Quadratic relations between Feynman integrals

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Feynman integrals come in two varieties: polylogarithmic, or **not**.
They are used in two ways: as contributions to an amplitude that is squared, or as contributions to an observable **matrix element**.
In the former case, products of integrals occur, in the latter they do **not**.
Perversely, I have studied **products** of **non-polylog** Feynman integrals related to the **magnetic moment** of the electron, finding an infinite set of **quadratic relations** between these integrals at **all loops** $L > 2$.

1 Four-loop sunrise: the electron's magnetic moment

The **magnetic moment** of the electron, in Bohr magnetons, has QED contributions $\sum_{L\geq 0} a_L(\alpha/\pi)^L$ given up to $L=4$ loops by

$$\begin{aligned}
a_0 &= 1 \quad [\textbf{LO}: \text{Dirac, 1928}]\\
a_1 &= 0.5 \quad [\textbf{NLO}: \text{Schwinger, 1947}]\\
a_2 &= -0.32847896557919378458217281696489239241111929867962\ldots\\
a_3 &= 1.18124145658720000627475398221287785336878939093213\ldots\\
a_4 &= -1.91224576492644557415264716743983005406087339065872\ldots
\end{aligned}$$

NNLO: in 1957, Petermann and Sommerfield obtained

$$a_2 = \frac{197}{144} + \frac{\zeta(2)}{2} + \frac{3\zeta(3) - 2\pi^2 \log 2}{4}\,.$$

NNNLO: in 1996, Laporta and Remiddi encountered the multiple polylog $U_{3,1} := \sum_{m>n>0}(-1)^{m+n}/(m^3 n)$ in

$$\begin{aligned}
a_3 \;=\; & \frac{28259}{5184} + \frac{17101\zeta(2)}{135} + \frac{139\zeta(3) - 596\pi^2 \log 2}{18}\\
& - \frac{39\zeta(4) + 400U_{3,1}}{24} - \frac{215\zeta(5) - 166\zeta(3)\zeta(2)}{24}\,.
\end{aligned}$$

1.1 NNNNLO: the first non-polylog

A **Bessel moment**

$$B = -\int_0^\infty \frac{27550138x + 35725423x^3}{48600} I_0(x) K_0^5(x) \mathrm{d}x$$
$$= -1483.6850591488252945905998518451083670050015263060781 0\ldots$$

occurs at weight 4 in the breath-taking evaluation by **Stefano Laporta** [arXiv:1704.06996] of **4800 digits** of

$$a_4 = P + B + E + U \approx 2650.565 - 1483.685 - 1036.765 - 132.027 \approx -1.912$$

where P comprises polylogs and E comprises integrals, with weights 5, 6 and 7, whose integrands contain logs and products of elliptic integrals. U comes from 6 light-by-light master integrals, still under investigation.

The weight-4 non-polylog term B has $N = 6$ Bessel functions, with 5 instances of $K_0(x)$, from 5-fermion intermediate states. The sibling of $K_0(x)$ is $I_0(x) = \sum_{k\geq 0}((x/2)^k/k!)^2$, from Fourier transformation.

Both master integrals in B occur in $D = 2$ spacetime dimensions.

1.2 A simple determinant of Bessel moments

Consider **Bessel moments** of the form

$$M(a,b,c) := \int_0^\infty I_0^a(x) K_0^b(x) x^c \mathrm{d}x.$$

$2^L M(1, L+1, 1)$ is an L-loop **sunrise integral** at $D = 2$, on shell:

$$S_L(t) := \int_0^\infty \frac{\mathrm{d}x_1}{x_1} \ldots \int_0^\infty \frac{\mathrm{d}x_L}{x_L} \frac{1}{(1+\sum_{j=1}^L x_j)(1+\sum_{k=1}^L 1/x_k) - t}$$

$$S_4(1) = 2^4 M(1,5,1) := 2^4 \int_0^\infty I_0(x) K_0^5(x) x \mathrm{d}x.$$

Laporta encountered $M(1,5,1)$ as a master integral at $D = 4$. He also encountered $M(1,5,3)$, which is obtained by differentiation of $S_4(t)$ before setting $t = 1$. Now look at the simple **determinant**

$$\det \begin{bmatrix} M(1,5,1) & M(1,5,3) \\ M(2,4,1) & M(2,4,3) \end{bmatrix} = \frac{\pi^4}{24^2}$$

$M(2,4,1)$ comes from cutting an internal line. It occurred at stages of Laporta's ε-expansions. $M(2,4,3)$ comes from a cut and differentiation.

2 Simple determinants up to $L = 6$ loops

At L loops, with $N = L + 2$ Bessel functions, there is a simple result for a $k \times k$ determinant with $k = \lfloor (L+1)/2 \rfloor$. The first non-trivial case is at $L = 3$ loops, where I discovered (and now **Yajun Zhou** has proved) that

$$M_3 := \begin{bmatrix} M(1,4,1) & M(1,4,3) \\ M(2,3,1) & M(2,3,3) \end{bmatrix} = \begin{bmatrix} \pi^2 C & \pi^2 \left(\frac{2}{15}\right)^2 \left(13C - \frac{1}{10C}\right) \\ \frac{\sqrt{15}\pi}{2} C & \frac{\sqrt{15}\pi}{2} \left(\frac{2}{15}\right)^2 \left(13C + \frac{1}{10C}\right) \end{bmatrix}$$

is determined by

$$C := \frac{\Gamma\left(\frac{1}{15}\right)\Gamma\left(\frac{2}{15}\right)\Gamma\left(\frac{4}{15}\right)\Gamma\left(\frac{8}{15}\right)}{240\sqrt{5}\pi^2}.$$

The presence of

$$\frac{1}{C} = \frac{75\,\Gamma\left(\frac{7}{15}\right)\Gamma\left(\frac{11}{15}\right)\Gamma\left(\frac{13}{15}\right)\Gamma\left(\frac{14}{15}\right)}{\sqrt{5}\pi^2}$$

in the second column ensures a simple determinant at 3 loops:

$$D_3 = \det M_3 = 2\pi^3/\sqrt{3^3 5^5}.$$

2.1 Hypergeometric identity at 4 loops

For the Laporta problem the Feynman integrals are combinations of

$$\begin{array}{lcll} F_a &=& {}_4F_3(1/2, 2/3, 2/3, 5/6; & 7/6, 7/6, 4/3;\ 1) \\ F_b &=& {}_4F_3(-1/2, 1/6, 1/3, 4/3; & -1/6, 5/6, 5/3;\ 1) \\ F_c &=& {}_4F_3(1/6, 1/3, 1/3, 1/2; & 2/3, 5/6, 5/6;\ 1) \\ F_d &=& {}_4F_3(-7/6, -1/2, -1/3, 2/3; & -5/6, 1/6, 1/3;\ 1) \end{array}$$

with a **quadratic** relation $7F_aF_b + 10F_cF_d = 40$ giving $D_4 = \pi^4/24^2$.

2.2 Hidden quadratic relations at 5 loops

$D_5 = 16\pi^6/\sqrt{3^3 5^5 7^7}$ involves products of three Feynman integrals. I shall show that this results from a substructure of several **quadratic** relations.

2.3 Quadratic relation at 6 loops

The 3×3 determinant $D_6 = 5\pi^8/(2^{19}3)$ comes from a **quadratic** relation

$$\det \begin{bmatrix} M(1,7,1) & 32M(1,7,3) - 64M(1,7,5) \\ M(2,6,1) & 32M(2,6,3) - 64M(2,6,5) \end{bmatrix} = \frac{5\pi^6}{192}.$$

3 Feynman integrals from modular forms

With $q := \exp(2\pi \mathrm{i} z)$ and $\Im(z) > 0$, the **Dedekind eta** function satisfies

$$\eta(z) := q^{1/24}\prod_{n=1}^{\infty}(1-q^n) = \sum_{n=-\infty}^{\infty}(-1)^n q^{(6n+1)^2/24} = \frac{\eta(-1/z)}{\sqrt{-\mathrm{i}z}}.$$

With $\eta_n := \eta(nz)$ I define the weight-3 level-15 **cuspform**

$$f_{3,15}(z) := (\eta_3\eta_5)^3 + (\eta_1\eta_{15})^3 = \sum_{n>0} A_5(n)q^n = -\frac{f_{3,15}(-1/(15z))}{(-15)^{3/2}z^3}.$$

If the **Kronecker** symbol $\left(\frac{p}{15}\right) = \left(\frac{p}{3}\right)\left(\frac{p}{5}\right)$ is negative, for **prime** p, then $A_5(p) = 0$. For $\Re s > 2$, there is a convergent **L-series**

$$L_5(s) = \sum_{n>0}\frac{A_5(n)}{n^s} = \prod_p \frac{1}{1 - A_5(p)p^{-s} + \left(\frac{p}{15}\right)p^{2-2s}}.$$

Its analytic continuation is provided by the **Eichler integral**

$$L_5(s) = \frac{(2\pi)^s}{\Gamma(s)}\int_0^\infty f_{3,15}(\mathrm{i}y)y^{s-1}\mathrm{d}y$$

with **critical** values

$$L_5(\mathbf{1}) = \frac{5}{\pi^2}\int_0^\infty I_0(x)K_0^4(x)x\mathrm{d}x, \quad L_5(\mathbf{2}) = \frac{4}{3}\int_0^\infty I_0^2(x)K_0^3(x)x\mathrm{d}x.$$

3.1 Modular L-series at 4 loops

Consider the **Fourier** expansion of the weight-4 level-6 **cuspform**

$$f_{4,6}(z) := (\eta_1\eta_2\eta_3\eta_6)^2 = \sum_{n>0} A_6(n)q^n = \frac{f_{4,6}(-1/(6z))}{6^2 z^4}.$$

For $\Re s > 5/2$, there is a convergent **L-series**

$$L_6(s) = \sum_{n>0} \frac{A_6(n)}{n^s} = \frac{1}{1+2^{1-s}} \frac{1}{1+3^{1-s}} \prod_{p>3} \frac{1}{1 - A_6(p)p^{-s} + p^{3-2s}}.$$

Its analytic continuation is provided by the **Eichler integral**

$$L_6(s) = \frac{(2\pi)^s}{\Gamma(s)} \int_0^\infty f_{4,6}(\mathrm{i}y) y^{s-1} \mathrm{d}y$$

with **critical** values related to **Bessel moments** as follows

$$\begin{aligned} L_6(\mathbf{2}) &= \frac{2}{\pi^2} M(1,5,1) = \frac{2}{3} M(3,3,1), \\ L_6(\mathbf{1}) &= \frac{2}{\pi^2} M(2,4,1) = \frac{3}{\pi^2} L_6(\mathbf{3}). \end{aligned}$$

3.2 Non-modular L-series at 5 loops

With 7 Bessel functions and $\Re s > 3$, the local factors at the primes in

$$L_7(s) = \prod_p \frac{1}{Z_7(p, p^{-s})}$$

are given, for p coprime to 105, by the **cubic**

$$Z_7(p,T) = \left(1 - \left(\frac{p}{105}\right) p^2 T\right) \left(1 + \left(\frac{p}{105}\right) (2p^2 - |\lambda_p|^2)T + p^4 T^2\right)$$

where λ_p is a complex Hecke eigenvalue of a weight-3 newform with level 525. For $p|105$, I obtained, from **Kloosterman** moments in **finite fields**,

$$Z_7(3,T) = 1 - 10T + 3^4T^2,\ Z_7(5,T) = 1 - 5^4T^2,\ Z_7(7,T) = 1 + 70T + 7^4T^2.$$

Then **Anton Mellit** suggested a **functional equation**

$$\Lambda_7(s) := \left(\frac{105}{\pi^3}\right)^{s/2} \Gamma\left(\frac{s-1}{2}\right) \Gamma\left(\frac{s}{2}\right) \Gamma\left(\frac{s+1}{2}\right) L_7(s) = \Lambda_7(5-s)$$

and **Tim Dokchitser**'s package ComputeL gave the empirical result

$$L_7(\mathbf{2}) \stackrel{?}{=} \frac{24}{5\pi^2} \int_0^\infty I_0^2(x) K_0^5(x) x \mathrm{d}x.$$

3.3 Modular L-series at 6 loops

Consider the **Fourier** expansion of the weight-6 level-6 **cuspform**

$$f_{6,6}(z) := \frac{\eta_2^9\eta_3^9}{\eta_1^3\eta_6^3} + \frac{\eta_1^9\eta_6^9}{\eta_2^3\eta_3^3} = \sum_{n>0} A_8(n)q^n = -\frac{f_{6,6}(-1/(6z))}{6^3 z^6}.$$

For $\Re s > 7/2$, there is a convergent **L-series**

$$L_8(s) = \sum_{n>0}\frac{A_8(n)}{n^s} = \frac{1}{1-2^{2-s}}\frac{1}{1+3^{2-s}}\prod_{p>3}\frac{1}{1-A_8(p)p^{-s}+p^{5-2s}}.$$

Its analytic continuation is provided by the **Eichler integral**

$$L_8(s) = \frac{(2\pi)^s}{\Gamma(s)}\int_0^\infty f_{6,6}(\mathrm{i}y)y^{s-1}\mathrm{d}y$$

with **critical** values related to **Bessel moments** as follows

$$L_8(\mathbf{4}) = \frac{4}{9\pi^2}M(1,7,1) = \frac{4}{9}M(3,5,1) = \frac{\pi^2}{9}L_8(\mathbf{2}),$$

$$L_8(\mathbf{5}) = \frac{4}{27}M(2,6,1) = \frac{2\pi^2}{21}M(4,4,1) = \frac{2\pi^2}{21}L_8(\mathbf{3}) = \frac{\pi^4}{54}L_8(\mathbf{1}).$$

4 Quasi-periods associated to modular forms

Francis Brown posted ideas [arXiv:1710.07912] on quasi-periods associated to modular forms. A definition of these has been strangely elusive at weights greater than 2. For the weight 12 level 1 cuspform

$$\Delta(z) := \eta_1^{24} = q\prod_{n>0}(1-q^n)^{24} = \frac{\Delta(-1/z)}{z^{12}}$$

with $q := \exp(2\pi \mathrm{i} z)$, **periods** are defined via $L(\Delta, s)$ which has 11 critical values at integers $s \in [1, 11]$. At odd integers these are given, up to rational multiples of powers of π, by ω_+, while at even integers we use ω_-. Specifically, in terms of $L(\Delta, 5)$ and $L(\Delta, 6)$, the **periods** are

$$\begin{aligned}
\omega_+ &:= -70(2\pi)^{11}\int_0^\infty \Delta(\mathrm{i}y)y^4\mathrm{d}y \\
&= -68916772.8095951947543101246553310304390699691\ldots \\
\omega_- &:= -6(2\pi)^{11}\int_0^\infty \Delta(\mathrm{i}y)y^5\mathrm{d}y \\
&= -5585015.37931040186687713926379627512963503343\ldots
\end{aligned}$$

To define **quasi-periods**, Brown considers the **weakly** holomorphic modular form $\Delta'(z)$, defined in terms of Klein's j-invariant by

$$\begin{aligned}\Delta'(z) &:= (j^2 - 1464j + 142236)\Delta(z) = 1/q + O(q^2),\\ j &:= \frac{1}{\Delta(z)}\left(1 + 240\sum_{n>0}\frac{n^3q^n}{1-q^n}\right)^3 = \frac{1}{q} + 744 + 196884q + O(q^2)\end{aligned}$$

The quasi-periods are

$$\begin{aligned}\eta_+ &= 127202100647.17709477731716129861087749404598 8\ldots\\ \eta_- &= 10276732343.64913275081719307240092090889939 90\ldots\end{aligned}$$

with numerical values obtainable from a **determinant** and **permanent**,

$$\begin{aligned}\omega_+\eta_- - \omega_-\eta_+ &= (2\pi)^{11}10!\\ \frac{\omega_+\eta_- + \omega_-\eta_+}{4\pi\omega_+\omega_-} &= -\sum_{c>0}\frac{I_{11}(4\pi/c)}{c}\sum_{r\in(\mathbf{Z}/\mathbf{Z}c)^*}\exp\left(\frac{2\pi\mathrm{i}(r-s)}{c}\right)\Bigg|_{rs=1\bmod c}\end{aligned}$$

Brown states that he obtained numerical values for $\eta_\pm$ from Eichler-type integrals, with an as yet unexplained regularization to deal with the exponential singularity at infinity in $\Delta'(z)$.

4.1 Quasi-periods from 4-loop sunrise

In terms of **Eichler** integrals,

$$\frac{D_2}{2} = \frac{M(1,5,1)}{\pi^4} = \frac{4M(1,5,3)}{\pi^4} + \frac{5E_2}{18}$$

$$\frac{3D_1}{5} = \frac{M(2,4,1)}{\pi^3} = \frac{4M(2,4,3)}{\pi^3} + \frac{E_1}{3}$$

$$\begin{bmatrix} D_s \\ E_s \end{bmatrix} := -\int_{1/\sqrt{3}}^{\infty} \begin{bmatrix} f_{4,6}\left(\frac{1+\mathrm{i}y}{2}\right) \\ g_{4,6}\left(\frac{1+\mathrm{i}y}{2}\right) \end{bmatrix} y^{s-1}\mathrm{d}y,$$

$$g_{4,6}(z) := \frac{(w^2-3)^2(w^4+9)}{8w^4} f_{4,6}(z) = 5q + 102q^2 + 945q^3 + O(q^4),$$

$$w := 3\,\frac{\eta_2^2\eta_3^4}{\eta_1^4\eta_6^2},$$

$$D_1E_2 - D_2E_1 = \frac{1}{24\pi^3}.$$

So I have satisfied the determinant criterion for quasi-periods. I have asked Francis Brown for a permanent condition at level 6.

4.2 Quasi-periods from six-loop sunrise

I have **empirical** relations to Eichler integrals for the **second** column of

$$\det\left[\begin{array}{cc} M(1,7,1) & 32M(1,7,3)-64M(1,7,5) \\ M(2,6,1) & 32M(2,6,3)-64M(2,6,5) \end{array}\right] = \frac{5\pi^6}{192},$$

$$\frac{F_2}{4} = \frac{M(1,7,1)}{\pi^6} \stackrel{?}{=} \frac{32M(1,7,3)-64M(1,7,5)}{\pi^6} + \frac{35G_2}{108},$$

$$\frac{9F_1}{28} = \frac{M(2,6,1)}{\pi^5} \stackrel{?}{=} \frac{32M(2,6,3)-64M(2,6,5)}{\pi^5} + \frac{5G_1}{12},$$

$$\left[\begin{array}{c} F_s \\ G_s \end{array}\right] := -\int_{1/\sqrt{3}}^{\infty} \left[\begin{array}{c} f_{6,6}\left(\frac{1+\mathrm{i}y}{2}\right) \\ g_{6,6}\left(\frac{1+\mathrm{i}y}{2}\right) \end{array}\right] (3y^2-1)y^{s-1}\mathrm{d}y,$$

$$g_{6,6}(z) := \frac{(w^2-3)^4}{16w^4} f_{6,6}(z) = q + 36q^2 + 567q^3 + 5264q^4 + O(q^5),$$

$$F_1G_2 - F_2G_1 \stackrel{?}{=} \frac{1}{4\pi^5},$$

with $(3y^2-1)$ inferred from the **dispersion relation** for a sub-diagram. Note that the integrand of G_s is of order $(3y^2-1)^6$ near its threshold. A link to **Francis Brown**'s concept of **quasi-periods** is forming, yet is not complete, since $g_{6,6}$ lacks a period polynomial enjoyed by $f_{6,6}$.

4.3 Quasi-periods from even Bessel moments

At weight 4 and level 8, I obtained

$$\left[\begin{array}{cc} 2M(0,4,0) & 4M(0,4,0)-16M(0,4,2) \\ 2M(1,3,0) & 4M(1,3,0)-16M(1,3,2) \end{array}\right] = \left[\begin{array}{cc} \pi^4 P_1 & 3\pi^4 Q_1 \\ \pi^3 P_2 & 3\pi^3 Q_2 \end{array}\right],$$

$$\left[\begin{array}{c} P_s \\ Q_s \end{array}\right] := -\mathrm{i}\int_1^\infty \left[\begin{array}{c} f_{4,8}\left(\frac{1+\mathrm{i}y}{4}\right) \\ g_{4,8}\left(\frac{1+\mathrm{i}y}{4}\right) \end{array}\right] \frac{y^s+y^{4-s}}{y}\mathrm{d}y,$$

$$f_{4,8}(z) := (\eta_2\eta_4)^4 = q - 4q^3 - 2q^5 + O(q^7),$$

$$g_{4,8}(z) := \left(1+64\frac{\eta_4^{24}}{\eta_2^{24}}\right) f_{4,8}(z) = q + 60q^3 + 1278q^5 + O(q^7),$$

$$P_1Q_2 - P_2Q_1 = -\frac{1}{2\pi^3},$$

with $g_{4,8}(z_0) = 0$ at $z_0 = (1+\mathrm{i})/4$, where $-\mathrm{i}f_{4,8}(z_0) = \Gamma^8(1/4)/(128\pi^6)$.

4.4 Periods at level 24

The **unique** weight-6 Hecke eigenform that is both a **newform** of level 24 and also has a **negative** sign in the functional equation for its **L-series** is

$$\begin{aligned}
f_{6,24}(z) &:= \frac{\eta_3^4\eta_4^2\eta_6^6\eta_8^2}{\eta_{24}^2} + \frac{\eta_1^4\eta_2^6\eta_{12}^2\eta_{24}^2}{3\eta_8^2} - \frac{16\eta_1^2\eta_2^2\eta_{12}^6\eta_{24}^4}{\eta_3^2} - \frac{16\eta_3^2\eta_4^6\eta_6^2\eta_8^4}{3\eta_1^2} \\
&\quad + \frac{64\eta_1^2\eta_3^2\eta_4\eta_8^4\eta_{12}\eta_{24}^4}{\eta_2\eta_6} - \frac{4\eta_1^4\eta_2\eta_3^4\eta_6\eta_8^2\eta_{24}^2}{\eta_4\eta_{12}} = -f_{6,24}(z+1/2) \\
&= \frac{f_{6,24}(-1/(24z))}{24^3 z^6} = -\frac{f_{6,24}((3z-1)/(12z-3))}{3^3(4z-1)^6} \\
&= q - 9q^3 - 34q^5 - 240q^7 + 81q^9 - 124q^{11} + 46q^{13} + O(q^{15}).
\end{aligned}$$

David Roberts and I related its **critical** L-series to **Bessel** moments:

$$\begin{aligned}
\widetilde{L}_6(4) &\stackrel{?}{=} \frac{M(0,6,0)}{108\pi^2}, \\
\widetilde{L}_6(5) &\stackrel{?}{=} \frac{M(1,5,0)}{144}, \\
\widetilde{L}_6(s) &:= \frac{(2\pi)^s}{\Gamma(s)} \int_0^\infty f_{6,24}(\mathrm{i}y) y^{s-1} \mathrm{d}y.
\end{aligned}$$

4.5 Striving for quasi-periods at level 24

After **intensive** experiment at high precision, I **conjecture** that

$$\det\begin{bmatrix} M(0,6,0) & 3M(0,6,2)-8M(0,6,4) \\ M(1,5,0) & 3M(1,5,2)-8M(1,5,4) \end{bmatrix} \stackrel{?}{=} \frac{5\pi^6}{16},$$

$$\frac{M(0,6,0)}{\pi^6} \stackrel{?}{=} \frac{R_1}{28} \stackrel{?}{=} \frac{3R_3}{4}, \qquad \frac{M(1,5,0)}{\pi^5} \stackrel{?}{=} \frac{R_2}{8},$$

$$R_s := -\mathrm{i}\int_0^\infty f_{6,24}\left(\frac{1+\mathrm{i}y}{4}\right) y^{s-1}\mathrm{d}y = 3^{3-s}R_{6-s}.$$

I strove to relate the **second** column of the determinant to Eichler integrals of a weakly holomorphic modular form. This is a very tough problem, to which I continue to devote much effort.

5 Quadratic relations between integrals for all loops

Conjecture: With the Feynaman, de Rham and Betti matrices below

$$F_N D_N F_N^{\mathtt{tr}} = B_N$$

The elements of F_N are given by **Feynman** as Bessel moments by

$$\begin{aligned} F_{2k+1}(u,a) &:= \frac{(-1)^{a-1}}{\pi^u} M(k+1-u, k+u, 2a-1) \\ F_{2k+2}(u,a) &:= \frac{(-1)^{a-1}}{\pi^{u+1/2}} M(k+1-u, k+1+u, 2a-1). \end{aligned}$$

with u and a running from 1 to k

The **Betti** matrices B_N have **rational** elements given by

$$\begin{aligned} B_{2k+1}(u,v) &:= (-1)^{u+k} 2^{-2k-2} (k+u)!(k+v)! Z(u+v) \\ B_{2k+2}(u,v) &:= (-1)^{u+k} 2^{-2k-3} (k+u+1)!(k+v+1)! Z(u+v+1) \\ Z(m) &= \frac{1+(-1)^m}{(2\pi)^m} \zeta(m) \end{aligned}$$

Construction of the rational **de Rham** matrices D_N was highly inductive. **David Roberts** and I have boiled it down to the following definition.

Let v_k and w_k be the rational numbers **generated** by

$$\begin{aligned}\frac{J_0^2(t)}{C(t)} &= \sum_{k\geq 0}\frac{v_k}{k!}\left(\frac{t}{2}\right)^{2k} = 1 - \frac{17t^2}{54} + \frac{3781t^4}{186624} + \ldots \\ \frac{2J_0(t)J_1(t)}{tC(t)} &= \sum_{k\geq 0}\frac{w_k}{k!}\left(\frac{t}{2}\right)^{2k} = 1 - \frac{41t^2}{216} + \frac{325t^4}{186624} + \ldots\end{aligned}$$

where $J_0(t) = I_0(\mathrm{i}t)$, $J_1(t) = -J_0'(t)$ and

$$C(t) \equiv \frac{32(1 - J_0^2(t) - tJ_0(t)J_1(t))}{3t^4} = 1 - \frac{5t^2}{27} + \frac{35t^4}{2304} - \frac{7t^6}{9600} + \ldots$$

We construct rational bivariate polynomials by the **recursion**

$$\begin{aligned}H_s(y,z) &= zH_{s-1}(y,z) - (s-1)yH_{s-2}(y,z) \\ &- \sum_{k=1}^{s-1}\binom{s-1}{k}\left(v_kH_{s-k}(y,z) - w_kzH_{s-k-1}(y,z)\right)\end{aligned}$$

for $s > 0$, with $H_0(y,z) = 1$. We use these to define

$$d_s(N,c) \equiv \frac{H_s(3c/2, N+2-2c)}{4^s s!}.$$

Finally, we construct rational **de Rham** matrices, with elements

$$D_N(a,b) \equiv \sum_{c=-b}^{a} d_{a-c}(N,-c)d_{b+c}(N,c)c^{N+1}.$$

Summary

1. Laporta's four-loop result contains a modular quasi-period.
2. Simple determinants occur for all loops.
3. The L-series are modular only for $L = 3$, 4 and 6 loops.
4. At 6 loops we also encounter quasi-periods.
5. There are quadratic relations of the form $F_N D_N F_N^{\rm tr} = B_N$ with Feynman, de Rham and Betti matrices that have been specified.

6 Postscript: Fourier coefficients of 14 eta quotients

For integers $n > 0$ and $N \in \{2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 16, 18, 25\}$ let

$$R_N(n) = \frac{2\pi}{\sqrt{nN}} \sum_{\substack{c \geq r > 0 \\ \gcd(c, Nr) = 1}} \frac{I_1(4\pi\sqrt{n/N}/c)}{c} \exp\left(\frac{2\pi {\rm i}(r - ns)}{c}\right)\Bigg|_{Nrs \equiv 1 \bmod c} .$$

I found that $R_N(n)/R_N(1)$ is the coefficient of q^n of an eta quotient T_N/B_N defining an OEIS sequence, as follows

N	$R_N(1)$	T_N	B_N	OEIS
2	4096	η_2^{24}	η_1^{24}	A014103
3	729	η_3^{12}	η_1^{12}	A121590
4	256	η_4^{8}	η_1^{8}	A092877
5	125	η_5^{6}	η_1^{6}	A121591
6	72	$\eta_2\eta_6^{5}$	$\eta_1^{5}\eta_3$	A128638
7	49	η_7^{4}	η_1^{4}	A121593
8	32	$\eta_2^{2}\eta_8^{4}$	$\eta_1^{4}\eta_4^{2}$	A107035
9	27	η_9^{3}	η_1^{3}	A121589
10	20	$\eta_2\eta_{10}^{3}$	$\eta_1^{3}\eta_5$	A095846
12	12	$\eta_2^{2}\eta_3\eta_{12}^{3}$	$\eta_1^{3}\eta_4\eta_6^{2}$	A187100
13	13	η_{13}^{2}	η_1^{2}	A121597
16	8	$\eta_2\eta_{16}^{2}$	$\eta_1^{2}\eta_8$	A123655
18	6	$\eta_2\eta_3\eta_{18}^{2}$	$\eta_1^{2}\eta_6\eta_9$	A128129
25	5	η_{25}	η_1	A092885

There is no eta quotient at $N = 1$. Instead, Rademacher obtained $j(z) = 1/q + 744 + \sum_{n>0} R_1(n)q^n$, with $R_1(1) = 108 \times 1823 = 196884$.