Five-Particle Phase-Space Integrals in QCD

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based on
OG, Magerya, Pikelner arXiv:1803.09084

II. Institut für Theoretische Physik
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Loops and Legs in Quantum Field Theory
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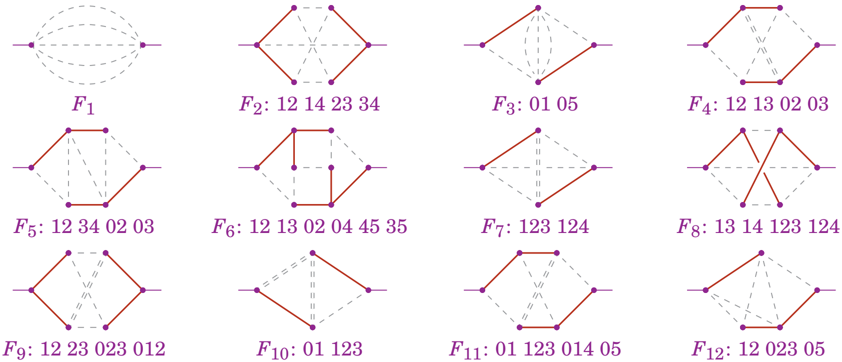
Agenda

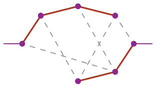
Five-Particle Masters

$$\boxed{F_i = S_\Gamma \int \mathrm{dPS}_5 \, \frac{1}{D_1^{(i)} \ldots D_{n_i}^{(i)}}} \qquad i = 1..31$$

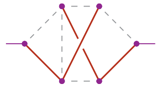
- phase-space $\mathrm{dPS}_5 = \prod_{i=1}^{5}[\mathrm{d}^D p_i] \; \delta^{(D)}\Big(q - p_1 - \ldots - p_5\Big)$ and $[\mathrm{d}p_i] = \mathrm{d}^D p_i \; \delta^+\big(p_i^2\big)$
- propagators $D_{n_i}^{(i)} = \big(p_k + p_l + \cdots + p_q\big)^2$



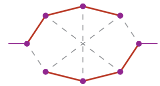
Five-Particle Masters



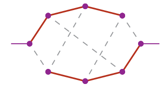
F_{13}: 12 13 124 05 02



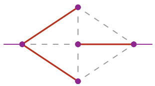
F_{14}: 02 12 023 05



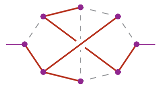
F_{15}: 12 13 124 134 02 03



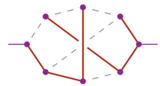
F_{16}: 12 34 02 03 124 134



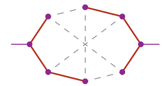
F_{17}: 12 34 013



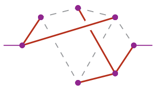
F_{18}: 12 13 25 124 05 04



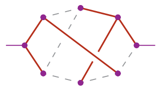
F_{19}: 12 13 25 124 013 05



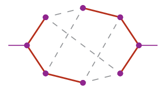
F_{20}: 12 13 124 134 35 25



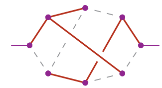
F_{21}: 12 13 24 012 05



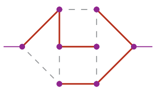
F_{22}: 12 13 24 012 45 05



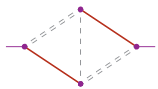
F_{23}: 12 13 25 45 134 012



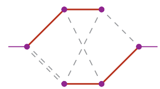
F_{24}: 12 13 45 134 05 03



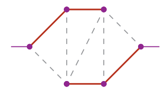
F_{25}: 12 13 45 134 012 05



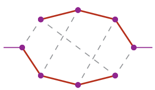
F_{26}: 012 123



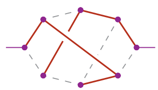
F_{27}: 12 01 123 013



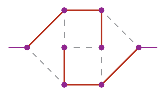
F_{28}: 123 012 05 01



F_{29}: 12 123 25 013 05 01



F_{30}: 12 13 034 134 05 03



F_{31}: 12 13 034 134 05 04

Five-Particle Masters

Some results truncated at ζ_6 are

$$
\begin{aligned}
F_{15} = {} & \frac{1}{6\epsilon^6} + \frac{41}{36\,\epsilon^5} + \frac{1}{\epsilon^4}\left(-\frac{311}{36} - \frac{73}{18}\zeta_2\right) + \left(\frac{445}{18} - \frac{563}{36}\zeta_2 - \frac{281}{9}\zeta_3\right)\frac{1}{\epsilon^3} + \frac{1}{\epsilon^2}\left(-\frac{689}{9}\right. \\
& \left. + \frac{5273}{36}\zeta_2 - \frac{907}{18}\zeta_3 - \frac{7103}{180}\zeta_2^2\right) + \frac{1}{\epsilon}\left(\frac{2024}{9} - \frac{7759}{18}\zeta_2 + \frac{13933}{18}\zeta_3 + \frac{10553}{120}\zeta_2^2\right. \\
& \left. + \frac{1489}{3}\zeta_3\zeta_2 - \frac{3257}{3}\zeta_5\right) - \frac{6158}{9} + \frac{12437}{9}\zeta_2 - \frac{22193}{9}\zeta_3 + \frac{28621}{120}\zeta_2^2 + \frac{14065}{18}\zeta_3\zeta_2 \\
& + \frac{3631}{6}\zeta_5 - \frac{134489}{420}\zeta_2^3 + 1189\zeta_3^2 + \mathcal{O}(\epsilon) \\
F_{28} = {} & -\frac{\zeta_2^3}{5} + 2\zeta_3^2 + \mathcal{O}(\epsilon)
\end{aligned}
$$

All results are available up to ζ_{12}.

Feynman Integrals Calculus

Numerical methods

- Sector Decomposition
- Mellin-Barnes
- Subtraction Schemes (e.g., CoLoRFulNNLO)
- Dimensional Recurrence Relations

Analytical methods

- Feynman/Schwinger parametrization
  - `HyperInt` Panzer '14
- Integration-By-Parts reduction Chetyrkin Tkachov '81
  - Laporta algorithm Laporta '00: `AIR`, `FIRE`, `Kira`, `Reduze`
  - Symbolic reduction: `LiteRed` Lee '12
  - private code: `Crusher` Marquard; Laporta; ...
- Method of Differential Equations Kotikov '91 Remiddi '97
  - Epsilon form Henn '13
    - leading singularity Henn '14
    - Lee algorithm '14: `Fuchsia` OG Magerya '16, `Epsilon` Prausa '17
    - Meyer algorithm '16: `Canonica` Meyer '17

Feynman Integrals Calculus

Construct System of ODE

- from IBP rules
  - `AIR`, `FIRE`, `Kira`, `LiteRed`, `Reduze`
- from definition
  - Holonomic Functions
- from Baikov representation

Solve System of ODE

- epsilon form Henn '13
  - `Fuchsia`, `Epsilon`, `Canonica`
  - works well with *hyperlogarithms*
  - with (potentially) *elliptic functions*
- by other means
  - series expansion
  - numerically

Find Constants of Integration

- no systematic approach
- challenging on its own

Motivation: Splitting Functions

Time-like splitting functions may be extracted form a semi-inclusive one-particle decay process

$$\frac{d^2\sigma}{dx\,d\cos\theta} = \frac{3}{8}(1+\cos^2\theta)\,\mathcal{F}_T(x,\epsilon) + \frac{3}{4}\sin^2\theta\,\mathcal{F}_L(x,\epsilon) + \frac{3}{4}\cos\theta\,\mathcal{F}_A(x,\epsilon)$$

- Transverse fragmentation functions

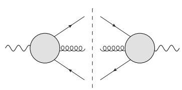
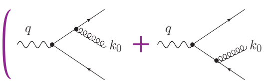
$$\mathcal{F}_T(x,\epsilon) \simeq \left(x^2 g^{\mu\nu} + 4k_0^\mu k_0^\nu\right) W_{\mu\nu}(x,\epsilon), \qquad x = 2q\cdot k_0$$

- Hadronic tensor

$$W_{\mu\nu}(x,\epsilon) \simeq \int \mathrm{d}^D\mathrm{PS}^{(n)}\, M_\mu^{(n)}\, M_\nu^{(n)*}$$

  where $\mathrm{d}^D\mathrm{PS}^{(n)}$ is n-particle phase-space in $D = 4-2\epsilon$ dimensions and $M_\mu^{(n)}$ is amplitude of the process

Example: LO contribution

$$\mathcal{F}_T^{(1)}(x,\epsilon) \equiv$$  $$\simeq \left(x^2 g^{\mu\nu} + 4k_0^\mu k_0^\nu\right) \int \mathrm{d}^D\mathrm{PS}^{(3)} \left(\quad \right)^2$$ 

$$\mathrm{d}^m\mathrm{PS}^{(3)} = \mathrm{d}^D k_0 \delta^+(k_0^2)\ \mathrm{d}^D k_1 \delta^+(k_1^2) \mathrm{d}^D k_2 \delta^+(k_2^2)\ \boxed{\delta(x - 2q\cdot k_0)}\ \delta^D(q - k_0 - k_1 - k_2)$$

Motivation: Splitting Functions

Vermaseren, Vogt, Moch '05

Splitting functions $P_{gq}^{(0)}, P_{gq}^{(1)}, P_{gq}^{(2)}$ on the RHS can be extracted from the mass factorization relations

$$\mathscr{F}_T^{(1)}(x,\epsilon) = \frac{1}{\epsilon}\boldsymbol{P_{gq}^{(0)}(x)} + c_{T,g}^{(1)}(x) + \epsilon\, a_{T,g}^{(1)}(x) + \epsilon^2 b_{T,g}^{(1)}(x)$$

$$\mathscr{F}_T^{(2)}(x,\epsilon) = \frac{1}{\epsilon^2}\left\{\frac{1}{2}P_{gi}^{(0)}P_{iq}^{(0)} + \frac{1}{2}\beta_0 P_{gq}^{(0)}\right\} - \frac{1}{\epsilon}\left\{\frac{1}{2}\boldsymbol{P_{gq}^{(1)}} + P_{gi}^{(0)}c_i^{(1)}\right\} + \left\{c_g^{(2)} - P_{gi}^{(0)}a_i^{(1)}\right\} + \epsilon\left\{a_g^{(2)} - P_{gi}^{(0)}b_i^{(1)}\right\}$$

$$\begin{aligned}\mathscr{F}_T^{(3)}(x,\epsilon) = {} & \frac{1}{\epsilon^3}\left\{\frac{1}{6}P_{gi}^{(0)}P_{ij}^{(0)}P_{jq}^{(0)} + \frac{1}{2}\beta_0 P_{gi}^{(0)}P_{iq}^{(0)} + \frac{1}{3}\beta_0^2 P_{gq}^{(0)}\right\} \\ & + \frac{1}{\epsilon^2}\left\{\frac{1}{6}P_{gi}^{(0)}P_{iq}^{(1)} + \frac{1}{3}P_{gi}^{(1)}P_{iq}^{(0)} + \frac{1}{3}\beta_1 P_{gq}^{(0)} + \frac{1}{2}P_{gi}^{(0)}P_{ij}^{(0)}c_j^{(1)} + \beta_0\left(\frac{1}{3}P_{gq}^{(1)} + \frac{1}{2}P_{gi}^{(0)}c_i^{(1)}\right)\right\} \\ & - \frac{1}{\epsilon}\left\{\frac{1}{3}\boldsymbol{P_{gq}^{(2)}} + \frac{1}{2}P_{gi}^{(1)}c_i^{(1)} + P_{gi}^{(0)}c_i^{(2)} - \frac{1}{2}P_{gi}^{(0)}P_{ij}^{(0)}a_j^{(1)} - \frac{1}{2}\beta_0 P_{gi}^{(0)}a_i^{(1)}\right\} \\ & + \left\{c_g^{(3)} - P_{gi}^{(0)}a_i^{(2)} - \frac{1}{2}P_{gi}^{(1)}a_i^{(1)} + \frac{1}{2}P_{gi}^{(0)}P_{ij}^{(0)}b_j^{(1)} + \frac{1}{2}\beta_0 P_{gi}^{(0)}b_i^{(1)}\right\}\end{aligned}$$

We construct a set of lowering dimensional recurrence relations [Tarasov '96] that are used to express masters in $D+2$ dimensions in terms of masters in D dimensions, as

$$F_i(D+2) = M_{ij}(D)\,F_j(D) \tag{1}$$

that has a block-diagonal form

$$F_i(D+2) = M_{ii}(D)\,F_i(D) + \sum_{k<i} M_{ik}(D) F_k(D). \tag{2}$$

We use `LiteRed` [Lee '12] and `FIRE` [Smirnov '14] to find $M(D)$ matrix.

Solutions of (2) has a form

$$F_i(D) = \omega_i(D)\,H_i(D) + R_i(D), \tag{3}$$

where

- $H_i(D)$ — homogeneous solution
- $R_i(D)$ — inhomogeneous solution
- $\omega_i(D)$ — "boundary conditions", some periodic function, i.e., $\omega_i(D+2) = \omega_i(D)$

Next, we argue that $\omega_i(D) = 0$ for $i > 1$.

$$F_i = S_\Gamma \left(\prod_{k=1}^{N-1} \Omega_{D-k} \right) \int \left(\prod_{l<m} \mathrm{d}s_{lm} \right) \left(\Delta_N \right)^{\frac{D-N-1}{2}} \Theta\left(\Delta_N\right) \delta\left(1 - s_{1\ldots N}\right) \frac{1}{D_1^{(i)} \ldots D_n^{(i)}}, \tag{4}$$

where Δ_N is the Gram determinant defined as

$$\Delta_N = \frac{(-1)^{N+1}}{2^N} \begin{vmatrix} s_{11} & s_{12} & \cdots & s_{1N} \\ s_{12} & s_{22} & \cdots & s_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ s_{1N} & s_{2N} & \cdots & s_{NN} \end{vmatrix}, \tag{5}$$

and Ω_k is the surface of a unit hypersphere in k-dimensional space

$$\Omega_k = 2\pi^{\frac{k}{2}}\, \Gamma\left(\frac{k}{2}\right)^{-1}. \tag{6}$$

If $\Delta_N(s_{ij})$ has a unique global maximum inside the integration region, we can apply Laplace's method to eq. (4) and find its asymptotic as

$$F_i(D \to \infty) = S_\Gamma \left(\prod_{k=1}^{N-1} \Omega_{D-k} \right) \left(\Delta_N^{max}\right)^{\frac{D}{2}} \left(\frac{2\pi}{D}\right)^{\frac{1}{2}\left(\frac{N(N-1)}{2}-1\right)} \left(\mathscr{C}_i + \mathscr{O}\left(D^{-1}\right)\right), \tag{7}$$

where $\mathscr{C}_i$ is a constant that depends on the location of the maximum and the denominators $D_j^{(i)}$, but not on D.

Asymptotic Behavior

The global maximum of Δ_N is reached when all s_{ij} ($i \neq j$) are identical and equal to $\frac{2}{N(N-1)}$. Geometrically this configuration corresponds to the vectors $\vec{p}_i$ pointing to the vertices of a regular N-hedron embedded into Euclidean space of $(N-1)$ dimensions. The maximum value is then

$$\Delta_N^{max} = \frac{1}{N^N (N-1)^{N-1}} \tag{8}$$

and explicitly we get

$$F_i(D \to \infty) = \pi^{\frac{7}{2}} 2^{\frac{25}{2}} \frac{\left(4^4 5^5\right)^{-\frac{D}{2}} \Gamma\left(\frac{3D}{2} - 3\right)}{D^{\frac{9}{2}} \Gamma\left(\frac{D-4}{2}\right) \Gamma\left(\frac{D-3}{2}\right) \Gamma\left(\frac{D-1}{2}\right)} \left(\mathscr{C}_i + \mathscr{O}\left(D^{-1}\right)\right). \tag{9}$$

It follows that all F_i have identical asymptotic behavior up to a constant $\mathscr{C}_i$.

As a confirmation, it can be shown that eq. (9) is asymptotically the same expression as we had for F_1.

Asymptotic Behavior

Next, we can find the asymptotics of the homogeneous parts, $H_i(D)$, using e.g. the routine `FindAsymptotics` from `DREAM` [Lee, Mingulov '17]. Comparing these to

$$F_i(D \to \infty) = \pi^{\frac{7}{2}} 2^{\frac{25}{2}} \frac{\left(4^4 5^5\right)^{-\frac{D}{2}} \Gamma\left(\frac{3D}{2} - 3\right)}{D^{\frac{9}{2}} \Gamma\left(\frac{D-4}{2}\right) \Gamma\left(\frac{D-3}{2}\right) \Gamma\left(\frac{D-1}{2}\right)} \left(\mathscr{C}_i + \mathscr{O}\left(D^{-1}\right)\right) \tag{10}$$

we determine that all $H_i(D)$ for $i > 1$ are growing exponentially faster than $F_i(D)$, which can only happen if the corresponding periodic functions $\omega_i(D)$ are zero.

Solution

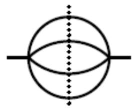
Thus, to find F_i we only need to find R_i, the inhomogeneous solutions. We compute them as a series in $\epsilon = (4 - D)/2$ using `DREAM` with 2000-digit accuracy and then restore the analytical form of the series coefficients in terms of MZVs using PSLQ method.

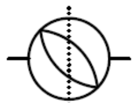
This way we obtain the analytical result for all master integrals up to MZVs of weight 12 using the corresponding bases from [Furusho '03; Blumlein, Broadhurst, Vermaseren '09] and the `SummerTime` package [Lee, Mingulov '16] for their numerical evaluation.

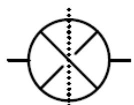
Four-Particle Masters

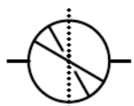
As the first consistency check of the calculation procedure we reproduce results for four-particle phase-space integrals reported in [Gehrmann – De Ridder et al. '03].

We perform all the steps described so far: generating the IBP rules with the help of `LiteRed` and then proceeding with `DREAM` we obtain the final result with 2000-digit accuracy and MZVs up to weight 12.

$$R_4 \equiv$$

$$= \int \mathrm{d}PS_4 = P_4$$

$$R_6 \equiv$$

$$= \int \mathrm{d}PS_4 \frac{1}{s_{134}s_{234}}$$

$$R_{8,a} \equiv$$

$$= \int \mathrm{d}PS_4 \frac{1}{s_{13}s_{23}s_{14}s_{24}}$$

$$R_{8,b} \equiv$$

$$= \int \mathrm{d}PS_4 \frac{1}{s_{13}s_{134}s_{23}s_{234}} \, .$$

The series reconstructed with PSLQ (using the original notation, and omitting S_Γ and q^2 factors) look as:

$$R_6 = -1+\zeta_2+\epsilon\Big(-12+5\zeta_2+9\zeta_3\Big)+\epsilon^2\Big(-91+27\zeta_2+45\zeta_3+\frac{61}{5}\zeta_2^2\Big)$$
$$+\epsilon^3\Big(-558+161\zeta_2+197\zeta_3+61\zeta_2^2-80\zeta_3\zeta_2+207\zeta_5\Big)$$
$$+\epsilon^4\Big(-3025+939\zeta_2+897\zeta_3+\frac{1157}{5}\zeta_2^2-400\zeta_3\zeta_2+1035\zeta_5+\frac{288}{5}\zeta_2^3-153\zeta_3^2\Big)$$
$$R_{8,a} = \frac{5}{\epsilon^4}-\frac{40\zeta_2}{\epsilon^2}-\frac{126\zeta_3}{\epsilon}+14\zeta_2^2+\epsilon\Big(1008\zeta_2\zeta_3-1086\zeta_5\Big)+\epsilon^2\Big(-\frac{272}{7}\zeta_2^3+1602\zeta_3^2\Big)$$
$$R_{8,b} = \frac{3}{4\epsilon^4}-\frac{17\zeta_2}{2\epsilon^2}-\frac{44\zeta_3}{\epsilon}-\frac{183}{5}\zeta_2^2+\epsilon\Big(376\zeta_2\zeta_3-790\zeta_5\Big)+\epsilon^2\Big(-\frac{19088}{105}\zeta_2^3+698\zeta_3^2\Big)$$

These results agree with hep-ph/0311276.

Numerical Verification

As another cross-check we have calculated the leading terms of F_i numerically using the direct way: through Monte-Carlo integration. While such a technique can not be easily applied to divergent integrals, we can sidestep such an issue by noting that our master integrals only suffer from IR divergences that disappear already at $D = 6$. In this way we can check several leading terms of the expansion at $D = 4-2\epsilon$ by calculating the corresponding integrals in $D = 6$ and $D = 8$ since both are connected by DRR.

To calculate a finite integral we choose a uniform mapping from a hypercube into momentum coordinates using an algorithm similar to RAMBO but extended into arbitrary D. Then we calculate the integrand from scalar products of the momenta, and finally we integrate over the hypercube using the Vegas implementation from CUBA [Hahn].

Note that although the integrals we are calculating are finite, the integrands are not. Exposing an integration algorithm like Vegas to such infinities may lead to unpredictable behaviour, so as a precaution we choose to regulate these infinities by adding a small parameter α to the denominator of the integrand, and then to calculate the integral with progressively smaller values of α (from 2^{-30} to 2^{-100}), checking if convergence was reached afterwards.

Numerical Verification

i	Numerical results			Analytic results		
	$D=4$	$D=6$	$D=8$	$D=4$	$D=6$	$D=8$
2	–	1708(2)00	4699(1)0	–	171085.62	47000.531
3	3.7823(4)	3.1704(2)	3.0221(1)	3.7823736	3.1704486	3.0221118
4	–	1504.7(8)	725.3(1)	–	1504.4507	725.26806
5	–	1007.4(5)	580.80(9)	–	1007.5235	580.76347
6	–	6191(5)00	14496(6)0	–	619633.25	144975.32
7	46.46(4)	18.533(2)	15.205(1)	46.435253	18.532303	15.205538
8	–	2031(2)0	5357(2)	–	20297.189	5355.3611
9	–	4313(3)	2406.7(4)	–	4312.8823	2406.7943
10	10.436(2)	7.1508(5)	6.5093(3)	10.435253	7.1507477	6.5092878
11	228.8(1)	62.67(1)	47.663(4)	229.11836	62.667046	47.663194
12	–	157.34(4)	102.26(1)	–	157.33521	102.26408
13	–	13729(8)	4000(1)	–	13732.166	4000.2779
14	–	268.46(8)	172.80(2)	–	268.45969	172.79805
15	–	6322(6)0	16048(5)	–	63316.356	16049.857

Table 1: Numerical results for the ratio F_i/F_1 with the corresponding uncertainties (standard deviations) indicated in the parenthesis. Missing entries correspond to divergent integrals.

Numerical Verification

i	Numerical results			Analytic results		
	$D=4$	$D=6$	$D=8$	$D=4$	$D=6$	$D=8$
16	–	4414(3)0	12952(4)	–	44117.898	12951.443
17	–	1243.4(7)	709.6(1)	–	1243.1369	709.52840
18	–	3002(2)00	5899(3)0	–	300402.99	58965.517
19	–	4982(4)00	10637(4)0	–	498329.79	106357.81
20	–	2360(2)000	5777(2)00	–	2362594.9	577686.64
21	–	6312(5)0	20642(5)	–	63147.876	20642.071
22	–	8402(7)00	24407(8)0	–	840453.94	244075.75
23	–	1443(1)000	4556(1)00	–	1443198.3	455543.43
24	–	1391(1)00	3997(1)0	–	139263.92	39966.878
25	–	3347(3)00	8526(3)0	–	335128.10	85254.217
26	25.563(6)	15.376(1)	13.6042(7)	25.564747	15.376404	13.604247
27	–	697.3(3)	397.84(6)	–	697.18948	397.83514
28	143.9(1)	52.855(7)	42.917(3)	143.97886	52.853837	42.917424
29	–	4409(3)0	13702(3)	–	44117.898	13700.597
30	–	6327(6)0	16181(5)	–	63316.356	16178.566
31	–	8955(8)0	19055(8)	–	89611.062	19051.115

Summary

- Five-Particle Phase-Space Integrals (up to ζ_{12})
- RAMBO-style integration code in D dimensions
- Easy Way to Find Single-Scale Integrals

Thank you!