Planar Two-Loop Five-Gluon Amplitudes from Numerical Unitarity

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based on [1712.03946]

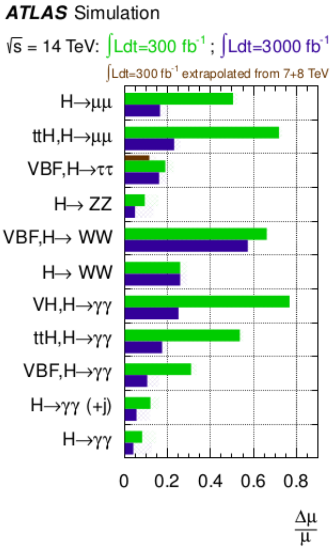
Outline

Introduction

Motivation

- Percent-level precision often requires NNLO QCD.
- Many 2-to-2 processes known @ NNLO.
- Can we add recoiling jet to signature final states to see kinematic dependence?
- Can we add mass effects?

Big effort. Many components.
Want higher-point amplitudes.



Progress in Higher Point NNLO Relevant Calculations

5-Point integrals:

- Planar massless [Gehrmann, Henn, lo Presti '15]
- Some one-mass [Papadopoulos, Tommasini, Wever '15] Costas' talk
- Non-planar massless examples [Chicherin, Henn, Mitev '17], [Chicherin, Henn, Sokatchev '18]. Johannes' talk

5-Point pure Yang-Mills amplitudes:

- Benchmark evaluation [Badger, Brønnum-Hansen, Bayu Hartanto, Peraro '17]. Simon's talk
- Numerical reduction [Abreu, Ita, Febres Cordero, B.P., Zeng '17] This talk

Related work: [Boels, Jin, Luo '18]

Numerical Unitarity at Two Loops

The Standard Approach to General Two-loop Amplitudes

Feynman diagrams

Tensor reduction
[Passarino, Veltman '79]
IBPs
[Tkachov, Chetyrkin '81]
↓

Sum of master integrals

$$A = \sum_{\Gamma \in \Delta} \sum_{i \in M_\Gamma} c_{\Gamma,i}(D) I_{\Gamma,i}.$$

Differential equations
[Kotikov '91; Remiddi '97; Gehrmann, Remiddi '01; Henn '13]
↓

Integrated form

General procedure, but:

- Large intermediate expressions.
- Generating IBP relations is practically difficult.

Two-loop numerical unitarity tries to avoid these issues by:

- Performing reduction and evaluation simultaneously.
- Working numerically.

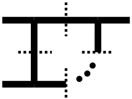
Two-Loop Reduction to Masters with Numerical Unitarity

- Take an ansatz for loop-amplitude integrand, decomposing into master (M_Γ) and surface (S_Γ) integrands.

$$\mathcal{A}(\ell_l) = \sum_{\text{Topologies}\,\Gamma} \; \sum_{i \in M_\Gamma \cup S_\Gamma} \frac{c_{\Gamma,i}\, m_{\Gamma,i}(\ell_l)}{\prod_{\text{props}\,j} \rho_j}.$$

[Ita '15]

- Numerically determine $c_{\Gamma,i}$ from on-shell information.



$$= \sum_{\substack{\Gamma' \geq \Gamma, \\ i \in M_{\Gamma'} \cup S_{\Gamma'}}} \frac{c_{\Gamma',i}\, m_{\Gamma',i}(\ell_l^\Gamma)}{\prod_{\text{props}\,j} \rho_j}.$$

[BDDK '94, '95]

- Insert master integrals, expand $\Rightarrow$ integrated amplitude.
- Applicable to both floating point and finite fields*.

*See also [Peraro '16], [von Manteuffel, Schabinger '14]

Master/Surface Decomposition

[Abreu, Febres-Cordero, Ita, B.P., Zeng '17]

- Surface terms naturally produced by IBPs.

$$0 = \int \prod_{l=1,2} d^D\ell_l \frac{\partial}{\partial \ell_j^\nu}\left[\frac{u_j^\nu}{\prod_{\text{props }k}\rho_k}\right].$$

- Generate complete set for a topology Γ by controlling powers of ρ_j.

$$u_i^\nu \frac{\partial}{\partial \ell_i^\nu}\rho_j = f_j\rho_j.$$

[Gluza, Kadja and Kosower '11]

- Write ansatz for u_k^ν, find polynomial solutions in ρ, α.
- Compute generating set using SINGULAR.
- Complement surface terms with master integrands.

$$u_k^\nu = u_{ka}^{\text{loop}}(\alpha,\rho)\ell_a^\nu + u_{kb}^{\text{ext}}(\alpha,\rho)p_b^\nu$$

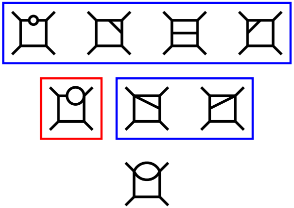
$$u_k^\nu(\rho,\alpha)\frac{\partial}{\partial \ell_k^\nu}\begin{pmatrix}\rho_{j(1)}\\ \rho_{j(2)}\\ \vdots\\ \rho_{j(|\Gamma|)}\end{pmatrix} - \begin{pmatrix}f_{j(1)}\rho_{j(1)}\\ f_{j(2)}\rho_{j(2)}\\ \vdots\\ f_{j(|\Gamma|)}\rho_{j(|\Gamma|)}\end{pmatrix} = 0\,,$$

Related [Georgoudis, Larsen, Zhang '16]

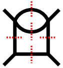
Sub-leading Poles

[Abreu, Febres-Cordero, Ita, Jaquier, B.P. '17]

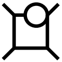
- Beyond one loop multiple poles can be associated to a given factorization limit.
- Only leading poles given by product of trees.



- Some numerators lack associated cut equation.
- Numerators determined from descendant cut equations.



$$- \sum_{\substack{\Gamma \in \Delta \setminus \widetilde{\Delta} \\ \Gamma > \Gamma'}} \frac{N(\Gamma, \ell_l^{\Gamma'})}{\prod_{k \in P_\Gamma \setminus P_{\Gamma'}} \rho_k(\ell_l^{\Gamma'})} \cdot$$

$$= N\left(\right.$$  $$\left.\right) + \frac{1}{\rho} N \left(\right.$$  $$\left.\right)$$

On-Shell Phase Spaces and Finite Fields

- On-shell parameterization naively requires solving quadratic equations over $\mathbb{F}_p$.
- Avoid with good basis choice.

$$\ell_1^2 = (\ell_1 - q_i)^2 \quad = \cdots = 0$$
$$\ell_2^2 = (\ell_2 - q_j)^2, \quad = \cdots = 0$$
$$(\ell_1 + \ell_2)^2 = (\ell_1 + \ell_2 - q_k)^2 = \cdots = 0.$$

- Aim: Algebraic momenta in controlled fashion.
- Use adapted coordinates.
- Take μ_k as basis vectors - coefficients non-algebraic.
- Affects scalar product and state sums.

$$\ell_l^\mu \to (\rho_i, \alpha_j, \mu_{ij}),$$

$$(\mu_l)^2 = \rho_{l0} - \sum_{\nu=0}^{3} \ell_l^\nu \ell_{l\,\nu}.$$

$$\ell_l^{(D-4)} = w_{l,1}\mu_1 + w_{l,2}\mu_2.$$

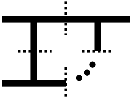
$$\ell_r \cdot \ell_s = \ell_r^4 \cdot \ell_s^4 - \sum_{i,j=1}^{2} w_i^r w_j^s \mu_{ij}.$$

Related work: [Peraro '16]

Solving for Master Integral Coefficients

- Build linear systems for master/surface coefficients,

$$\sum_{i \,\in\, M_\Gamma \cup S_\Gamma} c_{\Gamma,i}(D)\, m_{\Gamma,i}(\ell_l^\Gamma) =$$



$$- \sum_{\text{ancestors } \Gamma'} \frac{N(\Gamma', \ell_l^\Gamma)}{\prod_{k \in P_{\Gamma'} \setminus P_\Gamma} \rho_k(\ell_l^\Gamma)} .$$

- Sample randomly on-shell phase space ℓ_l^Γ to constrain $c_{\Gamma,i}$.
- Solve with standard linear algebra - PLU/QR factorization.
- Find coefficients for given numerical kinematics, D and D_s.

- Coefficients are rational functions of D,

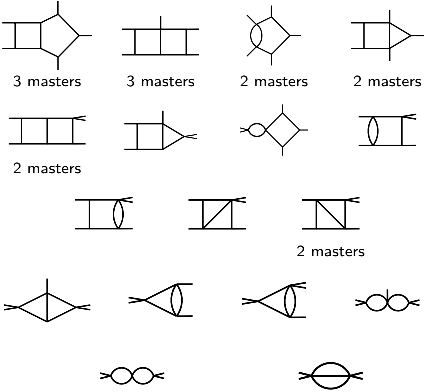
$$c(D) = \frac{P(D, s_{ij})}{Q(D)} = \frac{p_0(s_{ij}) + p_1(s_{ij})D + \ldots + p_i(s_{ij})D^i}{q_0 + q_1 D + \ldots + q_{j-1} D^{j-1} + D^j} .$$

- Reconstruct using [GKM] and Thiele's formula [Peraro '16].

Planar 5 Gluon Amplitudes at Two Loops

Ingredients for 5-Point Planar Gluonic Helicity Amplitudes

- Compute IBP generating vectors with SINGULAR.
- E.g. pentabox topology takes $<$ 1 second.
- Validate IBPs against FIRE5 at fixed s_{ij} ($\sim$ 1 day).
- Compute 155 coefficients in finite field, promote to $\mathbb{Q}$.
- Givaro* for finite fields.
- 5-point integrals from [Papdopoulos et al '15] ancillary. Analytic lower point integrals [Gehrmann, Remiddi '00].



*[Gauthier, Roch, Villard] - givaro.forge.imag.fr

Results @ 5-Point

- $\sim$ 2.5 core minutes per finite field for coefficients.
- Reproduce universal pole structure [Catani '98].
- Reproduce literature [Gehrmann et al '15], [Dunbar et al '16].
- Validated concurrent calculation of [Badger et al '17].

With, $g_s = 1$, $\mu = 1$, $s_{ij} = \{-1, -8, -10, -7, -3\}$:

$\mathcal{A}/(\mathcal{A}_0 N_c^2)(4\pi)^4$	ϵ^{-4}	ϵ^{-3}	ϵ^{-2}	ϵ^{-1}	ϵ^{0}
$(+,+,+,+,+)$	0	0	-5.00000	-3.89318	5.98109
$(-,+,+,+,+)$	0	0	-5.00000	-16.3220	-10.3838
$(-,-,+,+,+)$	12.5000	25.46247	-1152.84	-4072.94	-3637.25
$(-,+,-,+,+)$	12.5000	25.46247	-6.12163	-90.2218	-115.784

Conclusions

- Numerical unitarity provides a powerful organization of loop amplitude calculations, suitable for two-loop QCD amplitudes.
- We can numerically reduce leading-colour 5-gluon two-loop amplitudes to master integrals.
- Applicability over finite fields opens the door to future analytic reconstruction of integrated result.
- We look forward to computing new multi-scale amplitudes.

Finite Fields Crash Course (I)

- Take integers $\mathbb{F}_p = \{0, \ldots, p-1\}$, where p is prime.
- Perform multiplication/addition/subtraction modulo p,

$$5+7 \mod 11 = 1 \qquad 5\times 7 \mod 11 = 2 \qquad 5-7 \mod 11 = 9.$$

- Every $a \in \mathbb{F}_p$ has a multiplicative inverse, a^{-1} so $\mathbb{F}_p$ is a field,

$$5^{-1} \mod 11 = 9.$$

- All rational operations possible, but (e.g.) no square roots.
- From $\mathbb{Q}$ to $\mathbb{F}_p$

$$a = \frac{r}{s} \in \mathbb{Q} \quad \rightarrow \quad a \bmod p \equiv r \cdot (s^{-1} \bmod p) \bmod p$$

Finite Fields Crash Course (II)

- Is there an inverse map?
- Consider $a = \frac{r}{s}$ where $r, s < \sqrt{n}$ and we know $a \mod n$.
- Find r and s, using a "Rational Reconstruction" algorithm. [Wang, '81]
- p not big enough? Use Chinese Remainder Theorem.

$$\{a \mod n_1, a \mod n_2, \ldots\} \xrightarrow{CRT} a \mod (n_1 \cdot n_2 \cdots)$$

Exact numerics with speed of integer operations.

Avoiding Normalization

Problem: cannot normalize abitrary vectors over a finite field.

- Polarization vectors:
  - Consider constructing $\epsilon_+^\mu(p)$ from p^μ using $\epsilon_+^\mu = \frac{\langle\eta|\sigma^\mu|p]}{\sqrt{2}\langle\eta p\rangle}$.
  - Intermediate stages involve square roots, which always cancel.
  - Up to factors, gluon polarizations are rational functions of p^μ.

- Transverse space vectors:
  - Need to construct basis of space transverse to scattering plane.
  - Non-trivial norm affects "traceless completion" surface terms.
  - Explicit normalization now appears there, e.g.

$$\frac{(n_i \cdot l_1)^2}{n_i^2} - \frac{\mu_{11}}{D-4}$$