Generalised cuts and Wick Rotations

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The study of Unitarity/Discontinuity/Imaginary part of Feynman graph amplitudes brought to consider *cut graphs* (Dick Cutkosky, Tini Veltman), in which some of the propagators are replaced by the corresponding physical phase space δ-functions.

Evaluating cut graphs is surely easier than evaluating complete graphs, and can be a good starting point in that direction; once the imaginary part is known one can try to obtain the real part by a dispersion relation.

Cut graphs are currently used for establishing relations between closely related Feynman amplitudes (such as decomposition of tensor amplitudes in master integrals); *maximally cut graphs* (*i.e.* graphs in which *all* the propagators are cut) are obviously the most appealing possibility.

But the (physical) phase space for a maximally cut graph often vanishes, so that some change in the original integral representation (typically a contour deformation) is needed to get a non-zero result, and as a by-problem, the relation of the modified quantity to the original amplitude may arise.

It will be shown that switching from *Euclidean* to *Minkoskian* a suitable space component of the loop momentum, in close analogy to the familiar

Wick rotation

from *Minkoskian* to *Euclidean* of the energy component introduced by Giancarlo Wick in 1954.

summary:

- the definition of the Feynman propagator: real and imaginary parts
- maximally cut 1-loop vertex amplitude and Wick rotation
- as above, the (massive) 1-loop box
- the 1-loop bubble, real and imaginary parts (in *great detail*)

- conclusion:

one more Wick (counter)-rotation may help when dealing with generalised cuts

Some familiar formulas for x real, ϵ small and positive

$$\frac{1}{x - i\epsilon} = \frac{x + i\epsilon}{x^2 + \epsilon^2} = \frac{x}{x^2 + \epsilon^2} + i\,\frac{\epsilon}{x^2 + \epsilon^2}$$

$$\frac{x}{x^2 + \epsilon^2} = \frac{1}{2}\frac{d}{dx}\ln(x^2 + \epsilon^2) = P\left(\frac{1}{x}\right) \qquad \texttt{(principal value)}$$

$$\frac{\epsilon}{x^2 + \epsilon^2} = \pi\;\delta(x)$$

so that

$$\frac{1}{x - i\epsilon} = P\left(\frac{1}{x}\right) \qquad \texttt{(real part)}$$

$$+\,i\;\pi\;\delta(x) \qquad \texttt{(imaginary part)}$$

Accordingly, for a Feynman propagator one has

$$\frac{1}{k^2 + m^2 - i\epsilon} = P\left(\frac{1}{k^2 + m^2}\right) \qquad \texttt{(real part)}$$
$$+\, i\, \pi\, \delta(k^2 + m^2) \qquad \texttt{(imaginary part)}$$

In the above formula, m^2 is real (and positive), k^2 is a (real) quadratic expression in the *real* components of the d-dimensional vector k_μ, which in turn are a combination of the *real* components of the external momenta and of the loop vectors.
The integration on all the components of the loop momenta runs along the whole *real* axis from $-\infty$ to $+\infty$.

The Integration by Parts Identities are obtained by manipulations of the vectors occurring in the considered Feynman amplitude, while the masses and the accompanying $i\epsilon$ do not play any active role.
If an identity can be written, say, in the form

$$\int \mathfrak{D}k \ \frac{1}{k^2 + m^2 - i\epsilon} \times F(k, ...) = 0 \ ,$$

where $F(k, ...)$ is a suitable rational expression, involving some polynomials built with the scalar products of the occurring vectors and the other propagators of the considered graph, performing from the very beginning the replacement

$$\frac{1}{k^2 + m^2 - i\epsilon} \rightarrow \frac{1}{k^2 + m^2 + i\epsilon} \ ,$$

one would obtain a new Intagration by Parts Identity

$$\int \mathfrak{D}k \ \frac{1}{k^2 + m^2 + i\epsilon} \times F(k, ...) = 0 \ .$$

Obviously

$$\frac{1}{k^2 + m^2 + i\epsilon} = P\left(\frac{1}{k^2 + m^2}\right) - i\,\pi\,\delta(k^2 + m^2)$$

By subtracting the two IbP-identities one obtains (up to an overall factor $2\pi i$)

$$\int \mathfrak{D}k\ \delta(k^2 + m^2) \times F(k, ...) = 0\ ,$$

expressing the fact that the IbP-id valid for some propagator gives an identity with the same structure if the propagator is replaced by the corresponding δ-function.

Differential equations are a straightforward consequence of the IbP-id's; therefore one obtains a system of differential equations of the same structure if any propagator is replaced by the corresponding δ-function (or principal value).

Same structure however, does not implies exactly identical.

Indeed, the expression $F(k, ...)$ may contain, among its many terms, a numerator $(k^2 + m^2)$; and one has quite in general

$$\frac{1}{k^2 + m^2 - i\epsilon}\ (k^2 + m^2) = 1\ ,$$
$$\frac{1}{k^2 + m^2 + i\epsilon}\ (k^2 + m^2) = 1\ ,$$

but
$$\delta(k^2 + m^2)\ (k^2 + m^2) = 0\ .$$

The first line corresponds to a non vanishing term in the identity in which the propagator

$$\frac{1}{k^2 + m^2 - i\epsilon}$$

is absent (such term is usually referred to as a subtopology-amplitude).

The third line

$$\delta(k^2 + m^2)\ (k^2 + m^2) = 0$$

tells us that the term with the subtopology is absent when that propagator is replaced by the corresponding δ-function; the identity (or the differential equation) for the amplitude which involves a δ-function is obviously simpler than the original identity involving the full propagator.

The same applies to the differential equation, whose derivation relies on IBP identities;
the differential equation for the amplitude with a δ-function is simpler than the equation for the original amplitude, as some of the inhomogeneous terms are necessarily absent;
the simpler equation is easier to study, and can provide a (useful) hint for solving the original full equation.

Obvious idea: replace as many propagators as possible by the corresponding δ-functions.

A philological [?] digression.

The replacement of a full Propagator by its δ-function is represented, in the drawing of a Feynman graph, by cutting the line corresponding to that propagator. As cut propagators appears in the representation of imaginary parts of a Feynman amplitude as provided by the Cutkosky-Veltman cutting rules, cut propagators are often (but somewhat unproperly) called unitarity cuts
Indeed, a δ-function

$$\delta(k^2 + m^2) ,$$

as its argument is quadratic in the components of the vector k_μ, picks up, in general, two different values of one of the components of the vector, say for instance k_0,
while a unitarity cuts involves rather something like

$$\theta(k_0)\delta(k^2 + m^2) ,$$

which selects only one of the two values of k_0 (the socalled *proper* value).

Generalised cut is therefore more appropriate than unitarity cut.
End of the digression.

Using *cut* amplitudes (imaginary or generalised cuts):

The *imaginary part* of an amplitude is a solution of the *imaginary part* of the differential equation satisfied by that amplitude;

in the imaginary part of the complete equation the inhomogeneous terms not developping an imaginary part (such as the tadpoles) drop out, but other inhomogeneous terms can survive;

the *imaginary part* of a Feynman amplitude is given by the *unitarity cuts* of a suitable *subset* only of its propagator lines
(Cutkosky-Veltman rule);

the actual calculation of such an imaginary part is surely easier than the calculation of the whole amplitude

– but the task can remain highly non trivial.

In a *maximally cut* graph, *all* the propagators are replaced by the corresponding δ-functions;

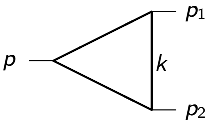
a *maximally cut* graph is a solution of the *homogeneous* part of the differential equation satisfied by the whole amplitude;

a *maximally cut* graph can be also relatively easy to evaluate; (in any case, it is the *easiest* thing to evaluate)

in general, *unitary cuts* are *not maximally cut* graphs; (with important exceptions; 1-loop bubble, sunrise, "banana", ...)

but one can consider, at least in principle, *maximally cut* graphs by considering *generalised cut* graphs.

To illustrate the difference between unitarity and maximally generalised-cut grpahs, consider the 1-loop vertex amplitude,



$p \to p_1 + p_2$, p timelike,
p_1 and p_2 massless, $p_1^2 = p_2^2 = 0$,
in the *c.m.s.* of p, $(p_0 = 2E, \vec{p} = 0)$,
$(p_{10} = E, p_{1x} = E)$, $(p_{20} = E, p_{2x} = -E)$,
internal masses equal to m,
loop momentum k in d-continuous dimensions

The denominators of the three propagators can be written as

$$D_k = k^2 + m^2 = -k_0^2 + k_x^2 + k_y^2 + k_d^2 + m^2 \ ,$$
$$D_1 = (p_1 + k)^2 + m^2 = -2Ek_0 + 2Ek_x + D_k \ ,$$
$$D_2 = (p_2 - k)^2 + m^2 = +2Ek_0 + 2Ek_x + D_k \ ,$$

where k_d^2 stands for the sum of the squares of all the other components of k besides k_0, k_x.

The unitarity cut is (apart from overall factors)

$$\int \mathfrak{D}k \ \theta(E + k_0)\delta(D_1) \ \theta(E - k_0)\delta(D_2) \ \frac{1}{D_k - i\epsilon}$$

where $\mathfrak{D}k$ is the integration measure

$$\mathfrak{D}k = dk_0 \ dk_x \ dk_y \ (d^d k_d)$$

and $(d^d k_d)$ stands for all the remaining $(d-3)$ (continuous) dimensions.

The two δ-functions, once integrate on k_0 and k_x, fix

$$k_0 = 0$$

and

$$k_x = -E \pm \sqrt{E^2 - k_y^2 - k_d^2 - m^2}\ ;$$

the imaginary part becomes

$$\int \frac{dk_y\ (d^d k_d)}{4E\sqrt{E^2 - k_y^2 - k_d^2 - m^2}}\ \frac{1}{D_k - i\epsilon}$$

and the k_y integration

$$\int \frac{dk_y}{\sqrt{E^2 - k_y^2 - k_d^2 - m^2}} \frac{1}{D_k - i\epsilon}$$

is still to be carried out.

Let us now look at the maximal generalised cut.

The maximally (generalised)cut amplitude is (apart from overall factors)

$$\int \mathfrak{D}k \ \delta(D_1) \ \delta(D_2) \ \delta(D_k)$$

The three δ-functions give the three equations

$$\begin{aligned} D_k &= -k_0^2 + k_x^2 + k_y^2 + k_d^2 + m^2 = 0 \ , \\ D_1 &= -2Ek_0 + 2Ek_x + D_k = 0 \ , \\ D_2 &= +2Ek_0 + 2Ek_x + D_k = 0 \ , \end{aligned}$$

whose solutions are

$$\begin{aligned} k_0 &= 0 \ , \\ k_x &= 0 \ , \\ k_y^2 + k_d^2 + m^2 &= 0 \ . \end{aligned}$$

The third equation

$$k_y^2 + k_d^2 + m^2 = 0$$

requires

$$k_y^2 = -(k_d^2 + m^2) \ ,$$

which is manifestly impossible to satisfy, as long as the integration variable k_y runs along the real axis from $-\infty$ and $+\infty$; therefore, tha maximal cut gives

$$\int \mathfrak{D}k \ \delta(D_1) \ \delta(D_2) \ \delta(D_k) = 0$$

To get a non-trivial result, a substitution such as, say,

$$k_y^2 \to -K_y^2$$

seems to be needed.

But that is a Wick rotation (or perhaps a *counter*-rotation), concerning not the component k_0 (as usually) but k_y.

The Wick (counter)-rotation for k_y consists of the substitutions

$$\int_{-\infty}^{\infty} dk_y \to \int_{-\infty}^{\infty} (-iK_y) \; ,$$
$$k_z^2 \to -K_z^2$$

correspondingly,

$$D_k = -k_0^2 + k_x^2 + k_y^2 + k_d^2 + m^2$$

becomes

$$D_k = -k_0^2 + k_x^2 - K_y^2 + k_d^2 + m^2$$

and the solution of the conditions imposed by the three δ-functions is

$$k_0 = 0 \; ,$$
$$k_x = 0 \; ,$$
$$K_y^2 = +k_d^2 + m^2 \; ,$$

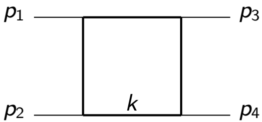
where k_0, k_x, and also K_y are all real.

The evaluation of the maximal is the immediate; up to an overall constants, the result is

$$\int dk_0\ dk_x\ dK_y\ (d^d k_d)\ \delta(D_1)\ \delta(D_2)\ \delta(D_k) = \frac{1}{4E^2} \int \frac{(d^d k_d)}{\sqrt{k_d^2 + m^2}}$$

(note that the remaining integral, independent of E, can be reabsorbed in the overall constant).

Another example, the maximal (quadruple) cut of the QED-like “light-light” 1-loop scalar box



$$Q(s,t) = \int dk_0 dk_x dk_y \, dK_z \, \delta(D_k)\delta(D_2)\delta(D_4)\delta(D_P)$$

$$\texttt{kinematics:} \quad s = -(p_1 + p_2)^2 = 4E^2 > 4m^2 \; ;$$

$$t = -(p_1 - p_3)^2 = -4E^2 \sin^2 \frac{\theta}{2}$$

$$p_1^2 = 0 \; , \quad p_1 = (p_{10} = E, p_{1x} = E, p_{1y} = 0, 0)$$

$$p_2^2 = 0 \; , \quad p_2 = (E, -E, 0, 0)$$

$$p_3^2 = 0 \; , \quad p_3 = (E, E\cos\theta, E\sin\theta, 0)$$

$$p_4^2 = 0 \; , \quad p_4 = p_1 + p_2 - p_3 = (E, -E\cos\theta, -E\sin\theta, 0)$$

The conditions imposed by the δ's are

$$
\begin{aligned}
D_k &= k^2 + m^2 = -k_0^2 + k_x^2 + k_y^2 - K_z{}^2 + m^2 = 0 \\
D_2 &= (p_2 - k)^2 + m^2 = 2Ek_0 + 2Ek_x + D_k = 0 \\
D_4 &= (p_4 - k)^2 + m^2 = 2Ek_0 + 2Ek_x \cos\theta + 2Ek_y \sin\theta + D_k = 0 \\
D_P &= (p_1 + p_2 - k)^2 + m^2 = -4E^2 + 4Ek_0 + D_k = 0
\end{aligned}
$$

whose solution is

$$
\begin{aligned}
k_0 &= E \\
k_x &= -E \\
k_y &= -E\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} \\
K_z &= \sqrt{k_y^2 + m^2}
\end{aligned}
$$

Correspondingly

$$\int dk_0 dk_x dk_y dK_z \delta(D_k)\delta(D_2)\delta(D_4)\delta(D_P) = \frac{1}{2}\frac{1}{\sqrt{(-st)[4m^2(s+t)-st]}}$$

For comparison, the Imaginary part of the "light-light" box graph in the standard (no extra Wick rotation) variables reads

$$\mathtt{Im}A(s,t) = \int dk_0 dk_x dk_y dk_z\ \theta(k_0)\delta(D_k)\ \theta(2E-k_0)\delta(D_k)\ \frac{1}{D_2 D_4}$$
$$= \frac{\pi}{2\sqrt{(-st)[4m^2(s+t)-st]}} \ln \frac{2m^2 s-(s-4m^2)t+\sqrt{-(s-4m^2)t[4m^2(s+t)-st]}}{2m^2 s-(s-4m^2)t-\sqrt{-(s-4m^2)t[4m^2(s+t)-st]}}$$

Note the appearance (in both formulas) of the *Mandelstam square root*

$$\sqrt{(-st)[4m^2(s+t)-st]}$$

To see in more details how the Wick rotation works, consider the simplest [?] possible integral

$$\int_{-\infty}^{\infty} \frac{dk_z}{-k_0^2 + k_z^2 + m^2 - i\epsilon} = \frac{\pi}{\sqrt{m^2 - k_0^2 - i\epsilon}}$$

$$= \frac{\pi}{\sqrt{m^2 - k_0^2}} \ , \quad \texttt{real} \qquad \texttt{if} \ \ m^2 - k_0^2 > 0 \ ,$$

$$= i \frac{\pi}{\sqrt{k_0^2 - m^2}} \ , \quad \texttt{imaginary} \ \texttt{if} \ \ k_0^2 - m^2 > 0 \ .$$

In both cases, the result is the analytic continuation of the other case.

But one has also

$$\frac{1}{-k_0^2+k_z^2+m^2-i\epsilon} = P\left(\frac{1}{-k_0^2+k_z^2+m^2}\right) \quad \texttt{real}$$
$$+\, i\pi\delta(-k_0^2+k_z^2+m^2) \quad \texttt{imaginary}$$

$$\int_{-\infty}^{\infty} dk_z P\left(\frac{1}{-k_0^2+k_z^2+m^2}\right) = \frac{\pi}{\sqrt{m^2-k_0^2}}\,, \quad \texttt{real} \quad \texttt{if} \;\; m^2-k_0^2>0\,,$$
$$= 0 \qquad \texttt{if} \;\; k_0^2-m^2>0\,,$$

$$i\pi\int_{-\infty}^{\infty} dk_z\delta(-k_0^2+k_z^2+m^2) = 0 \qquad \texttt{if} \;\; m^2-k_0^2>0\,,$$
$$= i\frac{\pi}{\sqrt{k_0^2-m^2}} \quad \texttt{imaginary} \;\; \texttt{if} \;\; k_0^2-m^2>0\,,$$

The two results for the principal value, *obviously*, are not the analytical contuation of a same quantity (and the same for the integral of the δ-function).

Wick (counter)rotation of k_z.

$$\int_{-\infty}^{\infty} dk_z \to \int_{-\infty}^{\infty} -i\ dK_z\ , \qquad\qquad k_z^2 \to -K_z^2$$

$$\int_{-\infty}^{\infty} \frac{dk_z}{-k_0^2 + k_z^2 + m^2 - i\epsilon} \to \int_{-\infty}^{\infty} \frac{-i\ dK_z}{-k_0^2 - K_z^2 + m^2 - i\epsilon}$$

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{-i\ dK_z}{-k_0^2 - k_z^2 + m^2 - i\epsilon} &= i\frac{\pi}{\sqrt{k_0^2 - m^2 + i\epsilon}} \\
&= i\frac{\pi}{\sqrt{k_0^2 - m^2}}\ , \quad \texttt{imaginary}\ \texttt{if}\ \ k_0^2 - m^2 > 0\ . \\
&= \frac{\pi}{\sqrt{m^2 - k_0^2}}\ , \quad \texttt{real}\ \qquad \texttt{if}\ \ m^2 - k_0^2 > 0\ ,
\end{aligned}$$

The result is (*obviously*) identical to the not(counter)rotated case.

Summarising:

$$\int_{-\infty}^{\infty} \frac{dk_z}{-k_0^2 + k_z^2 + m^2 - i\epsilon} = \int_{-\infty}^{\infty} \frac{-i\ dK_z}{-k_0^2 - K_z^2 + m^2 - i\epsilon}$$
$$= \frac{\pi}{\sqrt{m^2 - k_0^2 - i\epsilon}}$$

The Wick (counter)rotation $k_z^2 \to -K_z^2$ if fully harmless if the external vectors have no z-component;
otherwise, one might have something like

$$\frac{1}{-k_0^2 + (k_z - q_z)^2 + m^2 - i\epsilon} \to \frac{1}{-k_0^2 - (K_z - iq_z)^2 + m^2 - i\epsilon} [?] \,,$$

with the imaginary quantity $iK_z q_z$ overwriting the $-i\epsilon$ at the root of the properties of the Feynman propagator (to be recovered, perhaps, with a suitable integration contour deformation).

With (still) more details:

$$\frac{1}{-k_0^2+k_z^2+m^2-i\epsilon} = P\left(\frac{1}{-k_0^2+k_z^2+m^2}\right) + i\pi\delta(-k_0^2+k_z^2+m^2)\ ,$$

$$\frac{1}{-k_0^2-K_z^2+m^2-i\epsilon} = P\left(\frac{1}{-k_0^2-K_z^2+m^2}\right) + i\pi\delta(-k_0^2-K_z^2+m^2)\ ,$$

and the results for the integrals in (dk_z) or $(-idK_z)$ of principal values and δ-functions, taken separately, are *not* equal.

explicit results: 1)

$$k_0^2 - m^2 > 0$$

$$\int_{-\infty}^{\infty} (dk_z)\ \delta(-k_0^2 + k_z^2 + m^2) = \frac{1}{\sqrt{k_0^2 - m^2}}$$

$$\int_{-\infty}^{\infty} (-i\ dK_z)\ \delta(-k_0^2 - K_z^2 + m^2) = 0$$

$$\int_{-\infty}^{\infty} (dk_z)\ P\left(\frac{1}{-k_0^2 + k_z^2 + m^2}\right) = 0$$

$$\int_{-\infty}^{\infty} (-i\ dK_z)\ P\left(\frac{1}{-k_0^2 - K_z^2 + m^2}\right) = i\ \frac{\pi}{\sqrt{k_0^2 - m^2}}$$

explicit results: 2)

$$m^2 - k_0^2 > 0$$

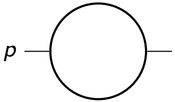
$$\int_{-\infty}^{\infty} (dk_z)\ \delta(-k_0^2 + k_z^2 + m^2) = 0$$

$$\int_{-\infty}^{\infty} (-i\ dK_z)\ \delta(-k_0^2 - K_z^2 + m^2) = -i \frac{1}{\sqrt{m^2 - k_0^2}}$$

$$\int_{-\infty}^{\infty} (dk_z)\ P\left(\frac{1}{-k_0^2 + k_z^2 + m^2}\right) = \frac{\pi}{\sqrt{m^2 - k_0^2}}$$

$$\int_{-\infty}^{\infty} (-i\ dK_z)\ P\left(\frac{1}{-k_0^2 - K_z^2 + m^2}\right) = 0$$

As a less trivial example, consider the 1-loop scalar *Bubble* amplitude with two different masses M, m, $M > m$, in $d = 2$ dimensions



$$= B(s) = -i \int_{-\infty}^{\infty} dk_0 \int_{-\infty}^{\infty} dk_z \, \frac{1}{(D_m - i\epsilon)(D_M - i\epsilon)} \ ,$$

where

$$s = -p^2 = E^2 \ , \quad p \ \ \texttt{timelike}, \quad p_0 = E > 0, \ p_z = 0 \ ,$$
$$D_m = -k_0^2 + k_z^2 + m^2 \ ,$$
$$D_M = -(k_0 - E)^2 + k_z^2 + M^2 \ .$$

After partial fractioning in k_z^2

$$\frac{1}{(D_m - i\epsilon)(D_M - i\epsilon)} = \frac{1}{2Ek_0 - E^2 + M^2 - m^2}\left(\frac{1}{D_m - i\epsilon} - \frac{1}{D_M - i\epsilon}\right)$$

and k_z integration

$$\int_{-\infty}^{\infty} \frac{dk_z}{D_m - i\epsilon} = \int_{-\infty}^{\infty} \frac{dk_z}{-k_0^2 + k_z^2 + m^2 - i\epsilon} = \frac{\pi}{\sqrt{m^2 - k_0^2 - i\epsilon}}$$

$$\int_{-\infty}^{\infty} \frac{dk_z}{D_M - i\epsilon} = \int_{-\infty}^{\infty} \frac{dk_z}{-(k_0 - E)^2 + k_z^2 + M^2 - i\epsilon} = \frac{\pi}{\sqrt{M^2 - (k_0 - E)^2 - i\epsilon}}$$

one arrives at

$$B(E^2) = \int_{-\infty}^{\infty} \frac{-i\ dk_0}{2Ek_0 - E^2 + M^2 - m^2} \times \left(\frac{\pi}{\sqrt{m^2 - k_0^2 - i\epsilon}} - \frac{\pi}{\sqrt{M^2 - (k_0 - E)^2 - i\epsilon}}\right)$$

The actual integration in k_0 is elementary (even if not *completely trivial...*) and the (known) result(s) are recovered

$$0 < U < M - m, \quad B(U^2) = \frac{2\pi}{\sqrt{(M+m)^2 - U^2}\sqrt{(M-m)^2 - U^2}} \times \ln \frac{\sqrt{(M+m)^2 - U^2} + \sqrt{(M-m)^2 - U^2}}{\sqrt{(M+m)^2 - U^2} - \sqrt{(M-m)^2 - U^2}}$$

$$M - m < W < M + m, \quad B(W^2) = \frac{2\pi \; i}{\sqrt{(M+m)^2 - W^2}\sqrt{W^2 - (M-m)^2}} \times \ln \frac{\sqrt{(M+m)^2 - W^2} - i\sqrt{W^2 - (M-m)^2}}{\sqrt{(M+m)^2 - W^2} + i\sqrt{W^2 - (M-m)^2}}$$

$$M + m < Z < \infty, \quad B(Z^2) = \frac{2\pi}{\sqrt{Z^2 - (M+m)^2}\sqrt{Z^2 - (M-m)^2}} \times \left[\ln \frac{\sqrt{Z^2 - (M+m)^2} - \sqrt{Z^2 - (M-m)^2}}{\sqrt{Z^2 - (M+m)^2} + \sqrt{Z^2 - (M-m)^2}} + i\,\pi \right]$$

The values of $B(W^2)$ and $B(Z^2)$ can be obtained (obviously) by analytical continuation from the value of $B(U^2)$.

One can also use the decompositions

$$\frac{1}{D_m - i\epsilon} = P\left(\frac{1}{D_m} + i\pi\delta(D_m)\right), \qquad \frac{1}{D_M - i\epsilon} = P\left(\frac{1}{D_M} + i\pi\delta(D_M)\right),$$

and then consider separately the four quantities

$$\begin{aligned}
B(P_m P_M, E^2) &= -i\int_{-\infty}^{\infty} dk_0 \int_{-\infty}^{\infty} dk_z P\left(\frac{1}{D_m}\right) P\left(\frac{1}{D_M}\right) \quad \textbf{imaginary} \\
B(P_m \delta_M, E^2) &= -i\int_{-\infty}^{\infty} dk_0 \int_{-\infty}^{\infty} dk_z P\left(\frac{1}{D_m}\right)\ i\pi\delta(D_M) \quad \textbf{real} \\
B(\delta_m P_M, E^2) &= -i\int_{-\infty}^{\infty} dk_0 \int_{-\infty}^{\infty} dk_z\ i\pi\delta(D_m)\ P\left(\frac{1}{D_M}\right) \quad \textbf{real} \\
B(\delta_m \delta_M, E^2) &= -i\int_{-\infty}^{\infty} dk_0 \int_{-\infty}^{\infty} dk_z\ i\pi\delta(D_m)\ i\pi\delta(D_M) \quad \textbf{imaginary}
\end{aligned}$$

whose sum is, by construction, $B(E^2)$

$$B(E^2) = B(P_m P_M, E^2) + B(P_m \delta_M, E^2) + B(\delta_m P_M, E^2) + B(\delta_m \delta_M, E^2)\ .$$

The same detailed decomposition can be carried out by using the Wick *counter*-rotation:

$$B^W(E^2) = -i\int_{-\infty}^{\infty} dk_0 \int_{-\infty}^{\infty} (-i dK_z) \frac{1}{(D_m^W - i\epsilon)(D_M^W - i\epsilon)} ,$$

$$D_m^W = -k_0^2 - K_z^2 + m^2, \qquad D_M^W = -(k_0 - E)^2 - K_z^2 + M^2 ,$$

$$\frac{1}{D_m^W - i\epsilon} = P\left(\frac{1}{D_m^W} + i\pi\delta(D_m^W)\right), \qquad \frac{1}{D_M^W - i\epsilon} = P\left(\frac{1}{D_M^W} + i\pi\delta(D_M^W)\right),$$

$$B^W(P_m P_M, E^2) = -i\int_{-\infty}^{\infty} dk_0 \int_{-\infty}^{\infty} (-i dK_z) P\left(\frac{1}{D_m^W}\right) P\left(\frac{1}{D_M^W}\right) \quad \texttt{real}$$

$$B^W(P_m \delta_M, E^2) = -i\int_{-\infty}^{\infty} dk_0 \int_{-\infty}^{\infty} (-i dK_z) P\left(\frac{1}{D_m^W}\right) \; i\pi\delta(D_M^W) \quad \texttt{imaginary}$$

$$B^W(\delta_m P_M, E^2) = -i\int_{-\infty}^{\infty} dk_0 \int_{-\infty}^{\infty} (-i dK_z) \; i\pi\delta(D_m^W) \; P\left(\frac{1}{D_M^W}\right) \quad \texttt{imaginary}$$

$$B^W(\delta_m \delta_M, E^2) = -i\int_{-\infty}^{\infty} dk_0 \int_{-\infty}^{\infty} (-i dK_z) \; i\pi\delta(D_m^W) \; i\pi\delta(D_M^W) \quad \texttt{real}$$

whose sum is

$$B^W(E^2) = B^W(P_m P_M, E^2) + B^W(P_m \delta_M, E^2) + B^W(\delta_m P_M, E^2) + B^W(\delta_m \delta_M, E^2) .$$

Not surprisingly, by using the Wick (counter)rotated variable K_z instead of k_z, after the dK_z integration one obtains the same representation of $B(E^2)$ as an integral in dk_0, so that

$$B^W(E^2) = B(E^2) \ ,$$

as (obviously) expected.
But what about the various separate contributions
$B(P_m P_M, E^2), B^W(P_m P_M, E^2), \ldots$ *etc.* ?

Let us list the results of the explicit calculation.

$$0 < U < M - m$$

$$\sqrt{R(U^2)} = \sqrt{(M+m)^2 - U^2}\sqrt{(M-m)^2 - U^2}$$

$$\sqrt{R(U^2)} \times B(P_m P_M, U^2) = -i\pi^2$$

$$\sqrt{R(U^2)} \times B^W(P_m P_M, U^2) = +2\pi \ln \frac{\sqrt{(M+m)^2 - U^2} + \sqrt{(M-m)^2 - U^2}}{\sqrt{(M+m)^2 - U^2} - \sqrt{(M-m)^2 - U^2}}$$

$$\sqrt{R(U^2)} \times B(P_m \delta_M, U^2) = -2\pi \ln \frac{\sqrt{(U+M)^2 - m^2} + \sqrt{(U-M)^2 - m^2}}{\sqrt{(U+M)^2 - m^2} - \sqrt{(U-M)^2 - m^2}}$$

$$\sqrt{R(U^2)} \times B^W(P_m \delta_M, U^2) = +i\pi^2$$

$$\sqrt{R(U^2)} \times B(\delta_m P_M, U^2) = +2\pi \ln \frac{\sqrt{M^2 - (U-m)^2} + \sqrt{M^2 - (U+m)^2}}{\sqrt{M^2 - (U-m)^2} - \sqrt{M^2 - (U+m)^2}}$$

$$\sqrt{R(U^2)} \times B^W(\delta_m P_M, U^2) = -i\pi^2$$

$$\sqrt{R(U^2)} \times B(\delta_m \delta_M, U^2) = +i\pi^2$$

$$\sqrt{R(U^2)} \times B^W(\delta_m \delta_M, U^2) = 0$$

$$M - m < W < M + m$$

$$\sqrt{R(W^2)} = \sqrt{(M+m)^2 - W^2}\sqrt{W^2 - (M-m)^2}$$

$$\sqrt{R(W^2)} \times B(P_m P_M, W^2) = 0$$

$$\sqrt{R(W^2)} \times B^{W}(P_m P_M, W^2) = -2\pi i \ln \frac{\sqrt{(M+m)^2 - W^2} + i\sqrt{W^2 - (M-m)^2}}{\sqrt{(M+m)^2 - W^2} - i\sqrt{W^2 - (M-m)^2}} - \pi^2$$

$$\sqrt{R(W^2)} \times B(P_m \delta_M, W^2) = -2\pi i \ln \frac{\sqrt{(W+M)^2 - m^2} - i\sqrt{m^2 - (W-M)^2}}{\sqrt{m^2 - (W-M)^2} - i\sqrt{(W+M)^2 - m^2}}$$

$$\sqrt{R(W^2)} \times B^{W}(P_m \delta_M, W^2) = 0$$

$$\sqrt{R(W^2)} \times B(\delta_m P_M, W^2) = -2\pi i \ln \frac{\sqrt{M^2 - (W+m)^2} + i\sqrt{(W+m)^2 - M^2}}{\sqrt{(W+m)^2 - M^2} + i\sqrt{M^2 - (W+m)^2}}$$

$$\sqrt{R(W^2)} \times B^{W}(\delta_m P_M, W^2) = 0$$

$$\sqrt{R(W^2)} \times B(\delta_m \delta_M, W^2) = 0$$

$$\sqrt{R(W^2)} \times B^{W}(\delta_m \delta_M, W^2) = +\pi^2$$

$$M + m < Z < \infty$$

$$\sqrt{R(Z^2)} = \sqrt{Z^2 - (M+m)^2}\sqrt{Z^2 - (M-m)^2}$$

$$\sqrt{R(Z^2)} \times B(P_m P_M, Z^2) = i\pi^2$$

$$\sqrt{R(Z^2)} \times B^W(P_m P_M, Z^2) = -2\pi \ln \frac{\sqrt{Z^2-(M-m)^2} + \sqrt{Z^2-(M+m)^2}}{\sqrt{Z^2-(M-m)^2} - \sqrt{Z^2-(M+m)^2}}$$

$$\sqrt{R(Z^2)} \times B(P_m \delta_M, Z^2) = -2\pi \ln \frac{\sqrt{(Z+M)^2-m^2} + \sqrt{(Z-M)^2-m^2}}{\sqrt{(Z+M)^2-m^2} - \sqrt{(Z-M)^2-m^2}}$$

$$\sqrt{R(Z^2)} \times B^W(P_m \delta_M, Z^2) = i\pi^2$$

$$\sqrt{R(Z^2)} \times B(\delta_m P_M, Z^2) = -2\pi \ln \frac{\sqrt{(Z+m)^2-M^2} + \sqrt{(Z-m)^2-M^2}}{\sqrt{(Z+m)^2-M^2} + \sqrt{(Z-m)^2-M^2}}$$

$$\sqrt{R(Z^2)} \times B^W(\delta_m P_M, Z^2) = i\pi^2$$

$$\sqrt{R(Z^2)} \times B(\delta_m \delta_M, Z^2) = i\pi^2$$

$$\sqrt{R(Z^2)} \times B^W(\delta_m \delta_M, Z^2) = 0$$

besides the general reshuffling of the separate contributions, when looking in particular at the *generalised cuts* of the Bubble amplitude one finds

$$0 < U < M - m \qquad \sqrt{R(U^2)} \times B(\delta_m \delta_M, U^2) = +i\pi^2$$
$$\sqrt{R(U^2)} \times B^W(\delta_m \delta_M, U^2) = 0$$

$$M - m < W < M + m \qquad \sqrt{R(W^2)} \times B(\delta_m \delta_M, W^2) = 0$$
$$\sqrt{R(W^2)} \times B^W(\delta_m \delta_M, W^2) = +\pi^2$$

$$M + m < Z < \infty \qquad \sqrt{R(Z^2)} \times B(\delta_m \delta_M, Z^2) = i\pi^2$$
$$\sqrt{R(Z^2)} \times B^W(\delta_m \delta_M, Z^2) = 0$$

showing that the "conventional" *generalised cuts* are different from zero in the two regions $0 < U < M - m, M + m < Z < \infty$, but vanish in the region $M - m < W < M + m$, while the *Wick generalised cuts*, somewhat complementarily, vanish in the regions $0 < U < M - m, M + m < Z < \infty$ and are different from zero in the region $M - m < W < M + m$.

- conclusion:

one more Wick (counter)-rotation may help when dealing with generalised cuts