Evaluating 'elliptic' master integrals at special kinematic values: using differential equations and their solutions via expansions near singular points

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It is a sequel of [R. Lee, A. Smirnov & V.S.'17]:

an algorithm to find a solution of differential equations for master integrals in the form of an ϵ-expansion series with numerical coefficients.

The algorithm is based on using generalized power series expansions near singular points of the differential system, solving difference equations for the corresponding coefficients in these expansions and using matching to connect series expansions at two neighbouring points.

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The ε-form is not always possible. The simplest counter example is the two-loop sunset diagram with three equal non-zero masses.

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E. Remiddi & L. Tancredi’17; M. Hidding & F. Moriello’17;
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Still we are far, even in lower loops orders, from answering the following question:
'What is the class of functions which can appear in results for Feynman integrals in situations where ϵ-form is impossible'?

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- Perspectives

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We imply that all the singular points of DE are regular, i.e. we can reduce the DE to a local Fuchsian form at any singular point, i.e. if x_i is a singular point then

$$M(x) = \frac{A_i(x)}{x - x_i}$$

where $A_i(x)$ is regular at $x = x_i$ and $A_i(x_i) \neq 0$.

General solution

$$\boldsymbol{J}(x) = U(x)\,\boldsymbol{C}\,,$$

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where $\boldsymbol{C}$ is a column of constants, and U is an evolution operator

$$U(x) = P\exp\left[\int M(x)\,dx\right]\,.$$

Expanding in a vicinity of each singular point.

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The expansion is

$$U(x) = \sum_{\lambda \in S} x^{\lambda} \sum_{n=0}^{\infty} \sum_{k=0}^{K_{\lambda}} \frac{1}{k!} C(n+\lambda, k)\, x^{n} \ln^{k} x\,,$$

where S is a finite set of powers of the form $\lambda = r\epsilon$ with integer r, $K_{\lambda} \geqslant 0$ is an integer number corresponding to the the maximal power of the logarithm.

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where S is a finite set of powers of the form $\lambda = r\epsilon$ with integer r, $K_\lambda \geqslant 0$ is an integer number corresponding to the the maximal power of the logarithm.
The goal is to determine S, K_λ, and the matrix coefficients $C(n+\lambda, k)$.

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In particular, the ‘elliptic’ cases, as a rule, can algorithmically be reduced to a global normalized Fuchsian form using, e.g., the algorithm of Lee [R.N. Lee’14].

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Boundary conditions are included at one of the singular points and then series expansions at other points can be obtained by matching, step by step, pairs of expansions at neighboring points.

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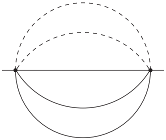
[X. Liu, Y.Q. Ma & C.Y. Wang'17]
(solving DE wrt η in propagators $1/(k^2 + i0) \to 1/(k^2 + i\eta)$)

Feynman integrals corresponding to the generalized sunset graph with two massless and three massive lines



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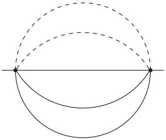

$$F_{a_1,\ldots,a_{14}} = \int \ldots \int \frac{\mathrm{d}^D k_1 \ldots \mathrm{d}^D k_4 \; (k_1 \cdot p)^{a_6} (k_2 \cdot p)^{a_7} (k_3 \cdot p)^{a_8} (k_4 \cdot p)^{a_9}}{(-k_1^2)^{a_1} (-k_2^2)^{a_2} (m^2 - k_3^2)^{a_3} (m^2 - k_4^2)^{a_4} (m^2 - (\sum k_i + p)^2)^{a_5}} \times (k_1 \cdot k_2)^{a_{10}} (k_1 \cdot k_3)^{a_{11}} (k_1 \cdot k_4)^{a_{12}} (k_2 \cdot k_3)^{a_{13}} (k_2 \cdot k_4)^{a_{14}} \,,$$

with $x = p^2/m^2$.

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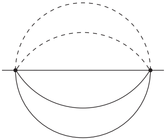

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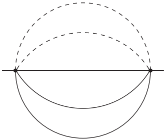

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There are four master integrals in this family. We choose

$$\{F_{1,1,1,1,1,0,\ldots,0},\; F_{1,1,2,1,1,0,\ldots,0},\; F_{1,2,1,1,1,0,\ldots,0},\; F_{1,2,1,1,2,0,\ldots,0}\} \, .$$

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The goal: to evaluate master integrals considered at threshold, $p^2 = 9m^2$,

$$\{J_1 = F_{1,1,1,1,1,0,\ldots,0},\ J_2 = F_{1,1,2,1,1,0,\ldots,0},\ J_3 = F_{1,2,1,1,1,0,\ldots,0}\}\,.$$

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The corresponding expansion is a large-momentum expansion [K.G. Chetyrkin'88, V.S.'90] where every term is a product of one-loop tadpoles and massless propagator integrals. It provides any required accuracy and any required number of terms in ε-expansions in the boundary conditions.

```
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```

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Using `DESS` we obtain numerical results for the threshold master integrals in an ε-expansion up to ε^2 with the accuracy of 6000 digits for the corresponding coefficients.
(Less than four hours on a desktop.)

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Apply the `PSLQ` algorithm
[H.R.P. Ferguson, D.H. Bailey & S. Arno'99]

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Results for the two-loop sunset diagram at threshold [F.A. Berends & A.I. Davydychev'97, A.I. Davydychev & V.S.'99]:
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Let us use multiple polylogarithm values at sixth roots of unity constructed up to weight 6
[J.M. Henn, A.V. Smirnov & V.S.'17] and $\sqrt{3}$.

$G(a_1, \ldots, a_w; 1)$,
where the indices a_i are equal to zero or a sixth root of unity, i.e. taken from the alphabet $\{0, r_1, r_3, -1, r_4, r_2, 1\}$ with

$$r_{1,2} = \frac{1}{2}\left(1 \pm \sqrt{3}\,\mathrm{i}\right) = \lambda^{\pm 1}\,, \quad r_{3,4} = \frac{1}{2}\left(-1 \pm \sqrt{3}\,\mathrm{i}\right) = \lambda^{\pm 2}\,,$$

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$$G(a_1, \ldots, a_w; z) = \int_0^z \frac{1}{t - a_1}\, G(a_2, \ldots, a_w; t)\, \mathrm{d}t$$

with $a_i, z \in \mathbb{C}$ and $G(z) = 1$.

$$G(0, \ldots, 0; z) = \frac{1}{n!} \log^n z\,.$$

$$G(a_1, \ldots, a_w; 1) = G_R(a_1, \ldots, a_w) + \mathrm{i}\, G_I(a_1, \ldots, a_w)$$

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Let us denote by $B_R(w)$ ($B_I(w)$) the bases generated by $G_R(a_1, \ldots, a_w)$ ($G_I(a_1, \ldots, a_w)$).

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[J.M. Henn, A.V. Smirnov & V.S.'17]:
$B_R(1) \cup B_I(1) =$

$$B_R(1) = \left\{ G_R(-1) = \log(2), \;\; G_R(r_4) = \frac{1}{2}\log(3) \right\},$$

$$B_I(1) = \left\{ G_I(r_2) = -\frac{\pi}{3} \right\}.$$

$B_R(2) =$

```
{GR[r2, -1],
 GR[-1]^2, GI[r2]^2, GR[-1] GR[r4], GR[r4]^2}
```

$B_I(2) =$

```
{GI[0, r2],
 GI[r2] GR[-1], GI[r2] GR[r4]}
```

$B_R(3) =$

```
{GR[0, 0, 1], GR[r2, 1, -1], GR[r2, 1, r3],
 GR[-1]^3, GI[r2]^2 GR[-1], GR[-1]^2 GR[r4], GI[r2]^2 GR[r4],
GR[-1] GR[r4]^2,  GR[r4]^3, GI[r2] GI[0, r2], GR[-1] GR[r2, -1],
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```

$B_I(3) =$

```
{GI[0, 1, r4], GI[0, r2, -1],
GI[r2] GR[-1]^2, GI[r2]^3, GI[r2] GR[-1] GR[r4], GI[r2]GR[r4]^2,
GI[0, r2] GR[-1], GI[0, r2] GR[r4], GI[r2] GR[r2, -1]}
```

$B_R(4) =$

```
{GR[0, 0, r2, -1], GR[0, 0, r4, 1], GR[r2, 1, 1, -1],
 GR[r2, 1, 1, r3], GR[r2, 1, r2, -1]}
```

and

```
{GR[-1]^4, GI[r2]^2 GR[-1]^2, GI[r2]^4, GR[-1]^3 GR[r4],
 GI[r2]^2 GR[-1] GR[r4], GR[-1]^2 GR[r4]^2, GI[r2]^2 GR[r4]^2,
 GR[-1] GR[r4]^3, GR[r4]^4, GI[r2] GI[0, r2] GR[-1],
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 GR[r2, -1]^2, GR[-1] GR[0, 0, 1], GR[r4] GR[0, 0, 1],
 GI[r2] GI[0, 1, r4], GI[r2] GI[0, r2, -1], GR[-1] GR[r2, 1, -1],
 GR[r4] GR[r2, 1, -1], GR[-1] GR[r2, 1, r3], GR[r4] GR[r2, 1, r3]}
```

$B_l(4) =$

```
{GI[0, 0, 0, r2], GI[0, 1, 1, r4], GI[0, 1, r2, -1], GI[0, 1, r2, r3],
  GI[0, r2, 1, -1]}
```

and

```
{GI[r2] GR[-1]^3, GI[r2]^3 GR[-1], GI[r2] GR[-1]^2 GR[r4],
 GI[r2]^3 GR[r4], GI[r2] GR[-1] GR[r4]^2, GI[r2] GR[r4]^3,
 GI[0, r2] GR[-1]^2, GI[r2]^2 GI[0, r2], GI[0, r2] GR[-1] GR[r4],
 GI[0, r2] GR[r4]^2, GI[r2] GR[-1] GR[r2, -1],
 GI[r2] GR[r4] GR[r2, -1], GI[0, r2] GR[r2, -1], GI[r2] GR[0, 0, 1],
 GI[0, 1, r4] GR[-1], GI[0, 1, r4] GR[r4], GI[0, r2, -1] GR[-1],
 GI[0, r2, -1] GR[r4], GI[r2] GR[r2, 1, -1], GI[r2] GR[r2, 1, r3]}
```

$B_R(5) =$

```
{GR[0, 0, 0, 0, 1], GR[0, 0, 1, 1, -1], GR[0, 0, 1, 1, r4],
 GR[0, 0, 1, r2, -1], GR[0, 0, 1, r2, r3], GR[0, 0, 1, r2, r4],
 GR[0, 0, r2, 1, -1], GR[r2, 1, 1, -1, r2], GR[r2, 1, 1, 1, -1],
 GR[r2, 1, 1, 1, r3], GR[r2, 1, 1, r2, -1], GR[r2, 1, 1, r2, r3],
 GR[r2, 1, 1, r4, -1]}
```

and

```
{GR[-1]^5, GI[r2]^2 GR[-1]^3, GI[r2]^4 GR[-1], GR[-1]^4 GR[r4],
 GI[r2]^2 GR[-1]^2 GR[r4], GI[r2]^4 GR[r4], GR[-1]^3 GR[r4]^2,
 GI[r2]^2 GR[-1] GR[r4]^2, GR[-1]^2 GR[r4]^3, GI[r2]^2 GR[r4]^3,
 GR[-1] GR[r4]^4, GR[r4]^5, GI[r2] GI[0, r2] GR[-1]^2,
 GI[r2]^3 GI[0, r2], GI[r2] GI[0, r2] GR[-1] GR[r4],
 GI[r2] GI[0, r2] GR[r4]^2, GI[0, r2]^2 GR[-1], GI[0, r2]^2 GR[r4],
 GR[-1]^3 GR[r2, -1], GI[r2]^2 GR[-1] GR[r2, -1],
 GR[-1]^2 GR[r4] GR[r2, -1], GI[r2]^2 GR[r4] GR[r2, -1],
 GR[-1] GR[r4]^2 GR[r2, -1], GR[r4]^3 GR[r2, -1],
 GI[r2] GI[0, r2] GR[r2, -1], GR[-1] GR[r2, -1]^2,
 GR[r4] GR[r2, -1]^2, GR[-1]^2 GR[0, 0, 1], GI[r2]^2 GR[0, 0, 1],
 GR[-1] GR[r4] GR[0, 0, 1], GR[r4]^2 GR[0, 0, 1],
 GR[r2, -1] GR[0, 0, 1], GI[r2] GI[0, 1, r4] GR[-1],
 GI[r2] GI[0, 1, r4] GR[r4], GI[0, r2] GI[0, 1, r4],
 GI[r2] GI[0, r2, -1] GR[-1], GI[r2] GI[0, r2, -1] GR[r4],
 GI[0, r2] GI[0, r2, -1], GR[-1]^2 GR[r2, 1, -1],
 GI[r2]^2 GR[r2, 1, -1], GR[-1] GR[r4] GR[r2, 1, -1],
 GR[r4]^2 GR[r2, 1, -1], GR[r2, -1] GR[r2, 1, -1],
 GR[-1]^2 GR[r2, 1, r3], GI[r2]^2 GR[r2, 1, r3],
 GR[-1] GR[r4] GR[r2, 1, r3], GR[r4]^2 GR[r2, 1, r3],
 GR[r2, -1] GR[r2, 1, r3], GI[r2] GI[0, 0, 0, r2],
 GR[-1] GR[0, 0, r2, -1], GR[r4] GR[0, 0, r2, -1],
 GR[-1] GR[0, 0, r4, 1], GR[r4] GR[0, 0, r4, 1],
 GI[r2] GI[0, 1, 1, r4], GI[r2] GI[0, 1, r2, -1],
 GI[r2] GI[0, 1, r2, r3], GI[r2] GI[0, r2, 1, -1],
 GR[-1] GR[r2, 1, 1, -1], GR[r4] GR[r2, 1, 1, -1],
 GR[-1] GR[r2, 1, 1, r3], GR[r4] GR[r2, 1, 1, r3],
 GR[-1] GR[r2, 1, r2, -1], GR[r4] GR[r2, 1, r2, -1]}
```

$B_l(5) =$

```
{GI[0, 0, 0, 1, r2], GI[0, 0, 0, 1, r4], GI[0, 0, 0, r2, -1],
GI[0, 1, 1, -1, r2], GI[0, 1, 1, -1, r4], GI[0, 1, 1, 1, r4],
GI[0, 1, 1, r2, r3], GI[0, 1, 1, r4, -1], GI[0, 1, 1, r4, r1],
GI[0, 1, r2, r3, r2], GI[0, r2, 1, 1, -1]}
```

and

```
{GI[r2] GR[-1]^4, GI[r2]^3 GR[-1]^2, GI[r2]^5, GI[r2] GR[-1]^3 GR[r4],
 GI[r2]^3 GR[-1] GR[r4], GI[r2] GR[-1]^2 GR[r4]^2, GI[r2]^3 GR[r4]^2,
 GI[r2] GR[-1] GR[r4]^3, GI[r2] GR[r4]^4, GI[0, r2] GR[-1]^3,
GI[r2]^2 GI[0, r2] GR[-1], GI[0, r2] GR[-1]^2 GR[r4],
GI[r2]^2 GI[0, r2] GR[r4], GI[0, r2] GR[-1] GR[r4]^2,
GI[0, r2] GR[r4]^3, GI[r2] GI[0, r2]^2, GI[r2] GR[-1]^2 GR[r2, -1],
GI[r2]^3 GR[r2, -1], GI[r2] GR[-1] GR[r4] GR[r2, -1],
GI[r2] GR[r4]^2 GR[r2, -1], GI[0, r2] GR[-1] GR[r2, -1],
GI[0, r2] GR[r4] GR[r2, -1], GI[r2] GR[r2, -1]^2,
GI[r2] GR[-1] GR[0, 0, 1], GI[r2] GR[r4] GR[0, 0, 1],
GI[0, r2] GR[0, 0, 1], GI[0, 1, r4] GR[-1]^2, GI[r2]^2 GI[0, 1, r4],
GI[0, 1, r4] GR[-1] GR[r4], GI[0, 1, r4] GR[r4]^2,
GI[0, 1, r4] GR[r2, -1], GI[0, r2, -1] GR[-1]^2,
GI[r2]^2 GI[0, r2, -1], GI[0, r2, -1] GR[-1] GR[r4],
GI[0, r2, -1] GR[r4]^2, GI[0, r2, -1] GR[r2, -1],
GI[r2] GR[-1] GR[r2, 1, -1], GI[r2] GR[r4] GR[r2, 1, -1],
GI[0, r2] GR[r2, 1, -1], GI[r2] GR[-1] GR[r2, 1, r3],
GI[r2] GR[r4] GR[r2, 1, r3], GI[0, r2] GR[r2, 1, r3],
GI[0, 0, 0, r2] GR[-1], GI[0, 0, 0, r2] GR[r4],
GI[r2] GR[0, 0, r2, -1], GI[r2] GR[0, 0, r4, 1],
GI[0, 1, 1, r4] GR[-1], GI[0, 1, 1, r4] GR[r4],
GI[0, 1, r2, -1] GR[-1], GI[0, 1, r2, -1] GR[r4],
GI[0, 1, r2, r3] GR[-1], GI[0, 1, r2, r3] GR[r4],
GI[0, r2, 1, -1] GR[-1], GI[0, r2, 1, -1] GR[r4],
GI[r2] GR[r2, 1, 1, -1], GI[r2] GR[r2, 1, 1, r3],
GI[r2] GR[r2, 1, r2, -1]}
```

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The numbers of elements are $3, 8, 21, 55, 144$ for weights $w = 1, 2, 3, 4, 5$, correspondingly.

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If a constant is expected to be uniformly transcendental one can use these bases. Otherwise, one uses

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The numbers of elements in these bases are 4, 12, 33, 88, 232 for weights $w = 1, 2, 3, 4, 5$, correspondingly.

The accuracy of 2000 digits was quite enough to obtain results with PSLQ in an ε-expansion up to the finite part in ε, or, in other words, up to weight 4, in a straightforward way.

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Let us look for uniformly transcendental threshold integrals. At $p^2 = m^2$, the integrals

$$\{J_4 = F_{1,2,2,2,2,0,\ldots,0},\, J_5 = F_{2,2,2,2,1,0,\ldots,0}\}\,.$$

are uniformly transcendental. Let us assume that these integrals at $p^2 = 9m^2$ also have this property. PSLQ with $B(w)$ confirms it and gives

$$
\begin{aligned}
J_4 \;=\; & \frac{1}{\epsilon}\left(-\frac{20}{3} G_I(r_2) G_I(0, r_2) - \frac{26}{9} G_R(0,0,1) \right) \\
& - 16 G_I(r_2) G_R(r_4) G_I(0, r_2) + 124 G_I(r_2) G_I(0, 1, r_4) \\
& + 72 G_I(r_2) G_I(0, r_2, -1) \\
& - \frac{100}{3} G_I(0, r_2)^2 + 8 G_R(0, 0, r_4, 1) + \frac{1153 G_I(r_2)^4}{15} + O(\varepsilon) \;,
\end{aligned}
$$

$$
\begin{aligned}
J_5 \quad = \quad & \frac{\sqrt{3}G_I(r_2)}{18\epsilon^3} + \frac{1}{\epsilon^2}\left(\frac{5}{9}\sqrt{3}G_I(0,r_2) - \frac{5}{9}\sqrt{3}G_I(r_2)G_R(r_4) - \sqrt{3}G_R(-1)G_I(r_2)\right) \\
& + \frac{1}{\epsilon}\Bigg(-\frac{52}{9}\sqrt{3}G_R(r_4)G_I(0,r_2) - 10\sqrt{3}G_R(-1)G_I(0,r_2) + \frac{40}{9}G_I(r_2)G_I(0,r_2) + 6\sqrt{3}G_I(0,r_2,-1) \\
& + \frac{26}{3}\sqrt{3}G_I(0,1,r_4) + \frac{52}{27}G_R(0,0,1) + \frac{25}{9}\sqrt{3}G_I(r_2)G_R(r_4)^2 + 10\sqrt{3}G_R(-1)G_I(r_2)G_R(r_4) \\
& + 9\sqrt{3}G_R(-1)^2G_I(r_2) + \frac{253}{36}\sqrt{3}G_I(r_2)^3\Bigg) \\
& + \frac{1060}{27}\sqrt{3}G_R(r_4)^2G_I(0,r_2) + \frac{32}{3}G_I(r_2)G_R(r_4)G_I(0,r_2) - 60\sqrt{3}G_R(r_4)G_I(0,r_2,-1) \\
& + 104\sqrt{3}G_R(-1)G_R(r_4)G_I(0,r_2) + \frac{5101}{324}\sqrt{3}G_R(0,0,1)G_I(r_2) + 90\sqrt{3}G_R(-1)^2G_I(0,r_2) \\
& - 54\sqrt{3}G_R(-1)G_I(0,r_2,-1) + 14\sqrt{3}G_I(0,r_2)G_R(r_2,-1) - \frac{530}{9}\sqrt{3}G_R(r_4)G_I(0,1,r_4) \\
& - 96\sqrt{3}G_R(-1)G_I(0,1,r_4) - 60\sqrt{3}G_I(0,1,r_2,r_3) - \frac{248}{3}G_I(r_2)G_I(0,1,r_4) + \frac{5695}{36}\sqrt{3}G_I(r_2)^2G_I(0,r_2) \\
& - \frac{7438}{81}\sqrt{3}G_I(0,0,0,r_2) - 48G_I(r_2)G_I(0,r_2,-1) + \frac{200}{9}G_I(0,r_2)^2 - 74\sqrt{3}G_I(0,1,r_2,-1) \\
& + 54\sqrt{3}G_I(0,r_2,1,-1) + \frac{250}{9}\sqrt{3}G_I(0,1,1,r_4) - \frac{16}{3}G_R(0,0,r_4,1) - \frac{1021}{9}\sqrt{3}G_I(r_2)^3G_R(r_4) \\
& - \frac{250}{27}\sqrt{3}G_I(r_2)G_R(r_4)^3 - 50\sqrt{3}G_R(-1)G_I(r_2)G_R(r_4)^2 - 90\sqrt{3}G_R(-1)^2G_I(r_2)G_R(r_4) \\
& - \frac{287}{2}\sqrt{3}G_R(-1)G_I(r_2)^3 - 54\sqrt{3}G_R(-1)^3G_I(r_2) - \frac{2306}{45}G_I(r_2)^4 + O(\varepsilon)\,.
\end{aligned}
$$

To evaluate the ε-term of J_1 let us construct the following linear combination:

$$J_6 = \left(1 + \frac{1}{2}\epsilon + \frac{95}{12}\epsilon^2 + \frac{2615}{144}\epsilon^3 + \frac{1154333}{1728}\epsilon^4\right) J_1 + 48\epsilon J_4 - 3024\epsilon^3 J_5 \ .$$

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The coefficients here are adjusted in such a way that the available result up to the finite part in ε is uniformly transcendental.
Moreover, analytical result for its ε-term can be revealed with the help of the basis

$$\tilde{B}(5) = B(5) \cup \left\{1, \sqrt{3}G_I(r_2), -\frac{20}{3}G_I(r_2)G_I(0,r_2) - \frac{26}{9}G_R(0,0,1)\right\}$$

which differs from the uniformly transcendental basis of weight 5 adding three elements proportional to the leading terms of J_1, J_5, J_4 in their ε-expansions.

$$
\begin{aligned}
J_1 \quad = \quad & -\frac{1}{4\epsilon^4}+\frac{1}{8\epsilon^3}+\frac{1}{\epsilon^2}\left(\frac{23}{12}-\frac{3G_I(r_2)^2}{4}\right)+\frac{1}{\epsilon}\left(-\frac{1}{3}G_R(0,0,1)+\frac{3G_I(r_2)^2}{8}+\frac{1493}{576}\right) \\
& -120G_I(r_2)G_R(r_4)G_I(0,r_2)+\frac{1941G_I(r_2)^4}{20}+\frac{23G_I(r_2)^2}{4}+180G_I(r_2)G_I(0,1,r_4)+320G_I(r_2) \\
& G_I(0,r_2)+72G_R(0,0,r_4,1)+\frac{833}{6}G_R(0,0,1)-56\sqrt{3}\pi+\frac{1024805}{6912} \\
+\epsilon\Bigg(& -1056G_I(r_2)G_R(r_4)^2G_I(0,r_2)-2592G_R(-1)G_I(r_2)G_I(0,1,r_4)+828G_I(r_2)G_R(r_4)G_I(0,r_2) \\
& +1584G_I(r_2)G_R(r_4)G_I(0,1,r_4)+2592G_I(r_2)G_R(r_4)G_I(0,r_2,-1)-\frac{15563}{9}G_R(0,0,1)G_I(r_2)^2 \\
& +1728G_I(r_2)G_I(0,r_2)G_R(r_2,-1)+2592G_I(r_2)G_I(0,1,r_2,r_3)-6042G_I(r_2)G_I(0,1,r_4) \\
& -2880G_I(r_2)G_I(0,1,1,r_4)+1704G_I(0,r_2)G_I(0,1,r_4)-\frac{72172}{9}G_I(r_2)^3G_I(0,r_2)+\frac{320}{9}G_I(r_2)G_I(0,r_2) \\
& -3456G_I(r_2)G_I(0,r_2,-1)+\frac{14816}{3}G_I(r_2)G_I(0,0,0,r_2)+864G_I(r_2)G_I(0,1,r_2,-1)+1600G_I(0,r_2)^2 \\
& +1680\sqrt{3}G_I(0,r_2)+1136G_R(0,0,1,r_2,r_4)+288G_R(r_4)G_R(0,0,r_4,1)-420G_R(0,0,r_4,1) \\
& -288G_R(0,0,1,1,r_4)+\frac{485}{27}G_R(0,0,1)-\frac{397811}{405}G_R(0,0,0,0,1)+\frac{15396}{5}G_I(r_2)^4G_R(r_4) \\
& -1680\sqrt{3}G_I(r_2)G_R(r_4)+1512G_R(-1)G_I(r_2)^4-3024\sqrt{3}G_R(-1)G_I(r_2)+\frac{28000}{9}\sqrt{3}G_I(r_2) \\
& -\frac{29905G_I(r_2)^4}{8}+\frac{1493G_I(r_2)^2}{192}+28\sqrt{3}\pi+\frac{232538063}{82944}\Bigg)+O(\varepsilon^2)\ .
\end{aligned}
$$

A similar procedure is applied to J_2 and J_3.
Two linear combinations

$$
\begin{aligned}
J_7 &= \left(1 + \frac{1}{3}\epsilon + \frac{37}{9}\epsilon^2 + \frac{571}{108}\epsilon^3 + \frac{139585}{324}\epsilon^4\right) J_2 \\
&\quad - 37\epsilon J_4 + 2112\epsilon^3 J_5 \ , \\
J_8 &= \left(1 + 8\epsilon^2 - \frac{277}{2}\epsilon^3 - \frac{29551}{12}\epsilon^4\right) J_3 \\
&\quad + 8(6\epsilon - 1) J_4 + 16(743\epsilon + 48)\epsilon^2 J_5 \ .
\end{aligned}
$$

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- Other applications of our algorithm are in progress.