \left(\frac{\alpha_s}{\pi}\right)^2 \overline{C}_n^{(2)} + \left(\frac{\alpha_s}{\pi}\right)^3 \overline{C}_n^{(3)}$$

Analysis in N^3LO

Algebraic reduction to 13 master integrals (Laporta algorithm); numerical and analytical evaluation of master integrals



: heavy quarks, : light quarks,

n_f: number of active quarks

$\Rightarrow$ About 700 Feynman-diagrams

$\overline{C}_0$ and $\overline{C}_1$ in order α_s^3 (four loops!) (2006)

$\Rightarrow$ Reduction to master integrals
(Chetyrkin, JK, Sturm; Boughezal, Czakon, Schutzmeier)

$\Rightarrow$ Evaluation of master integrals numerically
or analytically in terms of transcendentals.
All master integrals known analytically and double checked.
(Schröder + Vuorinen, Chetyrkin et al., Schröder + Steinhauser, Laporta, Broadhurst, Kniehl et al.)

$\Rightarrow \overline{C}_2, \overline{C}_3$
(Maier, Maierhöfer, Marquard, A. Smirnov, 2008)

$\Rightarrow \overline{C}_4 - \overline{C}_{10}$: extension to higher moments by Padé method, using analytic information from low energy ($q^2 = 0$), threshold ($q^2 = 4m^2$), high energy ($q^2 = -\infty$) (Kiyo, Maier, Maierhöfer, Marquard 2009)

New developments

$\Rightarrow \overline{C}_4$ in analytic form (Maier and Marquard 2017)
$\Rightarrow$ new data for charm (Chetyrkin, JK, Maier, Maierhöfer, Marquard, Steinhauser, Sturm 2017)

Relation to measurements:

$$\mathcal{M}_n^{\rm th} \equiv \frac{12\pi^2}{n!}\left(\frac{\mathrm{d}}{\mathrm{d}q^2}\right)^n \Pi_c(q^2)\Big|_{q^2=0} = \frac{9}{4}Q_c^2\left(\frac{1}{4m_c^2}\right)^n \overline{C}_n$$

Perturbation theory: $\overline{C}_n$ is function of α_s and $\ln \frac{m_c^2}{\mu^2}$
dispersion relation:

$$\Pi_c(q^2) = \frac{q^2}{12\pi^2}\int \mathrm{d}s \frac{R_c(s)}{s(s-q^2)} + \text{subtraction}$$

$$\Rightarrow \mathcal{M}_n^{\rm exp} = \int \frac{\mathrm{d}s}{s^{n+1}} R_c(s)$$

constraint: $\mathcal{M}_n^{\rm exp} = \mathcal{M}_n^{\rm th}$

$\Rightarrow m_c$

Ingredients (charm)

KIT Karlsruhe Institute of Technology

experiment:

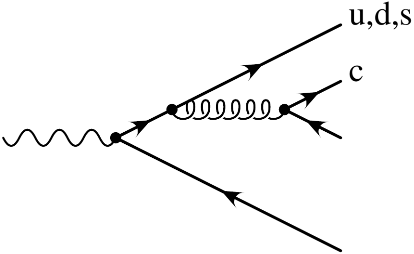
- $\Gamma_c(J/\Psi, \Psi')$ from BES & CLEO & BABAR
- $\Psi(3770)$ and $R(s)$ from BES
- $\alpha_s = 0.1181 \pm 0.0011$

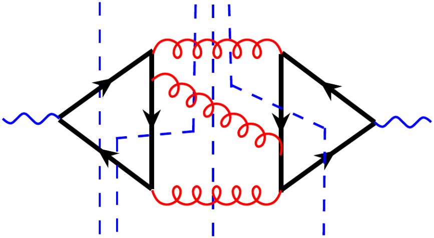
theory:

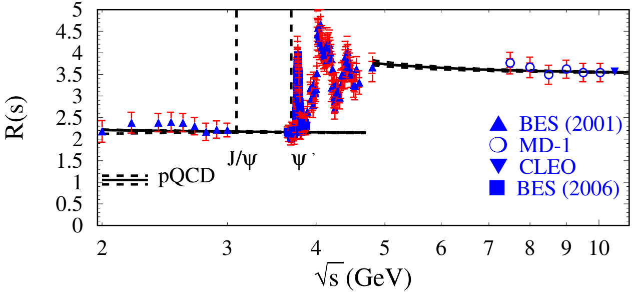
- N^3LO for $n = 1, 2, 3, 4$
- include condensates

$$\delta\mathcal{M}_n^{\mathrm{np}} = \frac{12\pi^2 Q_c^2}{(4m_c^2)^{(n+2)}} \left\langle \frac{\alpha_s}{\pi} G^2 \right\rangle a_n \left(1 + \frac{\alpha_s}{\pi} \bar{b}_n\right)$$

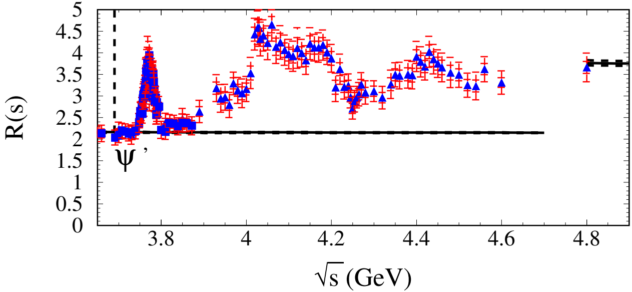
- estimate of non-perturbative terms (oscillations, based on Shifman)
- careful extrapolation of R_{uds}
- careful definition of R_c


u,d,s
c





R(s)
J/ψ
ψ'
pQCD
BES (2001)
MD-1
CLEO
BES (2006)
√s (GeV)



R(s)
ψ'
√s (GeV)

Contributions from

- narrow resonances: $R = \frac{9\Pi M_R \Gamma_e}{\alpha^2(s)} \delta(s - M_R^2)$ (PDG)
- threshold region ($2m_D - 4.8$ GeV) (BESS)
- perturvative continuum ($E \geq 4.8$ GeV) (Theory)

n	$\mathcal{M}_n^{\text{res}}$ $\times 10^{(n-1)}$	$\mathcal{M}_n^{\text{thresh}}$ $\times 10^{(n-1)}$	$\mathcal{M}_n^{\text{cont}}$ $\times 10^{(n-1)}$	$\mathcal{M}_n^{\text{exp}}$ $\times 10^{(n-1)}$	$\mathcal{M}_n^{\text{np}}$ $\times 10^{(n-1)}$
1	0.1191(14)	0.0318(15)	0.0646(11)	0.2154(23)	−0.0001(3)
2	0.1169(15)	0.0178(8)	0.0144(3)	0.1490(17)	−0.0002(5)
3	0.1165(15)	0.0101(5)	0.0042(1)	0.1308(16)	−0.0004(8)
4	0.1176(16)	0.0058(3)	0.0014(0)	0.1248(16)	−0.0006(12)

Different relative importance of resonances vs. continuum for $n = 1, 2, 3, 4$.

Results (m_c)

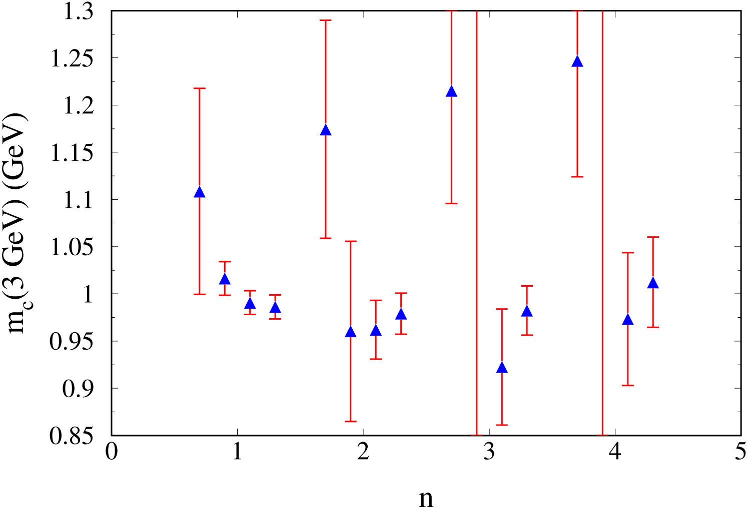
PRD 96 (2017) 116007

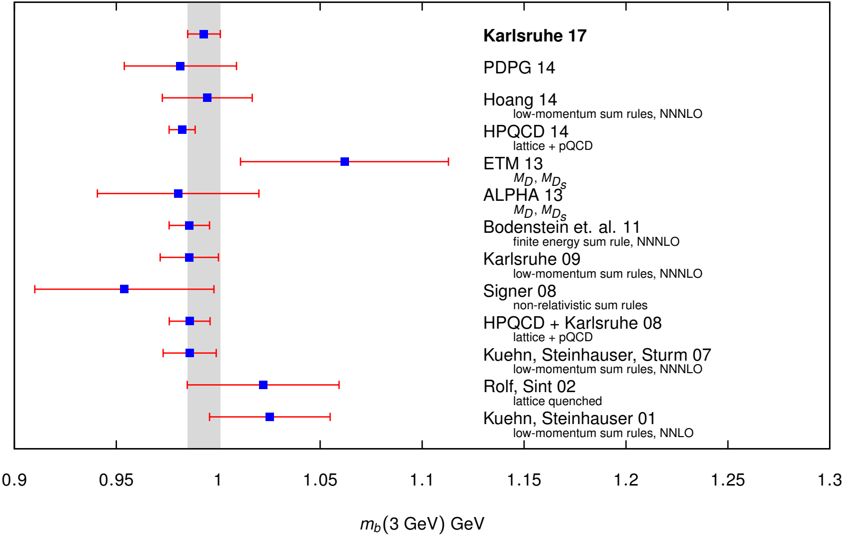
n	$m_c(3\text{ GeV})$	exp	α_s	μ	np	total
1	993	7	4	2	1	8
2	982	4	7	5	1	10
3	982	3	8	6	1	10
4	1003	2	5	28	1	29

Remarkable consistency between $n = 1, 2, 3, 4$
and stability ($\mathcal{O}(\alpha_s^2)$ vs. $\mathcal{O}(\alpha_s^3)$);
prefered scale: $\mu = 3$ GeV,

conversion to $m_c(m_c)$:

- $m_c(3\text{ GeV}) = 993 \pm 8$ MeV
- $m_c(m_c) = 1279 \pm 8$ MeV


1.3
1.25
1.2
1.15
1.1
1.05
1
0.95
0.9
0.85
0
1
2
3
4
5
m_c(3 GeV) (GeV)
n

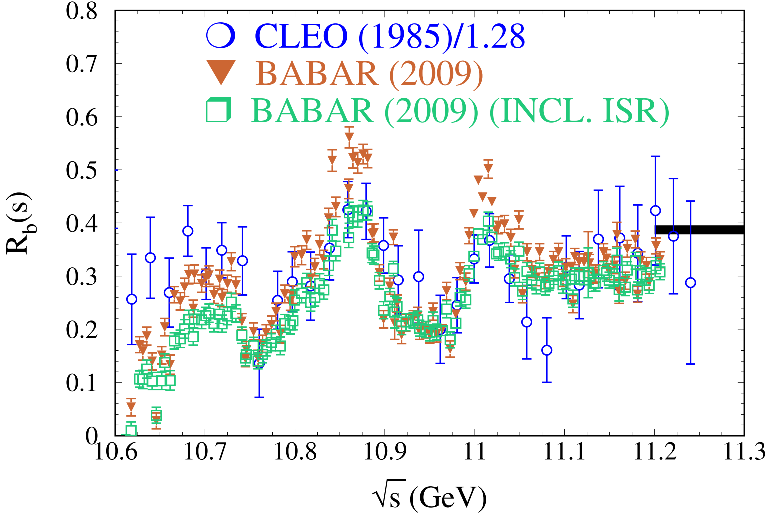

Karlsruhe 17
PDPG 14
Hoang 14
low-momentum sum rules, NNNLO
HPQCD 14
lattice + pQCD
ETM 13
M_D, M_{D_s}
ALPHA 13
M_D, M_{D_s}
Bodenstein et. al. 11
finite energy sum rule, NNNLO
Karlsruhe 09
low-momentum sum rules, NNNLO
Signer 08
non-relativistic sum rules
HPQCD + Karlsruhe 08
lattice + pQCD
Kuehn, Steinhauser, Sturm 07
low-momentum sum rules, NNNLO
Rolf, Sint 02
lattice quenched
Kuehn, Steinhauser 01
low-momentum sum rules, NNLO
0.9
0.95
1
1.05
1.1
1.15
1.2
1.25
1.3
m_b(3 GeV) GeV

Experimental Ingredients for m_b

Contributions from

- narrow resonances ($\Upsilon(1S) - \Upsilon(4S)$) (PDG)
- threshold region (10.618 GeV - 11.2 GeV) (BABAR 2009)
- perturvative continuum ($E \geq 11.2$ GeV) (Theory)
- different relative importance of resonances vs. continuum for $n = 1, 2, 3, 4$

n	$\mathcal{M}_n^{\mathrm{res},(1S-4S)}$ $\times 10^{(2n+1)}$	$\mathcal{M}_n^{\mathrm{thresh}}$ $\times 10^{(2n+1)}$	$\mathcal{M}_n^{\mathrm{cont}}$ $\times 10^{(2n+1)}$	$\mathcal{M}_n^{\mathrm{exp}}$ $\times 10^{(2n+1)}$
1	1.394(23)	0.287(12)	2.911(18)	4.592(31)
2	1.459(23)	0.240(10)	1.173(11)	2.872(28)
3	1.538(24)	0.200(8)	0.624(7)	2.362(26)
4	1.630(25)	0.168(7)	0.372(5)	2.170(26)



CLEO (1985)/1.28
BABAR (2009)
BABAR (2009) (INCL. ISR)
R_b(s)
√s (GeV)
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
10.6
10.7
10.8
10.9
11
11.1
11.2
11.3

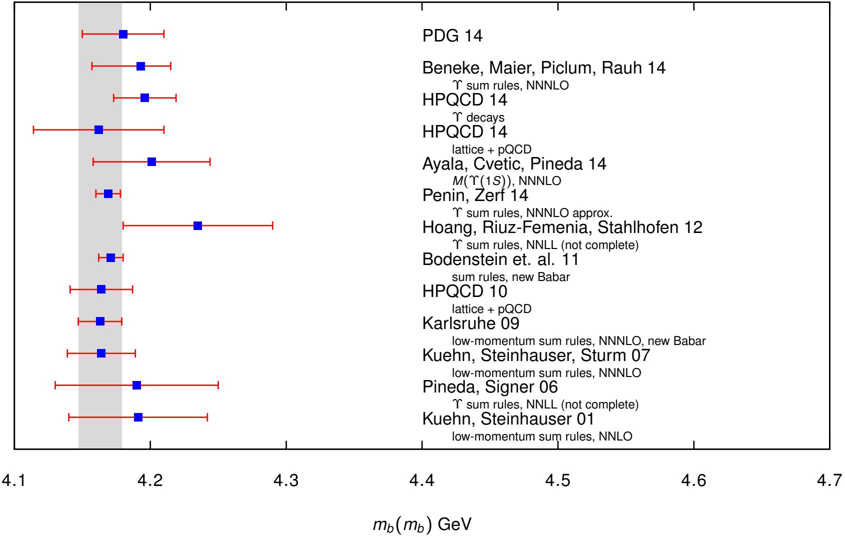
Results (m_b)

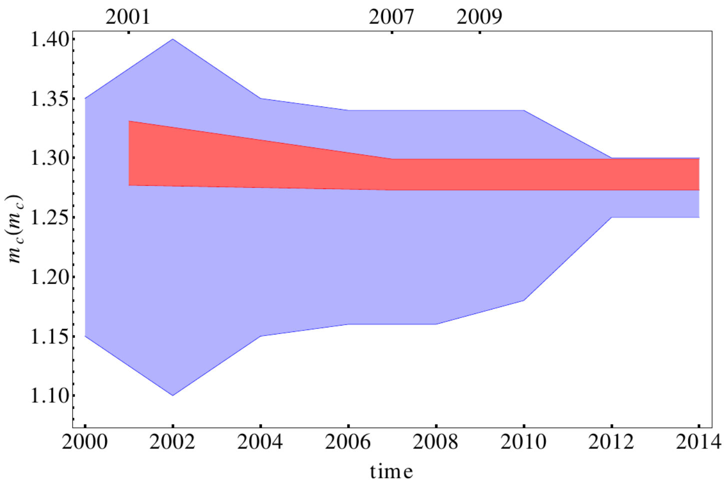
n	$m_b(10\ \text{GeV})$	exp	α_s	μ	total	$m_b(m_b)$
1	3597	14	7	2	16	4151
2	3610	10	12	3	16	4163
3	3619	8	14	6	18	4172
4	3631	6	15	20	26	4183

Consistency ($n = 1, 2, 3, 4$) and stability ($\mathcal{O}(\alpha_s^2)$ vs. $\mathcal{O}(\alpha_s^3)$);
(slight dependence on n could result from input into $\mathcal{M}^n_{\text{exp}}$)

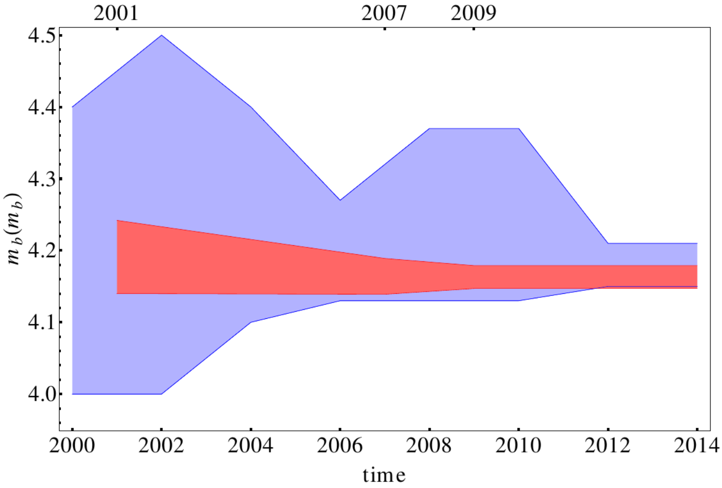
- $m_b(10\ \text{GeV}) = 3610 \pm 16\ \text{MeV}$
- $m_b(m_b) = 4163 \pm 16\ \text{MeV}$

well consistent with KSS 2007


PDG 14
Beneke, Maier, Piclum, Rauh 14
ϒ sum rules, NNNLO
HPQCD 14
ϒ decays
HPQCD 14
lattice + pQCD
Ayala, Cvetic, Pineda 14
M(ϒ(1S)), NNNLO
Penin, Zerf 14
ϒ sum rules, NNNLO approx.
Hoang, Riuz-Femenia, Stahlhofen 12
ϒ sum rules, NNLL (not complete)
Bodenstein et. al. 11
sum rules, new Babar
HPQCD 10
lattice + pQCD
Karlsruhe 09
low-momentum sum rules, NNNLO, new Babar
Kuehn, Steinhauser, Sturm 07
low-momentum sum rules, NNNLO
Pineda, Signer 06
ϒ sum rules, NNLL (not complete)
Kuehn, Steinhauser 01
low-momentum sum rules, NNLO
4.1
4.2
4.3
4.4
4.5
4.6
4.7
$m_b(m_b)$ GeV



2001
2007
2009
1.40
1.35
1.30
1.25
1.20
1.15
1.10
$m_c(m_c)$
2000
2002
2004
2006
2008
2010
2012
2014
time



2001
2007
2009
4.5
4.4
4.3
4.2
4.1
4.0
$m_b(m_b)$
2000
2002
2004
2006
2008
2010
2012
2014
time

α_s-dependence

$$m_c(3\ \mathrm{GeV}) = \left(993 - \frac{\alpha_s - 0.1181}{0.001} \cdot 5 \pm 16\right) \mathrm{MeV}$$

$$m_b(10\ \mathrm{GeV}) = \left(3610 - \frac{\alpha_s - 0.1189}{0.002} \cdot 12 \pm 11\right) \mathrm{MeV}$$

$$m_b(m_b) = \left(4163 - \frac{\alpha_s - 0.1189}{0.002} \cdot 12 \pm 11\right) \mathrm{MeV}$$

$$m_b(M_Z) = \left(2835 - \frac{\alpha_s - 0.1189}{0.002} \cdot 27 \pm 8\right) \mathrm{MeV}$$

$$m_b(161.8\ \mathrm{GeV}) = \left(2703 - \frac{\alpha_s - 0.1189}{0.002} \cdot 28 \pm 8\right) \mathrm{MeV}$$

Summary

$$m_c(3\ \mathrm{GeV}) = 993(8)\mathrm{MeV}$$

e^+e^- + pQCD

$$m_b(10\ \mathrm{GeV}) = 3610(16)\ \mathrm{MeV}$$
$$m_b(m_b) = 4163(16)\ \mathrm{MeV}$$

e^+e^- + pQCD

in excellent agreement with lattice results
nicely consistent with

HPQCD

$$m_c(3\ \mathrm{GeV}) = 985(6)\mathrm{MeV}$$
$$m_b(m_b) = 4162(48)\ \mathrm{MeV}$$

Fermilab Lattice, MILC & TUMQCD

$$m_c(3\ \mathrm{GeV}) = 984(6)\mathrm{MeV}$$
$$m_b(m_b) = 4197(14)\ \mathrm{MeV}$$