Two-mass, 3-loop effects for heavy flavor deep inelastic scattering

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Loops and Legs in Quantum Field Theory
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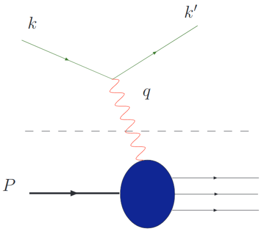
Content

- Theory of deep inelastic scattering
- The variable flavor number scheme
- Two mass contribution to $A_{gq,Q}^{(3)}$ and $A_{qq,Q}^{(3),\mathrm{NS}}$
- Two mass contribution to $A_{Qq}^{\mathrm{PS},(3)}$
- Two mass contribution to $A_{gg,Q}^{(3)}$
- Summary and Outlook

Based on:

- Ablinger, Blümlein, De Freitas, Goedicke Schneider, Wißbrock; Nucl.Phys. B921 (2017) 585-688
- Ablinger, Blümlein, De Freitas, Schneider, Schönwald; Nucl.Phys. B927 (2018) 339-367
- Ablinger, Blümlein, De Freitas, Goedicke Schneider, Schönwald; Nucl.Phys. B in print [arXiv:1804.02226[hep-ph]]
- Blümlein, De Freitas, Schneider, Schönwald; arXiv:1804.03129



- Kinematic invariants:

$$Q^2 = -q^2, \qquad x = \frac{Q^2}{2P.q}$$

- The cross section factorizes into leptonic and hadronic tensor:

$$\frac{d^2\sigma}{dQ^2 dx} \sim L_{\mu\nu} W^{\mu\nu}$$

- The hadronic tensor can be expressed through two structure functions:

$$W_{\mu\nu} = \frac{1}{4\pi} \int d^4\xi \exp(iq\xi) \langle P, | \left[J_\mu^{em}(\xi), J_\nu^{em}(\xi) \right] | P \rangle$$
$$= \frac{1}{2x} \left(g_{\mu\nu} + \frac{q_\mu q_\nu}{Q^2} \right) F_L(x, Q^2) + \frac{2x}{Q^2} \left(P_\mu P\nu + \frac{q_\mu P_\nu + q_\nu P_\mu}{2x} - \frac{Q^2}{4x^2} g_{\mu\nu} \right) F_2(x, Q^2)$$

- F_L and F_2 contain contributions from both, charm and bottom quarks.

Motivation

- Precision of DIS world data: $\sim 1\%$ for F_2
  $\rightarrow$ requires $\mathcal{O}(\alpha_s^3)$ corrections
- Heavy quarks yield essential contributions to the structure functions
  $\sim 20 - 30\%$ at small x
- Heavy quark contributions to the scaling violations have different shape than the massless ones

$\Rightarrow$ NNLO heavy quark contributions are important for a precise determination of

1. the strong coupling constant α_s
2. the heavy quark masses m_c, m_b
3. parton distribution functions for the heavy quarks

Operator Matrix Elements

- At $Q^2 \gg m^2$ the heavy flavor part can be calculated via massive operator matrix elements (OMEs) of local Operators O_i between partonic states j

$$A_{ij}\left(\frac{m_c^2}{\mu^2}, \frac{m_b^2}{\mu^2}, N\right) = \langle j|\, O_i \,|j\rangle \ .$$

[Buza, Matiounine, Smith, van Neerven, Nucl.Phys. B472 (1996) 611-658]

→ Additional Feynman rules with local operator insertions for the partonic matrix elements

- We will compute mainly in Mellin N-space:

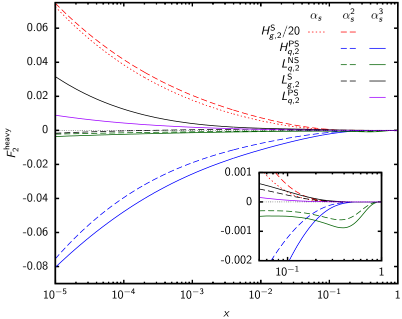
$$F(N) = \int_0^1 \mathrm{d}x\, x^{N-1} f(x)$$

- The OMEs can be split into parts which contain only contributions from one heavy quark and genuine two mass contributions

$$A_{ij}\left(\frac{m_c^2}{\mu^2}, \frac{m_b^2}{\mu^2}\right) = A_{ij}\left(\frac{m_c^2}{\mu^2}\right) + A_{ij}\left(\frac{m_b^2}{\mu^2}\right) + \tilde{A}_{ij}\left(\frac{m_c^2}{\mu^2}, \frac{m_b^2}{\mu^2}\right) .$$

General Status of the Calculation

- A large number of Mellin moments (1000) is available for all OMEs in the single mass case, as well as all logarithmic contributions.
- All single mass contributions, but for $A_{Qg}^{(3)}$, have been completed; all 1st order factorizing terms to $A_{Qg}^{(3)}$ are known.



- A series of moments is available for all OMEs in the two-mass case.
- The calculation of the two-mass OMEs for general N and x will be reported in the following.

Operator Matrix Elements

- Since the mass of the bottom quark m_b is not much larger than the charm mass m_c

$$\eta = \frac{m_c^2}{m_b^2} \sim 0.1 \, ,$$

  we have to take into account the contributions from both quarks simultaneously (the top quark decouples because of the heavy mass $m_t^2 \gg m_b^2$).

- Graphs with two massive quarks start contributing at $\mathcal{O}(\alpha_s^3)$, nevertheless two mass contributions can emerge from
  - reducible contributions
  - renormalization [Ablinger et al; Nucl.Phys. B921 (2017) 585-688]
  - scheme changes

$\Rightarrow$ this way two mass contributions even arise at $\mathcal{O}(\alpha_s^2)$

The variable flavor number scheme

- Matching conditions for parton distribution functions:

$$f_k(N_F+2) + f_{\overline{k}}(N_F+2) = A_{qq,Q}^{\rm NS}\left(N_F+2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2}\right) \cdot \left[f_k(N_F) + f_{\overline{k}}(N_F)\right] + \frac{1}{N_F} A_{qq,Q}^{\rm PS}\left(N_F+2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2}\right) \cdot \Sigma(N_F)$$
$$+ \frac{1}{N_F} A_{qg,Q}\left(N_F+2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2}\right) \cdot G(N_F)\,,$$

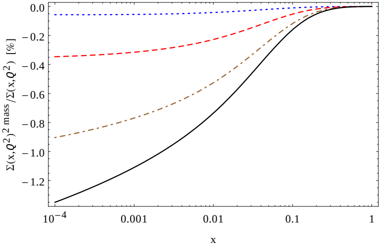
$$f_Q(N_F+2) + f_{\overline{Q}}(N_F+2) = A_{Qq}^{\rm PS}\left(N_F+2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2},\right) \cdot \Sigma(N_F) + A_{Qg}\left(N_F+2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2}\right) \cdot G(N_F)\,,$$

$$\Sigma(N_F+2) = \left[A_{qq,Q}^{\rm NS}\left(N_F+2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2}\right) + A_{qq,Q}^{\rm PS}\left(N_F+2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2}\right)\right.$$
$$\left. + A_{Qq}^{\rm PS}\left(N_F+2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2}\right)\right] \cdot \Sigma(N_F) + \left[A_{qg,Q}\left(N_F+2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2}\right)\right.$$
$$\left. + A_{Qg}\left(N_F+2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2}\right)\right] \cdot G(N_F)\,,$$

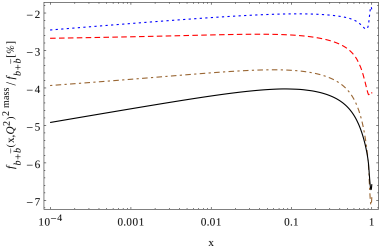
$$G(N_F+2) = A_{gq,Q}\left(N_F+2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2}\right) \cdot \Sigma(N_F) + A_{gg,Q}\left(N_F+2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2}\right) \cdot G(N_F)\,.$$

The variable flavor number scheme at NLO

$\Sigma(x, Q^2)^{\text{2-mass}} / \Sigma(x, Q^2)$



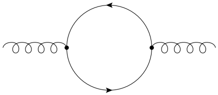
$f_{b+\bar{b}}(x, Q^2)^{\text{2-mass}} / f_{b+\bar{b}}(x, Q^2)$



- The ratio of the 2-mass contributions to the singlet parton distribution $\Sigma(x, Q^2)$ (left) and the heavy flavor parton distribution $f_{b+\bar{b}}(x, Q^2)$ (right) over their full form in percent for $m_c = 1.59$ GeV, $m_b = 4.78$ GeV in the on-shell scheme. Dash-dotted line: $Q^2 = 30$ GeV^2; Dotted line: $Q^2 = 30$ GeV^2; Dashed line: $Q^2 = 100$ GeV^2; Dash-dotted line: $Q^2 = 1000$ GeV^2; Full line: $Q^2 = 10000$ GeV^2.

Overview: Calculation

- No reduction to master integrals.
- Full Feynman parametrization of the diagrams.
- One fermion loop can be rendered effectively massless via a Mellin-Barnes integral:



$$= a_s T_F \frac{4}{\pi}(4\pi)^{-\varepsilon/2}(k_\mu k_\nu - k^2 g_{\mu\nu})(-k^2)^{\varepsilon/2} \times \int_{-i\infty}^{+\infty} d\sigma \left(\frac{m^2}{-k^2}\right)^\sigma \frac{\Gamma(\sigma-\varepsilon/2)\Gamma^2(2-\sigma+\varepsilon/2)\Gamma(-\sigma)}{\Gamma(4-2\sigma+\varepsilon)}$$

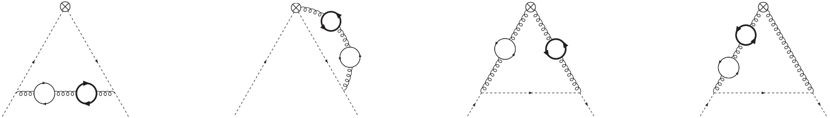
- Integration over loop momenta and Feynman parameters can be done in terms of Γ-functions.
- The complex contour integral can be done by closing the integration contour and summing up residues.

⇒ The integration problem has been transformed into a (multi-)summation, which can be handled with the packages `Sigma`, `EvaluateMultiSums` [Schneider '01-] , `SumProduction` [Ablinger, Blümlein, Hasselhuhn, Schneider '10-] and `HarmonicSums` [Ablinger, Blümlein, Schneider '10,'13] .

Overview: Mathematical Structures

$A_{qq,Q}^{(3),\mathrm{NS}}$, $A_{gq,Q}^{(3)}$	Harmonic Sums [Vermaseren '98; Blümlein, Kurth '98] $\sum_{i=1}^{N} \frac{1}{i^3} \sum_{j=1}^{i} \frac{1}{i}$	HPLs [Remiddi, Vermaseren '99] $\int_0^x \frac{d\tau_1}{1+\tau_1} \int_0^{\tau_1} \frac{d\tau_2}{1-\tau_2}$
$A_{gg,Q}^{(3)}$	Generalized harmonic and binomial sums [Ablinger, Blülein, Schneider '13] [Ablinger, Blülein, Raab, Schneider '14] $\sum_{i=1}^{N} \frac{4^i (1-\eta)^{-i}}{i\binom{2i}{i}} \sum_{j=1}^{i} \frac{(1-\eta)^j}{j^2}$	Iterated integrals over root and η valued letters [Ablinger, Blümlein, Raab, Schneider '14] $\int_0^x d\tau_1 \frac{\sqrt{\tau_1(1-\tau_1)}}{1-\tau_1(1-\eta)} \int_0^{\tau_1} \frac{d\tau_2}{\tau_2}$
$A_{Qq}^{(3),\mathrm{PS}}$	–	Iterated integrals over root valued letters with restricted support $\theta(x-\eta_+) \int_0^{x(1-x)/\eta} d\tau \frac{\sqrt{1-4\tau}}{\tau}$

Two mass contributions to $A_{qq,Q}^{(3),\mathrm{NS}}$ and $A_{gq,Q}^{(3)}$



- After introducing Feynman parameters the integrals can be represented as:

$$I \sim \mathcal{C}(\varepsilon, N) \int_{-i\infty}^{+i\infty} d\sigma \eta^{\sigma} \Gamma \left[\begin{matrix} g_1(\varepsilon)+\sigma, g_2(\varepsilon)+\sigma, g_3(\varepsilon)+\sigma, g_4(\varepsilon)-\sigma, g_5(\varepsilon)-\sigma \\ g_6(\varepsilon)+\sigma, g_7(\varepsilon)-\sigma \end{matrix} \right].$$

- The η and N dependence completely factorizes, and after closing the integration contour and summing residues a linear combination of hypergeometric ${}_4F_3$-functions is obtained

$$I = \sum_j \mathcal{C}_j(\varepsilon, N)\, {}_4F_3 \left[\begin{matrix} a_1(\varepsilon), a_2(\varepsilon), a_3(\varepsilon), a_4(\varepsilon) \\ b_1(\varepsilon), b_2(\varepsilon), b_3(\varepsilon) \end{matrix}, \eta \right]$$

- The ε-expansion gives rise to (poly)logarithmic functions with argument η, $\sqrt{\eta}$ and $-\sqrt{\eta}$.

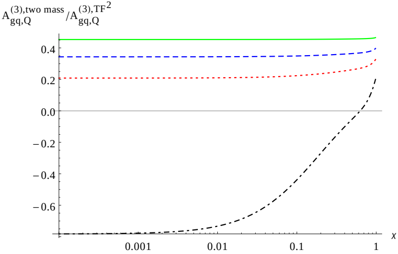
Results: $A_{qq,Q}^{(3),\mathrm{NS}}$

$$
\begin{aligned}
\tilde{a}_{qq,Q}^{(3),\mathrm{NS}} = {} & C_F T_F^2 \Biggl\{ \left(\frac{4}{9} S_1 - \frac{3N^2+3N+2}{9N(N+1)} \right) \Biggl[-24 \left({L_1}^3 + {L_2}^3 + (L_1 L_2 + 2\zeta_2 + 5)(L_1 + L_2) \right) \\
& + \frac{\eta+1}{\eta^{3/2}} \left(5\eta^2 + 22\eta + 5 \right) \left(-\frac{1}{4} \ln^2(\eta) \ln\left(\frac{1+\sqrt{\eta}}{1-\sqrt{\eta}} \right) + 2\ln(\eta) \mathrm{Li}_2\left(\sqrt{\eta}\right) - 4\mathrm{Li}_3\left(\sqrt{\eta}\right) \right) \\
& + \frac{\left(\sqrt{\eta}+1\right)^2}{2\eta^{3/2}} \left(-10\eta^{3/2} + 5\eta^2 + 42\eta - 10\sqrt{\eta} + 5 \right) \left[\mathrm{Li}_3(\eta) - \ln(\eta) \mathrm{Li}_2(\eta) \right] + \frac{64}{3}\zeta_3 \\
& + \frac{8}{3} \ln^3(\eta) - 16 \ln^2(\eta) \ln(1-\eta) + 10 \frac{\eta^2-1}{\eta} \ln(\eta) \Biggr] + \frac{16\left(405\eta^2 - 3238\eta + 405\right)}{729\eta} S_1 \\
& + \frac{4}{3} \left(\frac{3N^4 + 6N^3 + 47N^2 + 20N - 12}{3N^2(N+1)^2} - \frac{40}{3} S_1 + 8 S_2 \right) \left[\frac{4}{3}\zeta_2 + (L_1 + L_2)^2 \right] \\
& + \frac{8}{9} \left(\frac{130N^4 + 84N^3 - 62N^2 - 16N + 24}{3N^3(N+1)^3} - \frac{52}{3} S_1 + \frac{80}{3} S_2 - 16 S_3 \right) (L_1 + L_2) \\
& + \left[-\frac{R_1}{18N^2(N+1)^2\eta} + \frac{2\left(5\eta^2 + 2\eta + 5\right)}{9\eta} S_1 + \frac{32}{9} S_2 \right] \ln^2(\eta) - \frac{4R_2}{729N^4(N+1)^4\eta} \\
& + \frac{3712}{81} S_2 - \frac{1280}{81} S_3 + \frac{256}{27} S_4 \Biggr\}
\end{aligned}
$$

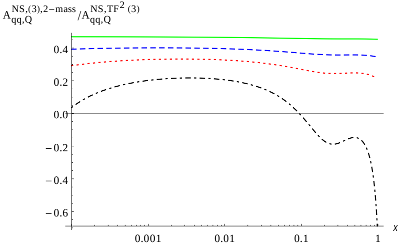
The R_i's are polynomials in N and η. For $\tilde{a}_{gq,Q}^{(3)}$ and $\tilde{a}_{qq,Q}^{(3),\mathrm{NS,\,Tr}}$ one finds similar expressions.

Results: $A_{qq,Q}^{(3),\mathrm{NS}}$ and $A_{gq,Q}^{(3)}$

$A_{gq,Q}^{(3),\text{two mass}} / A_{gq,Q}^{(3),T_F^2}$

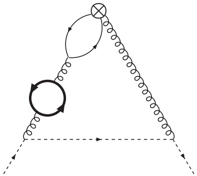


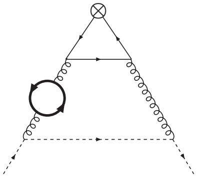
$A_{qq,Q}^{(3),\mathrm{NS},\text{two mass}} / A_{qq,Q}^{(3),\mathrm{NS},T_F^2}$



- The ratio of the genuine 2-mass contributions to $A_{qq,Q}^{(3),\mathrm{NS}}$ and $A_{gq,Q}^{(3)}$ to the respective complete T_F^2-part of massive 3-loop OMEs as a function of x and Q^2, for $m_c = 1.59$ GeV, $m_b = 4.78$ GeV in the on-shell scheme. Dash-dotted line: $\mu^2 = 30$ GeV^2; Dotted line: $\mu^2 = 50$ GeV^2; Dashed line: $\mu^2 = 100$ GeV^2; Full line: $\mu^2 = 1000$ GeV^2.

Two mass contribution to $A_{Qq}^{\mathrm{PS},(3)}$





- In general the same steps as before are applicable.
- Problem: the sums for the N-space solution are not first order factorizable
  $\Rightarrow$ Summation algorithms of `Sigma` cannot find the closed form solution
- Solution: aim directly for the momentum space solution, which is first order factorizable

Details: $A_{Qq}^{(3),\mathrm{PS}}$

- Leaving one Feynman parameter unintegrated one obtains representations like:

$$r(N) \int_0^1 dx\, x^N f(x) \int_{-i\infty}^{+i\infty} d\sigma\, \xi^\sigma g(\sigma)$$

- We now have two different cases for closing the contour:

o Case 1: $\xi = \frac{1}{\eta x(1-x)}$
We can close the contour to the left, since $\xi \geq 4/\eta$.

o Case 2: $\xi = \frac{\eta}{x(1-x)}$
We have to split the integration region into the three regions:

1: $x \in (0, \eta_-)$: close to the left
2: $x \in (\eta_-, \eta_+)$: close to the right
3: $x \in (\eta_+, 1)$: close to the left

with $\eta_\pm = \frac{1}{2}\left(1 \pm \sqrt{1-\eta}\right)$

- This shows that in momentum fraction space functions with different support contribute.

Details: $A_{Qq}^{(3),\mathrm{PS}}$

- The residue sums can be done with `Sigma`, `EvaluateMultiSums` and `HarmonicSums` in terms of generalized iterated integrals and there special values

$$G\left(\{f_1(\tau), f_2(\tau), \cdots, f_n(\tau)\}, z\right) = \int_0^z d\tau_1 \; f_1(\tau_1) G\left(\{f_2(\tau), \cdots, f_n(\tau)\}, \tau_1\right)$$

- Rational prefactors in N can be absorbed via integration by parts identities or convolution integrals, i.e.

$$\frac{1}{(n+a)^i} \int_0^1 \mathrm{d}z \; z^n \; f(z) = \int_0^1 \mathrm{d}z \; z^n \left\{ \int_z^1 \mathrm{d}y \; (-1)^{i-1} \left(\frac{y}{z}\right)^a \left[\mathrm{H}_0\left(\frac{y}{z}\right)\right]^{i-1} f(y) \right\}.$$

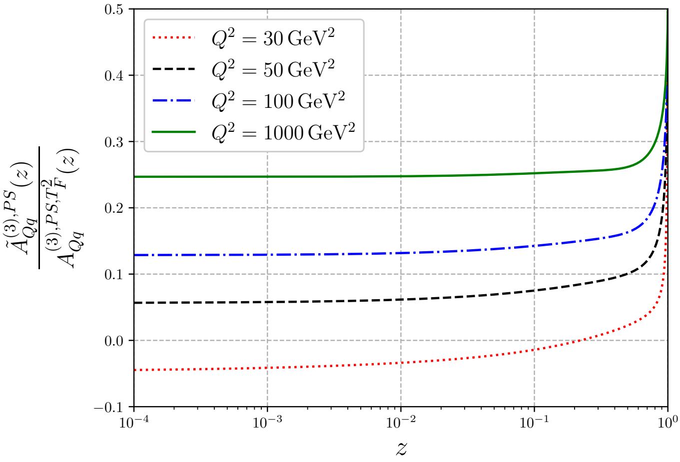
- The alphabet of the occurring iterated integrals contains only two new letters:

$$\frac{1}{x}, \qquad \frac{1}{1+x}, \qquad \frac{1}{1-x}, \qquad \sqrt{4-x}\sqrt{x}, \qquad \frac{\sqrt{1-4x}}{x}$$

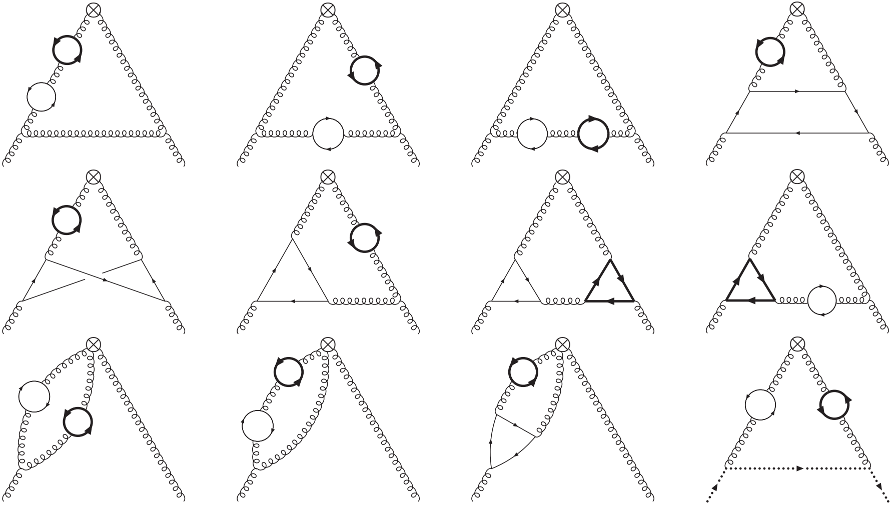
- For numerical evaluations we calculated all occurring iterated integrals as HPLs at involved and complex arguments.

Result: $A_{Qq}^{(3),\mathrm{PS}}$

- The contribution to the whole T_F^2-term is given by:



Two mass contribution to $A_{gg,Q}^{(3)}$



Details: $A_{gg,Q}^{(3)}$

- In general the structures in N and η do not factorize:

$$\int_{-\infty}^{+\infty} \eta^{\sigma}\Gamma\left[\begin{matrix}(2+\frac{\varepsilon}{2}-\sigma)^2, \varepsilon-\sigma, -\sigma, \sigma-\frac{3\varepsilon}{2}, (2-\varepsilon+\sigma)^2, N-\frac{\varepsilon}{2}+\sigma \\ N+2+\frac{\varepsilon}{2}, 4+\varepsilon-2\sigma, 4-2\varepsilon+2\sigma\end{matrix}\right]$$

- The gluonic Feynman rules introduce large numerator structures

$\rightarrow$ single diagrams can lead to big expressions after taking all residues, the most involved diagram $11b$ amounts to $\sim$ 100MB disk space

- Our approach to tackle these sums: (See also the talk by C. Schneider.)
  1. Crunch the expressions to a few master sums using `SumProduction`.
  2. Solve these master sums independently using the refined algorithms implemented in `EvaluateMultiSums` using `HarmonicSums` for limiting procedures.
  3. Reduce the occurring sums from the master sums to a smaller set of independent sums.
- This way the summation of diagram $11b$ can be tackled in $\sim$ 78 days and $\sim$ 33 days are needed to reduce the occurring sums to a basis
- The full summation amounted to around 5 months of calculation.

Results: $A_{gg,Q}^{(3)}$

$$
\begin{aligned}
\tilde{a}_{gg,Q}^{(3)}(N) &= \frac{1}{2}\left(1+(-1)^N\right)\Bigg\{T_F^3\Bigg\{\frac{32}{3}\left(L_1^3+L_2^3\right)+\frac{64}{3}L_1L_2\left(L_1+L_2\right)+32\zeta_2\left(L_1+L_2\right)+\frac{128}{9}\zeta_3\Bigg\}\\
&+ C_F T_F^2\Bigg\{\cdots+32\left(\mathrm{H}_0^2(\eta)-\frac{1}{3}S_2\right)S_1+\frac{128}{3}S_{2,1}-\frac{64}{3}S_{1,1,1}\left(\frac{1}{1-\eta},1-\eta,1,N\right)\\
&-\frac{4P_{41}}{3(N-1)N^3(N+1)^2(N+2)(2N-3)(2N-1)}\left(\frac{\eta}{1-\eta}\right)^N\Bigg[\mathrm{H}_0^2(\eta)\\
&-2\mathrm{H}_0(\eta)S_1\left(\frac{\eta-1}{\eta},N\right)-2S_2\left(\frac{\eta-1}{\eta},N\right)+2S_{1,1}\left(\frac{\eta-1}{\eta},1,N\right)\Bigg]+\ldots\Bigg\}\\
&+ C_A T_F^2\Bigg\{\cdots+\Bigg[\frac{8P_{65}}{3645\eta(N-1)N^3(N+1)^3(N+2)(2N-3)(2N-1)}\\
&+\frac{8P_{37}\mathrm{H}_0(\eta)}{45\eta(N-1)N^2(N+1)^2(N+2)}+\frac{2P_{23}\mathrm{H}_0^2(\eta)}{9\eta(N-1)N(N+1)^2}+\frac{32}{27}\mathrm{H}_0^3(\eta)+\frac{128}{9}\mathrm{H}_{0,0,1}(\eta)\\
&+\frac{64}{9}\mathrm{H}_0^2(\eta)\mathrm{H}_1(\eta)-\frac{128}{9}\mathrm{H}_0(\eta)\mathrm{H}_{0,1}(\eta)\Bigg]S_1\\
&+\frac{2^{-1-2N}P_{47}}{45\eta^2(N-1)N(N+1)^2(N+2)(2N-3)(2N-1)}\binom{2N}{N}\sum_{i=1}^{N}\frac{4^i\left(\frac{\eta}{\eta-1}\right)^i}{i\binom{2i}{i}}\Bigg\{\frac{1}{2}\mathrm{H}_0^2(\eta)\\
&S_{1,1}\left(\frac{\eta-1}{\eta},1,i\right)\Bigg\}+\ldots\Bigg\}
\end{aligned}
$$

Details: $A_{gg,Q}^{(3)}$

- The Mellin-inversion of the binomial sum structures can be handled with an improved algorithm implemented in HarmonicSums (see the talk by J. Ablinger)
- For $\tilde{A}_{gg,Q}^{(3)}$ we find the alphabet:

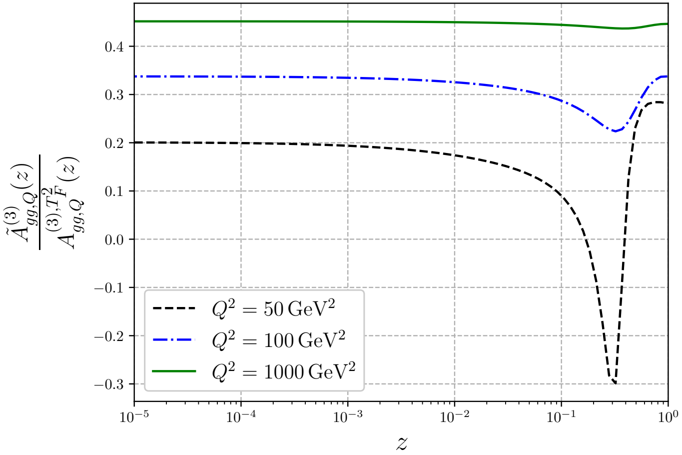
$$\frac{1}{x}\ ,\ \frac{1}{1+x}\ ,\ \frac{1}{1-x}\ ,\ \sqrt{x(1-x)}\ ,\ \frac{1}{x+\eta(1-x)}\ ,\ \frac{1}{1-x(1-\eta)}$$
$$\frac{1}{\eta+x(1-\eta)}\ ,\ \frac{\sqrt{x(1-x)}}{1-x(1-\eta)}\ ,\ \frac{\sqrt{x(1-x)}}{x+\eta(1-x)}\ ,\ \frac{\sqrt{x(1-x)}}{\eta+x(1-\eta)}$$

- Rational prefactors in N have to be included by convolution integrals.
- For numerical evaluations we calculated all occurring iterated integrals as HPLs at involved and complex arguments.

DESY

Results: $A_{gg,Q}^{(3)}$

The two mass contributions over the whole T_F^2- contributions to the OME $A_{gg,Q}^{(3)}$ for different values of Q^2 look like:



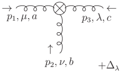
Summary and Outlook

- The 3-loop massive operator matrix elements $A_{qq,Q}^{(3),NS}$, $A_{qq,Q}^{(3),TR}$ $A_{gq,Q}^{(3)}$, $A_{Qq}^{(3),PS}$ and $A_{gg,Q}^{(3)}$ with two heavy quarks have been calculated analytically.
- For $A_{Qq}^{(3),PS}$ and $A_{gg,Q}^{(3)}$ iterative integral representations over square-root valued alphabets, containing the mass ratio, are obtained.
- The contribution of these terms are of comparable size to the complete $T_F^2 C_{F,A}$ contributions.
- Phenomenological implications have been studied to NLO. Here the bottom quark contributions receive the largest corrections of up to 5%.
- The calculation of $A_{Qg}^{(3)}$ still needs to be completed in the single mass case (here elliptic and possibly more involved structures contribute). The two-mass calculation, expanding in the mass ratio, is under way.

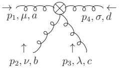
Backup

Feynman rules for the local operator

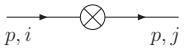
$$\frac{1+(-1)^N}{2}\delta^{ab}(\Delta\cdot p)^{N-2}$$
$$\Big[g_{\mu\nu}(\Delta\cdot p)^2-(\Delta_\mu p_\nu+\Delta_\nu p_\mu)\Delta\cdot p+p^2\Delta_\mu\Delta_\nu\Big]\ ,\quad N\geq 2$$



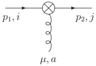
$$-ig\frac{1+(-1)^N}{2}f^{abc}\Bigg(\Big[(\Delta_\nu g_{\lambda\mu}-\Delta_\lambda g_{\mu\nu})\Delta\cdot p_1+\Delta_\mu(p_{1,\nu}\Delta_\lambda-p_{1,\lambda}\Delta_\nu)\Big](\Delta\cdot p_1)^{N-2}$$
$$+\Delta_\lambda\Big[\Delta\cdot p_1 p_{2,\mu}\Delta_\nu+\Delta\cdot p_2 p_{1,\nu}\Delta_\mu-\Delta\cdot p_1\Delta\cdot p_2 g_{\mu\nu}-p_1\cdot p_2\Delta_\mu\Delta_\nu\Big]$$
$$\times\textstyle\sum_{j=0}^{N-3}(-\Delta\cdot p_1)^j(\Delta\cdot p_2)^{N-3-j}$$
$$+\left\{\begin{smallmatrix}p_1\to p_2\to p_3\to p_1\\ \mu\to\nu\to\lambda\to\mu\end{smallmatrix}\right\}+\left\{\begin{smallmatrix}p_1\to p_3\to p_2\to p_1\\ \mu\to\lambda\to\nu\to\mu\end{smallmatrix}\right\}\Bigg)\ ,\quad N\geq 2$$



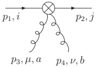
$$g^2\frac{1+(-1)^N}{2}\Bigg(f^{abe}f^{cde}O_{\mu\nu\lambda\sigma}(p_1,p_2,p_3,p_4)$$
$$+f^{ace}f^{bde}O_{\mu\lambda\nu\sigma}(p_1,p_3,p_2,p_4)+f^{ade}f^{bce}O_{\mu\sigma\nu\lambda}(p_1,p_4,p_2,p_3)\Bigg)\ ,$$
$$O_{\mu\nu\lambda\sigma}(p_1,p_2,p_3,p_4)=\Delta_\nu\Delta_\lambda\Bigg\{-g_{\mu\sigma}(\Delta\cdot p_3+\Delta\cdot p_4)^{N-2}$$
$$+[p_{4,\mu}\Delta_\sigma-\Delta\cdot p_4 g_{\mu\sigma}]\textstyle\sum_{i=0}^{N-3}(\Delta\cdot p_3+\Delta\cdot p_4)^i(\Delta\cdot p_4)^{N-3-i}$$
$$-[p_{1,\sigma}\Delta_\mu-\Delta\cdot p_1 g_{\mu\sigma}]\textstyle\sum_{i=0}^{N-3}(-\Delta\cdot p_1)^i(\Delta\cdot p_3+\Delta\cdot p_4)^{N-3-i}$$
$$+[\Delta\cdot p_1\Delta\cdot p_4 g_{\mu\sigma}+p_1\cdot p_4\Delta_\mu\Delta_\sigma-\Delta\cdot p_4 p_{1,\sigma}\Delta_\mu-\Delta\cdot p_1 p_{4,\mu}\Delta_\sigma]$$
$$\times\textstyle\sum_{i=0}^{N-4}\sum_{j=0}^{i}(-\Delta\cdot p_1)^{N-4-i}(\Delta\cdot p_3+\Delta\cdot p_4)^{i-j}(\Delta\cdot p_4)^j\Bigg\}$$
$$-\left\{\begin{smallmatrix}p_1\leftrightarrow p_2\\ \mu\leftrightarrow\nu\end{smallmatrix}\right\}-\left\{\begin{smallmatrix}p_3\leftrightarrow p_4\\ \lambda\leftrightarrow\sigma\end{smallmatrix}\right\}+\left\{\begin{smallmatrix}p_1\leftrightarrow p_2,\ p_3\leftrightarrow p_4\\ \mu\leftrightarrow\nu,\ \lambda\leftrightarrow\sigma\end{smallmatrix}\right\}\ ,\quad N\geq 2$$



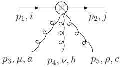
$$\delta^{ij}\not\Delta\gamma_\pm(\Delta\cdot p)^{N-1}\ ,\quad N\geq 1$$



$$gt^a_{ji}\Delta^\mu\not\Delta\gamma_\pm\textstyle\sum_{j=0}^{N-2}(\Delta\cdot p_1)^j(\Delta\cdot p_2)^{N-j-2}\ ,\quad N\geq 2$$



$$g^2\Delta^\mu\Delta^\nu\not\Delta\gamma_\pm\textstyle\sum_{j=0}^{N-3}\sum_{l=j+1}^{N-2}(\Delta p_2)^j(\Delta p_1)^{N-l-2}$$
$$\Big[(t^at^b)_{ji}(\Delta p_1+\Delta p_4)^{l-j-1}+(t^bt^a)_{ji}(\Delta p_1+\Delta p_3)^{l-j-1}\Big]\ ,$$
$$N\geq 3$$



$$g^3\Delta_\mu\Delta_\nu\Delta_\rho\not\Delta\gamma_\pm\textstyle\sum_{j=0}^{N-4}\sum_{l=j+1}^{N-3}\sum_{m=l+1}^{N-2}(\Delta.p_2)^j(\Delta.p_1)^{N-m-2}$$
$$\Big[\ (t^at^bt^c)_{ji}(\Delta.p_4+\Delta.p_5+\Delta.p_1)^{l-j-1}(\Delta.p_5+\Delta.p_1)^{m-l-1}$$
$$+(t^at^ct^b)_{ji}(\Delta.p_4+\Delta.p_5+\Delta.p_1)^{l-j-1}(\Delta.p_4+\Delta.p_1)^{m-l-1}$$
$$+(t^bt^at^c)_{ji}(\Delta.p_3+\Delta.p_5+\Delta.p_1)^{l-j-1}(\Delta.p_5+\Delta.p_1)^{m-l-1}$$
$$+(t^bt^ct^a)_{ji}(\Delta.p_3+\Delta.p_5+\Delta.p_1)^{l-j-1}(\Delta.p_3+\Delta.p_1)^{m-l-1}$$
$$+(t^ct^at^b)_{ji}(\Delta.p_3+\Delta.p_4+\Delta.p_1)^{l-j-1}(\Delta.p_4+\Delta.p_1)^{m-l-1}$$
$$+(t^ct^bt^a)_{ji}(\Delta.p_3+\Delta.p_4+\Delta.p_1)^{l-j-1}(\Delta.p_3+\Delta.p_1)^{m-l-1}\Big]\ ,$$
$$N\geq 4$$

$$\gamma_+=1\ ,\quad\gamma_-=\gamma_5\ .$$

[Bierenbaum, Blümlein and Klein, Nucl.Phys. B820 (2009), 417-482]

Results: $A_{Qq}^{(3),\mathrm{PS}}$

$$
\begin{aligned}
a_{Qq}^{(3),\mathrm{PS}}(N) \quad = \quad & \int_0^1 dx\; x^{N-1} \Big\{ K(\eta,x) + \big(\theta(\eta_- - x) + \theta(x-\eta_+)\big) x\, g_0(\eta,x) \\
& + \theta(\eta_+ - x)\theta(x-\eta_-) \left[x\, f_0(\eta,x) - \int_{\eta_-}^{x} dy \left(f_1(\eta,y) + \frac{y}{x} f_2(\eta,y) + \frac{x}{y} f_3(\eta,y) \right) \right] \\
& + \theta(\eta_- - x) \int_x^{\eta_-} dy \left(g_1(\eta,y) + \frac{y}{x} g_2(\eta,y) + \frac{x}{y} g_3(\eta,y) \right) \\
& - \theta(x - \eta_+) \int_{\eta_+}^{x} dy \left(g_1(\eta,y) + \frac{y}{x} g_2(\eta,y) + \frac{x}{y} g_3(\eta,y) \right) \\
& + x\, h_0(\eta,x) + \int_x^1 dy \left(h_1(\eta,y) + \frac{y}{x} h_2(\eta,y) + \frac{x}{y} h_3(\eta,y) \right) \\
& + \theta(\eta_+ - x) \int_{\eta_-}^{\eta_+} dy \left(f_1(\eta,y) + \frac{y}{\eta_+ x} f_2(\eta,y) + \eta_+ \frac{x}{y} f_3(\eta,y) \right) \\
& + \int_{\eta_+}^{1} dy \left(g_1(\eta,y) + \frac{y}{x} g_2(\eta,y) + \frac{x}{y} g_3(\eta,y) \right) \Big\}
\end{aligned}
$$

The integrals $\int_{\eta_-}^{x} dy$, $\int_{\eta_+}^{x} dy$, $\int_{x}^{1} dy$, $\int_{\eta_-}^{\eta_+} dy$ and $\int_{\eta_+}^{1} dy$ arise from the absorption of N dependant factors.

These are two of the functions:

$$
\begin{aligned}
f_2(\eta, y) \quad = \quad & -\frac{64P_1\left(\eta+4y^2-4y\right)^{3/2}}{9\eta^{3/2}(1-y)y^2} G\left(\left\{\frac{1}{\tau}, \sqrt{4-\tau}\sqrt{\tau}\right\}, \frac{\eta}{y(1-y)}\right) \\
& + G\left(\left\{\sqrt{4-\tau}\sqrt{\tau}\right\}, \frac{\eta}{y(1-y)}\right)\left\{\frac{128}{3}(1-y)G\left(\left\{\frac{1}{\tau}, \sqrt{4-\tau}\sqrt{\tau}\right\}, \frac{\eta}{y(1-y)}\right)\right. \\
& \left.-\frac{32P_1\left(\eta+4y^2-4y\right)^{3/2}}{9\eta^{3/2}(1-y)y^2}\left[1-2\ln\left(\frac{\eta}{y(1-y)}\right)\right]\right\} + \frac{1280}{9}(1-y)\ln^2\left(\frac{\eta}{y(1-y)}\right) \\
& -\frac{128}{3}(1-y)G\left(\left\{\frac{1}{\tau}, \sqrt{4-\tau}\sqrt{\tau}, \sqrt{4-\tau}\sqrt{\tau}\right\}, \frac{\eta}{y(1-y)}\right) - \frac{256}{9}(1-y)\ln^3\left(\frac{\eta}{(1-y)y}\right) \\
& + \frac{32}{3}(1-y)\left[1-2\ln\left(\frac{\eta}{y(1-y)}\right)\right] G\left(\left\{\sqrt{4-\tau}\sqrt{\tau}\right\}, \frac{\eta}{y(1-y)}\right)^2 + \frac{4P_2}{9(1-y)^3y^4} \\
& -\left(\frac{16P_3}{9(1-y)^3y^4} + \frac{512}{3}(1-y)\zeta_2\right)\ln\left(\frac{\eta}{y(1-y)}\right) + \frac{2560}{9}(1-y)\zeta_2 - \frac{1024}{3}(1-y)\zeta_3
\end{aligned}
$$

$$
\begin{aligned}
h_3(\eta, y) \quad = \quad & (1-y)\left\{\frac{512}{9}(1-4\eta y(1-y))^{3/2} G\left(\left\{\frac{1}{\tau}, \frac{\sqrt{1-4\tau}}{\tau}\right\}, \eta y(1-y)\right)\right. \\
& + \frac{1024}{3} G\left(\left\{\frac{\sqrt{1-4\tau}}{\tau}, \frac{\sqrt{1-4\tau}}{\tau}\right\}, \eta y(1-y)\right) + \frac{512}{3} G\left(\left\{\frac{\sqrt{1-4\tau}}{\tau}, \frac{\sqrt{1-4\tau}}{\tau}, \frac{1}{\tau}\right\}, \eta y(1-y)\right) \\
& + \left(\frac{512}{3}\zeta_2 - \frac{1024}{9}\left(4\eta y^2 - 4\eta y - 1\right)^2\right)\ln(\eta y(1-y)) + \frac{4096}{9}\eta y(1-y) + \frac{256}{9}\ln^3(\eta y(1-y)) \\
& + \frac{1280}{9}\ln^2(\eta y(1-y)) + \frac{512}{9}\zeta_2(1-4\eta y(1-y))^{3/2} - \frac{512}{9}\zeta_2 \\
& \left.+\left(\frac{512}{3}\zeta_2 - \frac{2}{9}(1-4\eta y(1-y))^{3/2}\left[2+\ln(\eta y(1-y))\right]\right) G\left(\left\{\frac{\sqrt{1-4\tau}}{\tau}\right\}, \eta y(1-y)\right)\right\}
\end{aligned}
$$