Hopf algebra for Feynman diagrams and integrals

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Loops and Legs in Quantum Field Theory, 2018

Based on work with Samuel Abreu, Claude Duhr, Einan Gardi, and James Matthew

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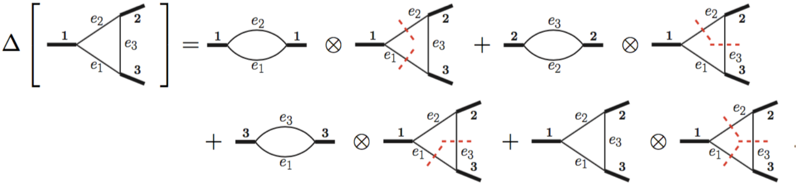
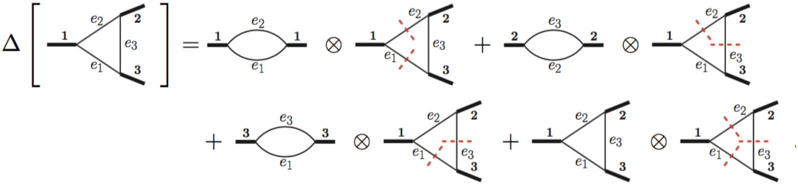
- Feynman integrals have a Hopf algebra structure that maps the functions to simpler ones in a way that exposes their behavior in differential equations and discontinuities
- At 1 loop, this Hopf algebra is consistent with the Hopf algebra of multiple polylogarithms, in the ϵ-expansion of Feynman integrals in dimensional regularization
- Also consistent with emerging Hopf algebras on hypergeometric functions in closed forms
- Seek general Hopf algebra on families of integrals

Hopf algebra operations



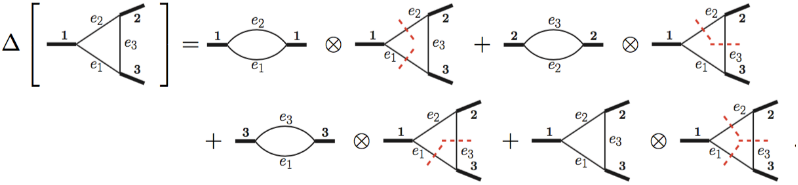
Hopf algebra operations



$$
\begin{aligned}
\Delta(\log z) &= 1 \otimes \log z + \log z \otimes 1 \\
\Delta(\log^2 z) &= 1 \otimes \log^2 z + 2 \log z \otimes \log z + \log^2 z \otimes 1 \\
\Delta(\mathrm{Li}_2(z)) &= 1 \otimes \mathrm{Li}_2(z) + \mathrm{Li}_2(z) \otimes 1 + \mathrm{Li}_1(z) \otimes \log z
\end{aligned}
$$

$$
\mathrm{Li}_2(z) = -\int_0^z \frac{\log(1-t)}{t}\,dt
$$

Hopf algebra operations



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\end{aligned}
$$

Discontinuities and cuts:

$$\Delta \ \mathrm{Disc} = (\mathrm{Disc} \otimes 1)\, \Delta$$

Differential operators:

$$\Delta\, \partial = (1 \otimes \partial)\, \Delta$$

Master formula for Hopf algebra on integrals

Coactions of the following form:

$$\Delta\left(\int_\gamma \omega\right) = \sum_i \int_\gamma \omega_i \otimes \int_{\gamma_i} \omega$$

with a duality condition

$$P_{ss} \int_{\gamma_i} \omega_j = \delta_{ij} \, .$$

Master formula for Hopf algebra on integrals

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$$\Delta\left(\int_\gamma \omega\right) = \sum_i \int_\gamma \omega_i \otimes \int_{\gamma_i} \omega$$

with a duality condition

$$P_{ss} \int_{\gamma_i} \omega_j = \delta_{ij}\,.$$

P_{ss} is semi-simple projection ("drop logarithms but not π").

The master formula coaction is like inserting a complete set of states ("ω_i are a set of master integrands for ω").

An algebra H is a ring (addition group & multiplication)
which has a multiplicative unit (1)
and which is also a vector space over a field K.

Example: $n \times n$ matrices with entries in K.
In this talk, the field is always $K = \mathbb{Q}$.

Algebras and bialgebras

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and which is also a vector space over a field K.

Example: $n \times n$ matrices with entries in K.
In this talk, the field is always $K = \mathbb{Q}$.

A bialgebra is an algebra H with two maps, the coaction $\Delta : H \to H \otimes H$, and the counit $\varepsilon : H \to \mathbb{Q}$, satisfying the following axioms.

- Coassociativity: $(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta$
- Δ and ε are algebra homomorphisms:
  $\Delta(a \cdot b) = \Delta(a) \cdot \Delta(b) \;\text{ and }\; \varepsilon(a \cdot b) = \varepsilon(a) \cdot \varepsilon(b)$
- The counit and the coaction are related by $(\varepsilon \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \varepsilon)\Delta = \mathrm{id}$

The "incidence" bialgebra [Joni, Rota]

A simple combinatorial algebra: let $[\mathbf{n}] = \{\mathbf{1}, \mathbf{2}, \ldots, \mathbf{n}\}$.
Elements: pairs of nested subsets $S \subseteq T$, where $S \subseteq T \subseteq [\mathbf{n}]$.

$\{\mathbf{1}\} \subseteq \{\mathbf{1}, \mathbf{2}\}$ represented by $\mathbf{1}\,\mathbf{2}$
$\emptyset \subseteq \{\mathbf{1}, \mathbf{2}\}$ represented by $\mathbf{1}\,\mathbf{2}$
$\emptyset \subset \emptyset$ represented by $*$

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$\emptyset \subseteq \{\mathbf{1}, \mathbf{2}\}$ represented by $\mathbf{1}\,\mathbf{2}$
$\emptyset \subset \emptyset$ represented by $\color{purple}{*}$

Multiplication is free, and the coaction is defined by

$$\Delta(S \subseteq T) = \sum_{S \subseteq X \subseteq T} (S \subseteq X) \otimes (X \subseteq T).$$

For example:

$$\begin{aligned}
\Delta(\mathbf{1}\,\mathbf{2}) &= \mathbf{1}\,\mathbf{2} \otimes \color{purple}{\mathbf{1}\,\mathbf{2}}\color{black} + \mathbf{1} \otimes \color{purple}{\mathbf{1}}\color{black}\,\mathbf{2} + \mathbf{2} \otimes \mathbf{1}\,\color{purple}{\mathbf{2}}\color{black} + * \otimes \mathbf{1}\,\mathbf{2} \\
\Delta(\mathbf{1}\,\color{purple}{\mathbf{2}}\color{black}) &= \mathbf{1}\,\color{purple}{\mathbf{2}}\color{black} \otimes \color{purple}{\mathbf{1}\,\mathbf{2}}\color{black} + \color{purple}{\mathbf{2}}\color{black} \otimes \mathbf{1}\,\color{purple}{\mathbf{2}}\color{black} \\
\Delta(\mathbf{2}) &= \mathbf{2} \otimes \color{purple}{\mathbf{2}}\color{black} + * \otimes \mathbf{2} \\
\Delta(\color{purple}{\mathbf{2}}\color{black}) &= \color{purple}{\mathbf{2}}\color{black} \otimes \color{purple}{\mathbf{2}} \\
\Delta(S \subseteq S) &= (S \subseteq S) \otimes (S \subseteq S)
\end{aligned}$$

The "incidence" bialgebra [Joni, Rota]

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Multiplication is free, and the coaction is defined by

$$\Delta(S \subseteq T) = \sum_{S \subseteq X \subseteq T} (S \subseteq X) \otimes (X \subseteq T).$$

The counit is

$$\varepsilon(S \subseteq T) = \begin{cases} 1\,, & \text{if } S = T\,, \\ 0\,, & \text{otherwise}\,. \end{cases}$$

e.g. $\varepsilon(\mathbf{2}) = 0, \quad \varepsilon(\mathbf{1}\,\mathbf{2}) = 0, \quad \varepsilon(\mathbf{2}) = 1, \quad \varepsilon(*) = 1$

Illustration of axioms

$$\begin{aligned}\Delta(1\,2) &= 1\,2 \otimes 1\,2 + 1 \otimes 1\,2 + 2 \otimes 1\,2 + * \otimes 1\,2\\ \Delta(2) &= 2 \otimes 2 + * \otimes 2\\ \Delta(2) &= 2 \otimes 2\\ \Delta(*) &= * \otimes *\end{aligned}$$

- Coassociativity of the coaction, $(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta$

$$\begin{aligned}(\Delta \otimes \mathrm{id})\Delta(2) &= \Delta(2) \otimes 2 + \Delta(*) \otimes 2\\ &= 2 \otimes 2 \otimes 2 + * \otimes 2 \otimes 2 + * \otimes * \otimes 2\\ &= 2 \otimes \Delta(2) + * \otimes \Delta(2)\\ &= (\mathrm{id} \otimes \Delta)\Delta(2)\end{aligned}$$

- Counit, $(\varepsilon \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \varepsilon)\Delta = \mathrm{id}$

$$\varepsilon(2) \otimes 2 + \varepsilon(*) \otimes 2 = 2 \otimes \varepsilon(2) + * \otimes \varepsilon(2) = 2$$

Hopf algebras

A Hopf algebra is a bialgebra H with an *antipode* map $S : H \to H$ that satisfies

$$\begin{tikzcd}
& H \otimes H \arrow[rr, "S \otimes \mathrm{id}"] & & H \otimes H \arrow[dr, "\mu"] & \\
H \arrow[ur, "\Delta"] \arrow[dr, "\Delta"'] \arrow[rr, "\varepsilon"] & & K \arrow[rr, "\eta"] & & H \\
& H \otimes H \arrow[rr, "\mathrm{id} \otimes S"'] & & H \otimes H \arrow[ur, "\mu"'] &
\end{tikzcd}$$

(Here μ, η denote multiplication and inclusion, respectively.)

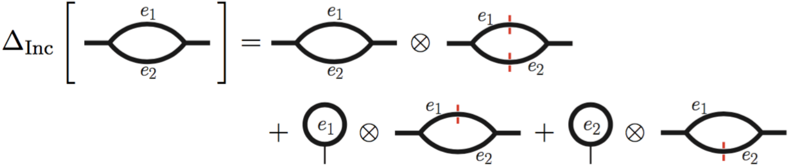
The incidence bialgebra becomes a Hopf algebra if we adjoin inverse elements $(S \subseteq S)^{-1}$.

Example of the incidence algebra: edges of graphs

$$\Delta(1\,2) \quad = \quad 1\,2 \otimes 1\,2 + 1 \otimes 1\,2 + 2 \otimes 1\,2 + * \otimes 1\,2$$

For graphs, set $* = (\emptyset \subseteq \emptyset) = 0$.

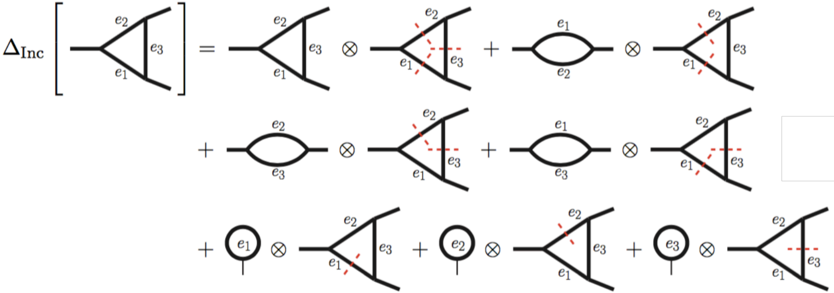
Pinch and cut *complementary* subsets of edges:



Example of the incidence algebra: edges of graphs

$$
\begin{aligned}
\Delta(1\,2\,3) \quad = \quad & 1\,2\,3 \otimes 1\,2\,3 + 1\,2 \otimes 1\,2\,3 + 2\,3 \otimes 1\,2\,3 + 1\,3 \otimes 1\,2\,3 \\
& + 1 \otimes 1\,2\,3 + 2 \otimes 1\,2\,3 + 3 \otimes 1\,2\,3 + * \otimes 1\,2\,3
\end{aligned}
$$

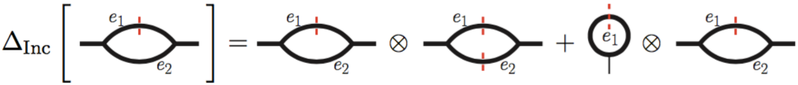
Pinch and cut *complementary* subsets of edges:



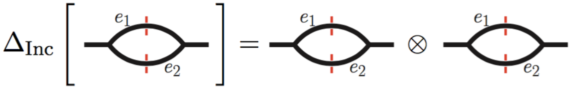
Example of the incidence algebra: edges of graphs

Can also start with a cut diagram.

$$\Delta(1\,2) \quad = \quad 1\,2 \otimes 1\,2 + 1 \otimes 1\,2$$



$$\Delta(1\,2) \quad = \quad 1\,2 \otimes 1\,2$$



Multiple polylogarithms (MPL)

A large class of iterated integrals are described by multiple polylogarithms:

$$G(a_1, \ldots, a_n; z) = \int_0^z \frac{dt}{t - a_1} G(a_2, \ldots, a_n; t)$$

Examples:

$$G(0; z) = \log z, \quad G(a; z) = \log\left(1 - \frac{z}{a}\right)$$
$$G(\vec{a}_n; z) = \frac{1}{n!} \log^n\left(1 - \frac{z}{a}\right), \quad G(\vec{0}_{n-1}, a; z) = -\mathrm{Li}_n\left(\frac{z}{a}\right)$$

Harmonic polylog if all $a_i \in \{-1, 0, 1\}$.
n is the *transcendental weight*.

Many Feynman integrals can be written in terms of polylogs in the Laurent expansion of dimensional regularization.

Hopf algebra of MPL [Goncharov; Duhr]

Closure under multiplication via the shuffle product:

$$G(\vec{a}_1; z)\, G(\vec{a}_2; z) = \sum_{\vec{a} \in \vec{a}_1 \,\text{Ш}\, \vec{a}_2} G(\vec{a}; z)\,,$$

where $\vec{a}_1 \,\text{Ш}\, \vec{a}_2$ are the permutations preserving the relative orderings of $\vec{a}_1$ and $\vec{a}_2$.

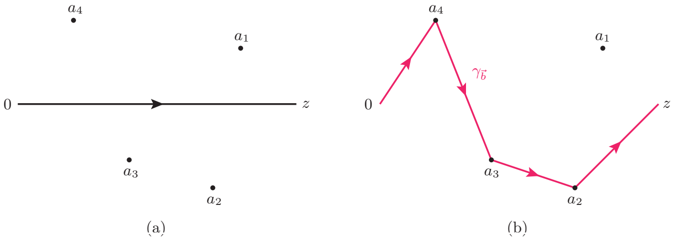
There is a coaction on MPLs. It is graded by weight, and thus breaks MPLs into simpler functions (lower weight).

$$\begin{aligned}
\Delta(\log z) &= 1 \otimes \log z + \log z \otimes 1 \\
\Delta(\log x \log y) &= 1 \otimes (\log x \log y) + \log x \otimes \log y + \log y \otimes \log x + (\log x \log y) \otimes 1 \\
\Delta(\mathrm{Li}_2(z)) &= 1 \otimes \mathrm{Li}_2(z) + \mathrm{Li}_2(z) \otimes 1 + \mathrm{Li}_1(z) \otimes \log z
\end{aligned}$$

Contour integrals

The coaction is a pairing of contours and integrands. Recalls the incidence algebra.

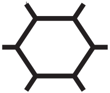
$$\Delta_{\mathrm{MPL}}(G(\vec{a};z)) = \sum_{\vec{b}\subseteq\vec{a}} G(\vec{b};z)\otimes G_{\vec{b}}(\vec{a};z)$$



Contour (b) takes a subset of residues in a given order.

A useful basis for all 1-loop integrals:

$$J_n = \frac{ie^{\gamma_E \epsilon}}{\pi^{D_n/2}} \int d^{D_n}k \prod_{j=1}^{n} \frac{1}{(k-q_j)^2 - m_j^2}$$



- k is the loop momentum
- q_j are sums of external momenta, m_j are internal masses
- Dimensions:

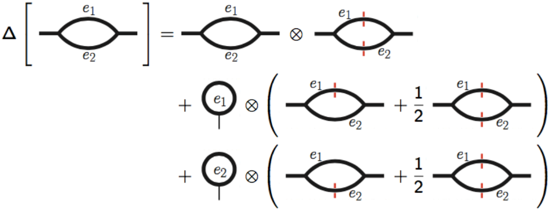
$$D_n = \begin{cases} n - 2\epsilon\,, & \text{for } n \text{ even}\,, \\ n + 1 - 2\epsilon\,, & \text{for } n \text{ odd}\,. \end{cases}$$

  e.g. tadpoles and bubbles in $2 - 2\epsilon$ dimensions, triangles and boxes in $4 - 2\epsilon$ dimensions, etc.
- Each J_n has uniform transcendental weight and satisfies nice differential equations.

2 equivalent Hopf algebras

The combinatorial algebra agrees with the Hopf algebra on the MPL of evaluated diagrams!

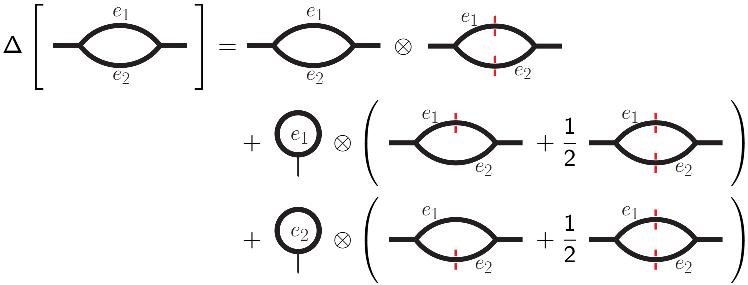
- The graph with n edges is interpreted as J_n, i.e. in D_n dimensions, no numerator.
- Need to insert extra terms in the diagrammatic equation:



Isomorphic to the more basic construction. (For any value of $1/2$.)

[related work: Brown; Bloch and Kreimer]

The master formula for 1-loop integrals



$$
\begin{aligned}
\Delta\left(\int_{\Gamma_\emptyset}\omega_{12}\right) &= \int_{\Gamma_\emptyset}\omega_{12}\otimes\int_{\Gamma_{12}}\omega_{12}+\int\omega_1\otimes\left(\int_{\Gamma_1}\omega_{12}+\frac{1}{2}\int_{\Gamma_{12}}\omega_{12}\right)+\cdots\\
&= \int_{\Gamma_\emptyset}\omega_{12}\otimes\int_{\Gamma_{12}}\omega_{12}+\int_{\Gamma_\emptyset}\omega_1\otimes\int_{-\frac{1}{2}\Gamma_{1\infty}}\omega_{12}+\int_{\Gamma_\emptyset}\omega_2\otimes\int_{-\frac{1}{2}\Gamma_{2\infty}}\omega_{12}
\end{aligned}
$$

The master formula for 1-loop integrals

$$\Delta\left(\int_{\Gamma_\emptyset}\omega_{12}\right) = \int_{\Gamma_\emptyset}\omega_{12}\otimes\int_{\Gamma_{12}}\omega_{12} + \int\omega_1\otimes\left(\int_{\Gamma_1}\omega_{12}+\frac{1}{2}\int_{\Gamma_{12}}\omega_{12}\right)+\cdots$$

$$= \int_{\Gamma_\emptyset}\omega_{12}\otimes\int_{\Gamma_{12}}\omega_{12} + \int_{\Gamma_\emptyset}\omega_1\otimes\int_{-\frac{1}{2}\Gamma_{1\infty}}\omega_{12} + \int_{\Gamma_\emptyset}\omega_2\otimes\int_{-\frac{1}{2}\Gamma_{2\infty}}\omega_{12}$$

Basis of integrands and corresponding contours:

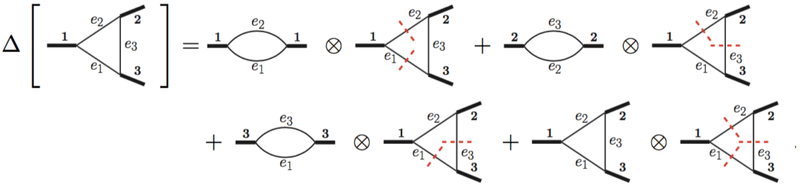
ω_j :	ω_{12}	ω_1	ω_2
γ_j :	Γ_{12}	$-\frac{1}{2}\Gamma_{1\infty}$	$-\frac{1}{2}\Gamma_{2\infty}$
	$\mathcal{C}_{12}$	$\mathcal{C}_1+\frac{1}{2}\mathcal{C}_{12}$	$\mathcal{C}_2+\frac{1}{2}\mathcal{C}_{12}$

They satisfy

$$\int_{\gamma_i}\omega_j \sim \delta_{ij} \qquad \text{after dropping logs}$$

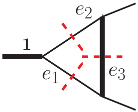
Odd number of cut propagators $\rightarrow$ pick up residue at infinity. Understood through homology theory.

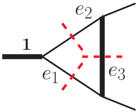
More examples of the graphical conjecture

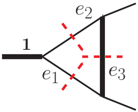


Terms with 1/2 are always present in principle, but vanished here due to massless propagators.

More examples of the graphical conjecture

$\Delta \Big[$ 

$\Big] =$ 

$\otimes$ 

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Statement of the graphical conjecture

The coaction on 1-loop graphs defined by pinching and cutting subsets of propagators,

when evaluated by Feynman rules,
if expanded order by order in ϵ,

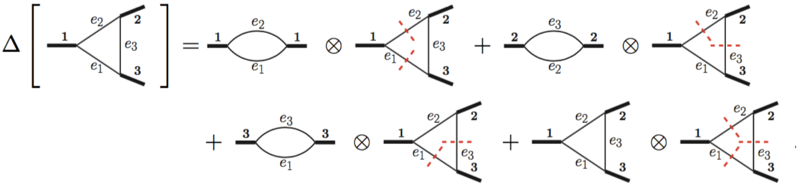
is consistent with the coaction on MPLs!

Evidence for the graphical conjecture

- all tadpoles and bubbles
- triangles and boxes with several combinations of internal and external masses
- consistency checks for more complicated boxes, 0m pentagon, 0m hexagon
- diagrammatic groupings emerging in 2-loop integrals

Checked to at least several orders in ϵ, or for closed forms with hypergeometric functions.

Motivation for the diagrammatic coaction



Second entries are discontinuities; first entries have discontinuities.

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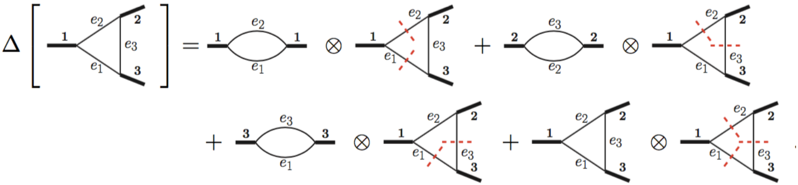


Second entries are discontinuities; first entries have discontinuities.

Motivated by the identity

$$\Delta\ \mathrm{Disc} = (\mathrm{Disc} \otimes 1)\, \Delta.$$

The companion relation

$$\Delta\, \partial = (1 \otimes \partial)\, \Delta$$

produces differential equations.

Application: cuts and discontinuities

$$\Delta\ \mathrm{Disc} = (\mathrm{Disc} \otimes 1)\, \Delta$$

$$\Delta\, (\mathrm{Disc}\, I_n) = (\mathrm{Disc} \otimes 1)\ (\Delta I_n)$$

Since $\Delta\, (\mathrm{Disc}\, I_n) = 1 \otimes (\mathrm{Disc}\, I_n) + \cdots$, it is enough to look at the terms $\Delta_{1,w-1} I_n$.

The basis integrals of weight 1 are precisely the tadpoles and bubbles. The corresponding cut diagrams have 1 or 2 propagators cut.

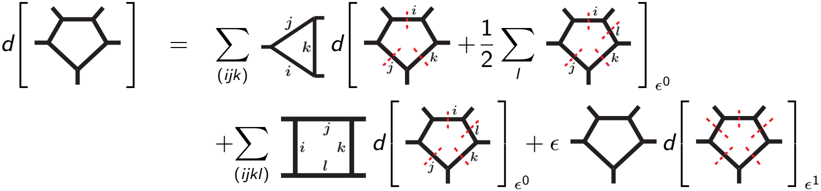
Therefore: the discontinuities are precisely the unitarity cut diagrams (momentum discontinuities) and the single-cut diagrams (mass discontinuities).

Generalized cuts can be interpreted as well.

Application: differential equations

$$\Delta\,\partial = (1 \otimes \partial)\,\Delta$$

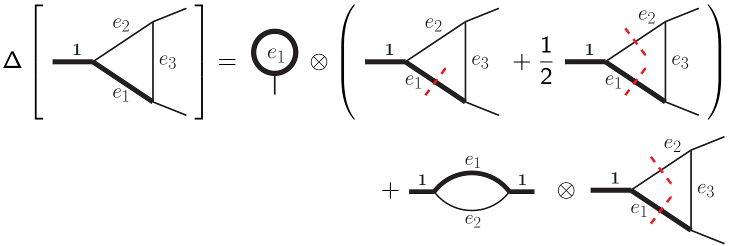
Likewise, we get differential equations by focusing on nearly-maximal cuts in the second factor:



This same relation shows that generalized cuts are related to symbol alphabet letters.

The master formula for the ${}_2F_1$ family

Consider the diagrammatic coaction



There is a coaction on ${}_2F_1$ that gives

$$
\begin{aligned}
\Delta\,{}_2F_1\left(1, 1+\epsilon, 2-\epsilon, x\right) \quad &= \quad {}_2F_1\left(1, \epsilon, 1-\epsilon, x\right) \otimes {}_2F_1\left(1, 1+\epsilon, 2-\epsilon, x\right) \\
&\quad + {}_2F_1\left(1, 1+\epsilon, 2-\epsilon, x\right) \otimes {}_2F_1\left(1, \epsilon, 1-\epsilon, \frac{1}{x}\right)
\end{aligned}
$$

without expanding in ϵ!

Master formula for Hopf algebra on integrals

Coaction of the form

$$\Delta\left(\int_\gamma \omega\right) = \sum_i \int_\gamma \omega_i \otimes \int_{\gamma_i} \omega$$

with a duality condition

$$P_{ss} \int_{\gamma_i} \omega_j = \delta_{ij}\,.$$

P_{ss} is semi-simple projection ("drop logarithms but not π").

To be precise, P_{ss} projects onto the space of semi-simple numbers x satisfying $\Delta(x) = x \otimes 1$.

The master formula for the ${}_2F_1$ family

Consider the family of integrands

$$\omega(\alpha_1, \alpha_2, \alpha_3) = x^{\alpha_1}(1-x)^{\alpha_2}(1-zx)^{\alpha_3}\ dx$$

where $\alpha_i = n_i + \epsilon_i$ and $n_i \in \mathbb{Z}$.

$$\int_0^1 \omega(\alpha_1, \alpha_2, \alpha_3) = \frac{\Gamma(\alpha_1)\Gamma(\alpha_2 - \alpha_1)}{\Gamma(\alpha_2)}\, {}_2F_1(-\alpha_3, \alpha_1 + 1; \alpha_2 + \alpha_1 + 2; z)$$

Basis of master integrands:

$$\int_0^1 \omega = c_0 \int_0^1 \omega_0 + c_1 \int_0^1 \omega_1$$

where

$$\begin{aligned} \omega_0 &= x^{\epsilon_1}(1-x)^{-1+\epsilon_2}(1-zx)^{\epsilon_3} \\ \omega_1 &= x^{\epsilon_1}(1-x)^{\epsilon_2}(1-zx)^{-1+\epsilon_3} \end{aligned}$$

With the two contours $\gamma_0 = [0,1]$ and $\gamma_1 = [0, 1/z]$, we have $P_{ss} \int_{\gamma_i} \omega_j \sim \delta_{ij}$.

Master formula for Appell F_1

Family of integrands for F_1.

$$\omega(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \quad = \quad x^{\alpha_1}(1-x)^{\alpha_2}(1-z_1x)^{\alpha_3}(1-z_2x)^{\alpha_4}\, dx$$

where $\alpha_i = n_i + \epsilon_i$ and $n_i \in \mathbb{Z}$.

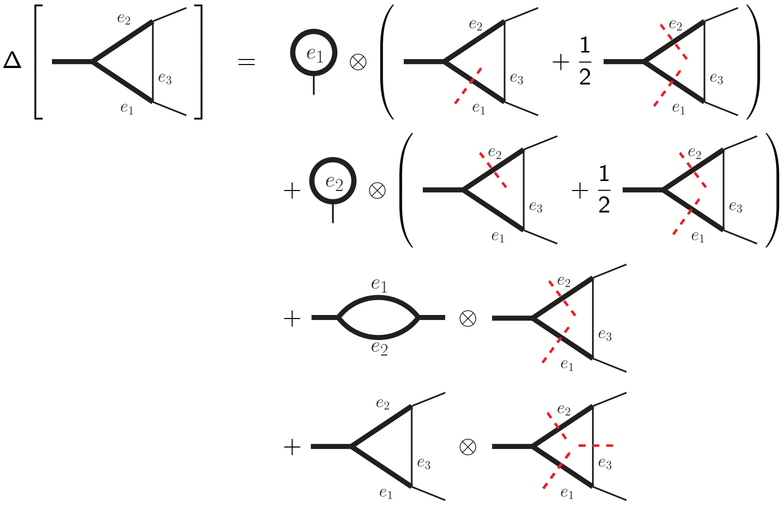
$$\int_0^1 \omega(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \frac{\Gamma(\alpha_1)\Gamma(\alpha_2 - \alpha_1)}{\Gamma(\alpha_2)}\, F_1(\alpha_1, \alpha_3, \alpha_4, \alpha_2; z_1, z_2)$$

Master integrands:

$$\begin{aligned}
\omega_0 &= x^{\epsilon_1}(1-x)^{-1+\epsilon_2}(1-z_1x)^{\epsilon_3}(1-z_2x)^{\epsilon_4} \\
\omega_1 &= x^{\epsilon_1}(1-x)^{\epsilon_2}(1-z_1x)^{-1+\epsilon_3}(1-z_2x)^{\epsilon_4} \\
\omega_2 &= x^{\epsilon_1}(1-x)^{\epsilon_2}(1-z_1x)^{\epsilon_3}(1-z_2x)^{-1+\epsilon_4}
\end{aligned}$$

Master contours: $\gamma_0 = [0,1]$, $\gamma_1 = [0, z_1^{-1}]$, $\gamma_2 = [0, z_2^{-1}]$.

Diagrammatic example with F_1



Master formula for ${}_{p+1}F_p$

Family of integrands for ${}_3F_2$.

$$\omega(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5) = x^{\alpha_1}(1-x)^{\alpha_2}y^{\alpha_3}(1-y)^{\alpha_4}(1-zxy)^{\alpha_5}\,dx\,dy$$

where $\alpha_i = n_i + \epsilon_i$ and $n_i \in \mathbb{Z}$.
Then

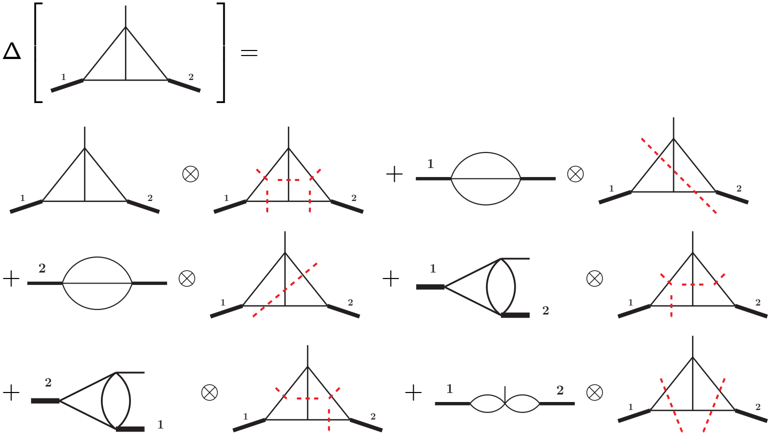
$$\int_0^1 \omega(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5) =$$
$$\frac{\Gamma()\Gamma()\Gamma()\Gamma()}{\Gamma()\Gamma()}\,{}_3F_2(\alpha_1+1,\alpha_3+1,-\alpha_5;2+\alpha_1+\alpha_2,2+\alpha_3+\alpha_4;z)$$

Basis of master integrands:

$$\begin{aligned}
\omega_0 &= x^{\epsilon_1}(1-x)^{-1+\epsilon_2}y^{\epsilon_3}(1-y)^{-1+\epsilon_4}(1-zxy)^{\epsilon_5}\\
\omega_1 &= x^{\epsilon_1}(1-x)^{-1+\epsilon_2}y^{\epsilon_3}(1-y)^{\epsilon_4}(1-zxy)^{-1+\epsilon_5}\\
\omega_2 &= x^{\epsilon_1}(1-x)^{\epsilon_2}y^{\epsilon_3}(1-y)^{-1+\epsilon_4}(1-zxy)^{-1+\epsilon_5}
\end{aligned}$$

With the master contours $\gamma_0 = \int_0^1 dx \int_0^1 dy$, $\gamma_1 = \int_0^1 dx \int_0^{1/zx} dy$, $\gamma_1 = \int_0^1 dy \int_0^{1/zy} dx$, we find that $P_{ss}\int_{\gamma_i}\omega_j \sim \delta_{ij}$

Diagrammatic example with ${}_3F_2$

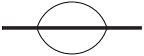


(with various prefactors and dimension shifts inserted to produce pure integrals)

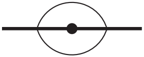
Features of diagrammatic coaction at two loops

Matrix of integrands and contours for *each topology*.

Example: sunrise with one internal mass. 2 master integrands in top topology.



$$= \int_{\Gamma_\emptyset} \omega_{111} \sim {}_2F_1\left(1+2\epsilon, 1+\epsilon, 1-\epsilon, p^2/m^2\right)$$



$$= \int_{\Gamma_\emptyset} \omega_{121} \sim {}_2F_1\left(2+2\epsilon, 1+\epsilon, 1-\epsilon, p^2/m^2\right)$$

For each, only two of the generalized cuts are linearly independent!

Thus 2 independent integration contours, e.g. $\Gamma_\emptyset$ and Γ_{123}.  

Diagonalize the matrix: $\int_{\gamma_i} \omega_j \sim \delta_{ij}$ with

$$\omega_1 = a\epsilon^2 \omega_{111}, \qquad \omega_2 = b\epsilon\omega_{111} + c\epsilon\omega_{121}$$

$$\gamma_1 = \Gamma_\emptyset, \qquad \gamma_2 = -\frac{1}{6\epsilon}\Gamma_{123} + \frac{2}{3}\Gamma_\emptyset$$

Coaction $\Delta\left(\int_\gamma \omega\right) = \sum_i \int_\gamma \omega_i \otimes \int_{\gamma_i} \omega$ is expressible in terms of diagrams.

Summary & Outlook

- We observe a Hopf algebra structure on Feynman diagrams, related to pinch and cut operations.
- Corresponds to Goncharov's Hopf algebra on MPLs, with prospects for extensions to hypergeometric integrals and beyond.
- The master formula is a Hopf algebra on integrals, based on matched pairs of integrands and contours
- Deep connections to discontinuities and differential equations, which are tools for computation.
- To explore further: systematic description beyond 1 loop, full range of hypergeometric functions, applications to integral and amplitude computations.

Coproduct and coaction

If H is a Hopf algebra, then a H (right-) comodule is a vector space A with a map $\rho : A \to A \otimes H$ such that

$$(\rho \otimes \mathrm{id})\rho = (\mathrm{id} \otimes \Delta)\rho \quad \text{and} \quad (\mathrm{id} \otimes \varepsilon)\rho = \mathrm{id}\,.$$

Here Δ is a coproduct on H. ρ is a coaction on A.

MPLs modulo $i\pi$ form a Hopf algebra H. For the full space of MPLs, we need the comodule $\mathbb{Q}[i\pi] \otimes H$, with a coaction Δ where $\Delta(i\pi) = i\pi \otimes 1$.

[Goncharov, Duhr, Brown]