Anomalous dimensions & splitting fct's beyond NNLO

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Jos Vermaseren (Nikhef) and Sven Moch (Hamburg University)

- Introduction, fixed Mellin-N diagram calculations: methods & results
- Structural properties, all-N results: large-n_c/n_f non-singlet, ζ_n parts
- Numerical size of N^{3}LO and N^{4}LO corrections, small-x expansions
- Soft limit: results for the cusp (and virtual) anomalous dimensions

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Hard processes in perturbative QCD

Observables in ep and pp scattering, up to power corrections

$$O^{ep} = f_i \otimes c_i^{\mathrm{o}} \; , \quad O^{pp} = f_i \otimes f_k \otimes c_{ik}^{\mathrm{o}}$$

c^{o}: coefficient functions, scale $\mu \leftrightarrow$ physical hard scale, e.g., Q^2 or M_{Higgs}

Evolution of parton densities (PDFs) f_i , $\otimes =$ Mellin convolution

$$\frac{\partial}{\partial \ln \mu^2} f_i(\xi, \mu^2) \; = \; \left[P_{ik}(\alpha_s(\mu^2)) \otimes f_k(\mu^2) \right](\xi)$$

Splitting functions ($\leftrightarrow$ twist-2 anomalous dimensions) & coefficient funct's:

$$\begin{aligned} P \; &= \; \alpha_{\mathsf{S}} P^{(0)} + \alpha_{\mathsf{S}}^2 P^{(1)} + \alpha_{\mathsf{S}}^3 P^{(2)} + \alpha_{\mathsf{S}}^4 P^{(3)} + \ldots \\ c_a \; &= \; \underbrace{\alpha_{\mathsf{S}}^{n_a} \left[c_a^{(0)} + \alpha_{\mathsf{S}} c_a^{(1)} + \alpha_{\mathsf{S}}^2 c_a^{(2)} \right.}_{\text{NNLO: now the standard approximation for many processes}} \left. + \alpha_{\mathsf{S}}^3 c_a^{(3)} + \ldots \right] \end{aligned}$$

N$^{n>2}$LO: for high precision (e.g., DIS); slow convergence (e.g., Higgs in pp)

MVV ('05) [incl. DIS]; Anastasiou, Duhr, . . . ('15) [$gg \to H{+}X$]; Currie, Gehrmann, . . . ('18) [DIS jets]

Non-singlet (ns) and singlet splitting functions

NS: $2(n_f-1)$ **flavour asymmetries of** $q_i \pm \bar{q}_i$ **+ one total valence distribution**

$$q^{\pm}_{\mathrm{ns},ik} \;=\; q_i \pm \bar{q}_i - (q_k \pm \bar{q}_k) \;\;, \quad q^{\mathrm{v}}_{\mathrm{ns}} \;=\; \sum_{r=1}^{n_f} (q_r - \bar{q}_r)$$

Splitting fct's: $P^{\pm}_{\mathrm{ns}}$**, same in the large-**n_c **limit,** $P^{\mathrm{s}}_{\mathrm{ns}} = P^{\mathrm{v}}_{\mathrm{ns}} - P^{-}_{\mathrm{ns}} \propto d_{abc}d^{abc}$

Evolution of flavour-singlet quark & gluon distributions, $P_{\mathrm{qq}} = P^{+}_{\mathrm{ns}} + P_{\mathrm{ps}}$

$$q_{\mathrm{s}} \;=\; \sum_{r=1}^{n_f} (q_r + \bar{q}_r) \;, \qquad \frac{d}{d\ln\mu^2}\begin{pmatrix} q_{\mathrm{s}} \\ g \end{pmatrix} \;=\; \begin{pmatrix} P_{\mathrm{qq}} & P_{\mathrm{qg}} \\ P_{\mathrm{gq}} & P_{\mathrm{gg}} \end{pmatrix} \otimes \begin{pmatrix} q_{\mathrm{s}} \\ g \end{pmatrix}$$

N**-space calc. (OPE, 'NIKHEF method'): even (**$+$**, singlet) or odd (**$-$**, v)** N

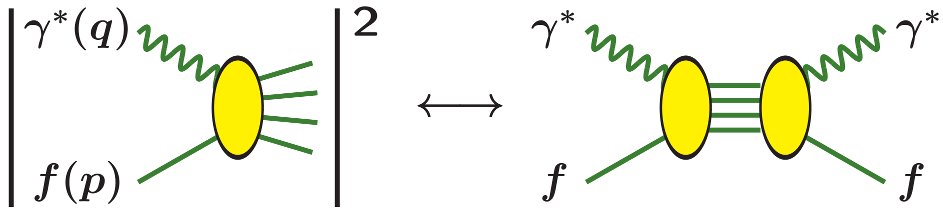
Splitting fct's in N**, so far: harmonic sums** $S_{\vec{w}}$ **and simple denominators**

$$P^{(n)}_{ik}(N) \;=\; \delta_{ik} \sum_{\vec{w},w=0}^{2n+1} c_{00\vec{w}}\, S_{\vec{w}}(N) \;+\; \sum_{a} \sum_{k=1}^{2n+1} \sum_{\vec{w},w=0}^{2n+1-k} c_{ak\vec{w}}\, D^{k}_{a} S_{\vec{w}}(N)$$

$D^{k}_{a} \equiv (N+a)^{-k}$**. Coeff's** c_{00w}**,** c_{akw}**: integer mod low powers of** $\frac{1}{2}$ **and** $\frac{1}{3}$

Splitting functions via calculations of DIS and OMEs

Inclusive DIS



forward amplitude

$p^2 = 0, \; q^2 < 0$

Coefficient of $(2p\cdot q)^N \;\rightarrow\; N$**-th Mellin moment** $A(N) = \int_0^1 dx\, x^{N-1} A(x)$

$D = 4 - 2\varepsilon$, n-**loop:** $\varepsilon^{-1} \rightarrow$ **splitting fct's** $P_{ik}^{(n)}(N)$, $\varepsilon^0 \rightarrow$ **coefficient fct's**

2 and 3 loops: Larin, Vermaseren ('91); Larin, van Ritbergen, Vermaseren [, Nogueira] ('93, '96); . . .

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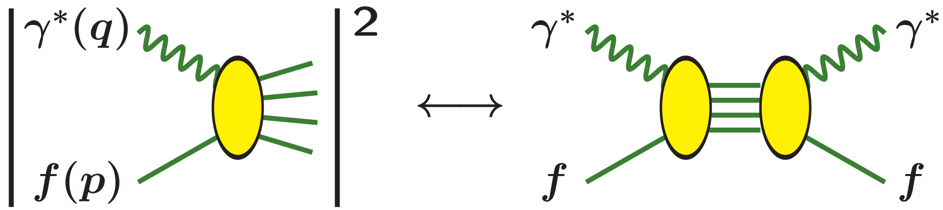

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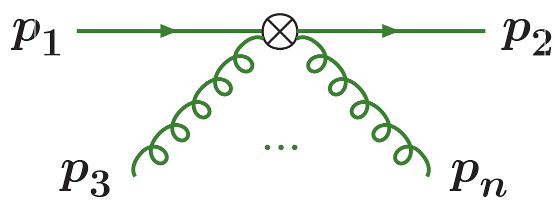
Twist-two quark operators (symm., traceless)

$$O^{(\mathrm{ns})}_{\{\mu_1,\ldots,\,\mu_N\}} = \overline{\psi}\,(\lambda^{\alpha})\,\gamma_{\{\mu_1} D_{\mu_2} \ldots D_{\mu_N\}}\psi \qquad \rightarrow$$



Here: off-shell operator matrix elements (OMEs), $\;p_1 = p_2 = p,\; p^2 < 0$

$$A^{\mathrm{ns}}(N) \;=\; \Delta^{\mu_1} \;\ldots\; \Delta^{\mu_N} \langle p | O^{\mathrm{ns}}_{\{\,\mu_1,\ldots,\,\mu_N\}} | p \rangle \;, \quad \Delta^2 = 0$$

Renormalization constants $\rightarrow$ **anomalous dimensions** $\;\gamma(N) = -P(N)$

2-loop: Floratos, Ross, Sachrajda ('77); 3-loop [heavy quarks, $p^2 = 0$]: Blümlein et al. ('09), . . .

Four-loop calculations up to now

Fixed N: harmonic projection $\rightarrow$ self-energy integrals $\rightarrow$ FORCER program

DIS: $+$ phys. process: conceptually simpler, obtain also coefficient fct's
$-$ extra denom's with p $\Rightarrow$ $N \rightarrow N+2$: integral complexity $c \rightarrow c+4$

- $\gamma q,\ \phi q$ **cases** $\rightarrow P^{(3)}_{\mathrm{qq,gq}}$: $N \leq 8$, $\gamma g,\ \phi g$ **cases** $\rightarrow P^{(3)}_{\mathrm{qg,gg}}$: $N \leq 6$
- **High-n_f (ns: n_f^2, singlet n_f^3 terms): $N > 40$ $\rightarrow$ all-N forms for $P^{(3)}_{\mathrm{ik}}|_{(\mathrm{N})\mathrm{L}n_f}$**

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OPE: $+$ max. one extra denom. $\Rightarrow$ $N \rightarrow N+2$ 'only' leads to $c \rightarrow c+2$
$-$ off-shell OMEs conceptually harder, especially the gluon operator

Floratos, Ross, Sachrajda ('79) . . . , Matiounine, Smith, vN ('98)

- **NS: $N \leq 16 \rightarrow$ approx. in x. Large-n_c of SU(n_c) to $N = 20$: all-N $P^{(3)}_{\rm ns}|_{{\rm L}n_c}$**

$N = 2, 3, 4$: Velizhanin ('12, '14); Baikov, Chetyrkin [, Kühn] ('06, '15)

- **Singlet (in progress): $N \leq 16$ for quartic Casimirs: all-N ζ_5 parts, large N**

One more N very hard, two more N virtually impossible with present means

Towards all-N non-singlet expressions

$\gamma_{\rm ns}(N)$: constrained by ‘self-tuning’ (conjecture, conformal symmetry)

$$\gamma_{\rm ns}(N) = \gamma_{\rm u}\,(N + \sigma\,\gamma_{\rm ns}(N) - \beta(a_{\rm s})/a_{\rm s}))$$

PDFs/fragm. fct’s: $\sigma = \mp 1$; universal kernel $\gamma_{\rm u}$: reciprocity-respecting (RR), invariant under $N \rightarrow (1-N)$ Dokshitzer, Marchesini (’06); Basso, Korchemsky (’06); ...

$\Rightarrow$ Non-RR parts, PDF/fragm. difference ‘inherited’, need to find ‘only’ $\gamma_{\rm u}$

$D_a^k \equiv (N+a)^{-k}$. Denominators at calculated N values: $a = 0, 1$ for $\gamma_{\rm ns}^{\pm}$

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Limit of a large number of colours n_c: $\gamma_{\rm ns}^{+} = \gamma_{\rm ns}^{-}$, no alternating sums,

1, 1, 2, 3, 5, 8, 13 = Fibonacci(w) sums at $w = 1, \ldots, 7$. List to $w = 9$: Velizhanin [website] ('10)

$\Rightarrow$ Sums + powers of $1/[N(N+1)]$ (RR): 87 basis functions at $w \leq 7$

Large-N and small-x limits: more than 40 constraints. Large-N:

$$\gamma_{\rm ns}^{(n-1)}(N) = A_n \ln \widetilde{N} - B_n + N^{-1}\{C_n \ln \widetilde{N} - \widetilde{D}_n + \tfrac{1}{2} A_n\} + O(N^{-2})$$

C_n, $\widetilde{D}_n$: fixed by lower-order information Dokshitzer, Marchesini, Salam ('05), ...

$N \leq 18$ Diophantine eqs. $\Rightarrow$ remaining large-n_c coeff's. Check: $N = 19, 20$

Large-N coefficients in the large-n_c limit

Cusp anomalous dimension, expansion in $a_s = \alpha_s/(4\pi)$

$$
\begin{aligned}
A_{L,4} \;=\;& C_F n_c^3 \Big(\frac{84278}{81} - \frac{88832}{81}\,\zeta_2 + \frac{20992}{27}\,\zeta_3 + 1804\,\zeta_4 - \frac{352}{3}\,\zeta_2\zeta_3 - 352\,\zeta_5 \\
& \quad -32\,\zeta_3^2 - 876\,\zeta_6 \Big) \\
& - C_F n_c^2 n_f \Big(\frac{39883}{81} - \frac{26692}{81}\,\zeta_2 + \frac{16252}{27}\,\zeta_3 + \frac{440}{3}\,\zeta_4 - \frac{256}{3}\,\zeta_2\zeta_3 - 224\,\zeta_5 \Big) \\
& + C_F n_c n_f^2 \Big(\frac{2119}{81} - \frac{608}{81}\,\zeta_2 + \frac{1280}{27}\,\zeta_3 - \frac{64}{3}\,\zeta_4 \Big) - C_F n_f^3 \Big(\frac{32}{81} - \frac{64}{27}\,\zeta_3 \Big)
\end{aligned}
$$

Agreement with result obtained from the large-n_c photon-quark form factor

Henn, Smirnov, Smirnov, Steinhauser [, Lee] (n_f: Apr '16, n_f^0: Dec '16)
ζ_3^2, ζ_6 ($\mathcal{N} = 4$ SYM): Bern, Czakon, Dixon, Kosower, Smirnov ('06); . . .

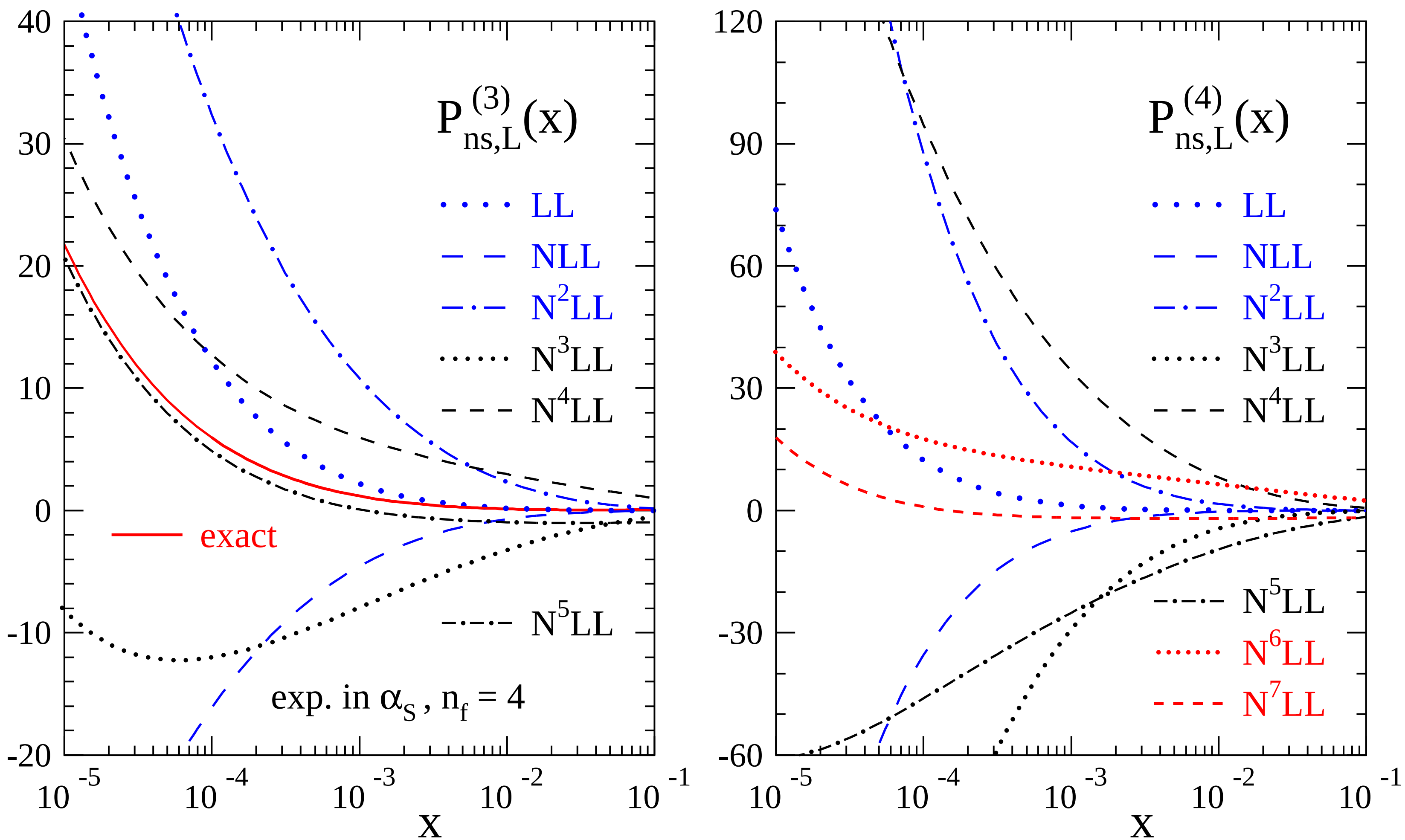
Further non-trivial check of the determination of the all-N form of $\gamma_{\rm ns}(N)$
(+ same for complete n_f^2 contributions)

Also relevant beyond PDFs: $\delta(1-x)$ coeff. $B_{L,4}$, 'virtual anomalous dim.'

Ravindran, Smith, van Neerven ('04); Dixon, Magnea, Sterman ('08); . . . , Dixon ('17)

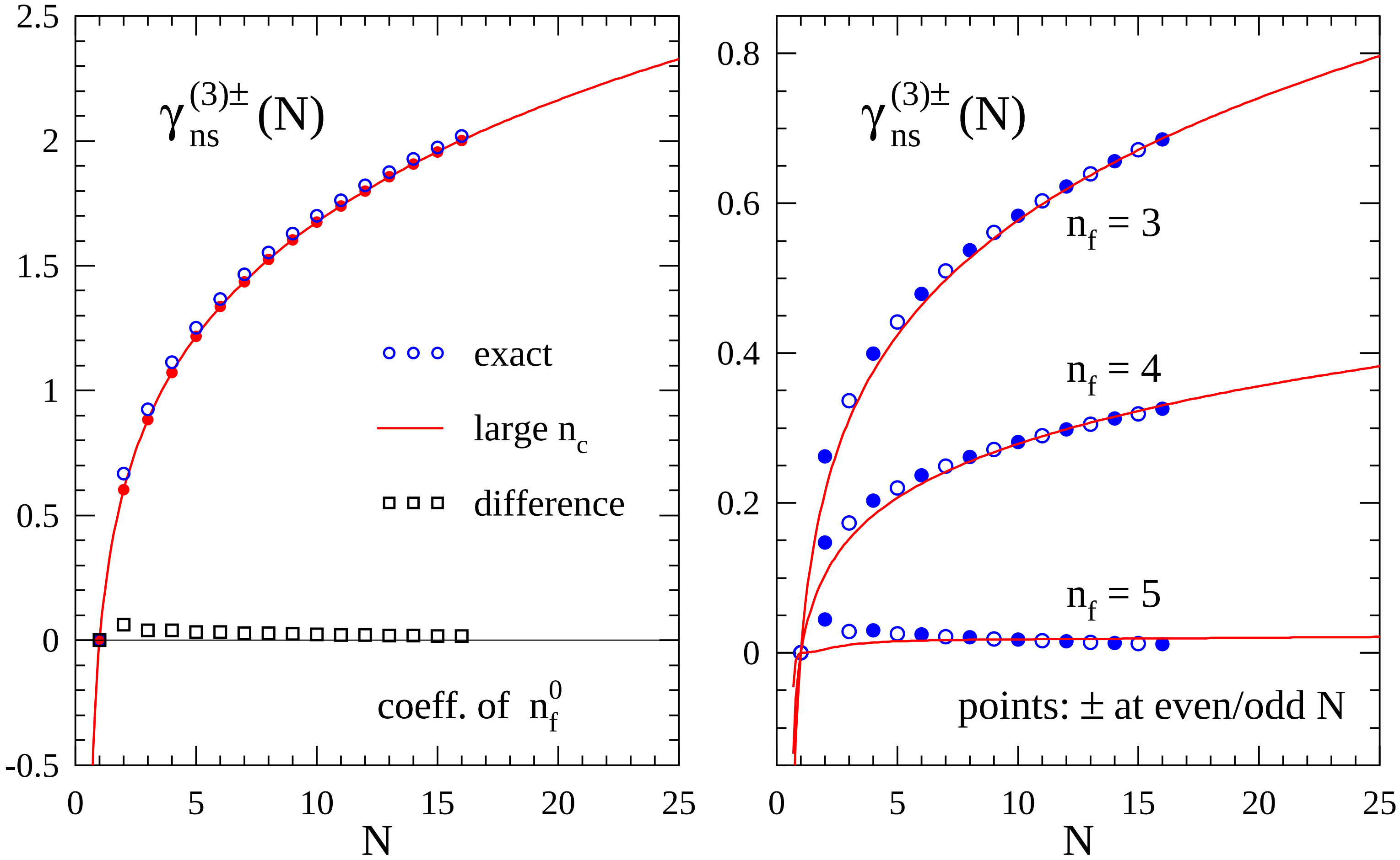
4- and 5-loop large-n_c limit of $P_{\mathrm{ns}}^{\pm}(x)$ at small x

4-loop all-N/all-x expression $\rightarrow$ all $x^0 \ln x$ terms at 5 loops Velizhanin ('14)



Fixed depth ℓ (N$^\ell$LL): not a phenomenologically useful approx. (not only here)

$\gamma_{\rm ns}^{(3)\pm}(N)$: large-$n_c$ limit vs QCD

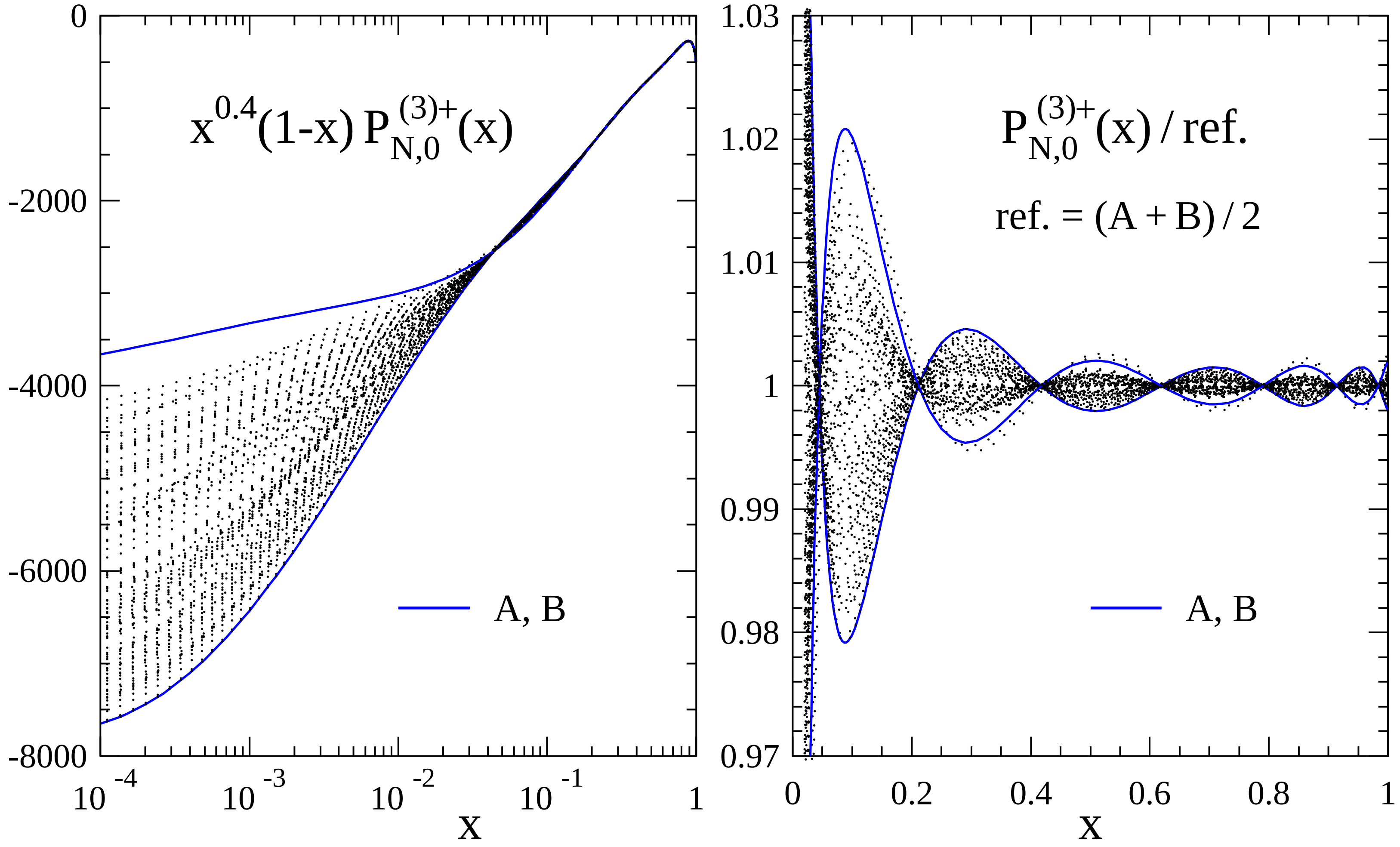


Cancellations between n_f-powers: non large-n_c needed for phenomenology

Approximations analogous to those used before 2004 at 3 loops (but better)

van Neerven, A.V. ('99, '00); MVV [photon structure] ('01), . . .

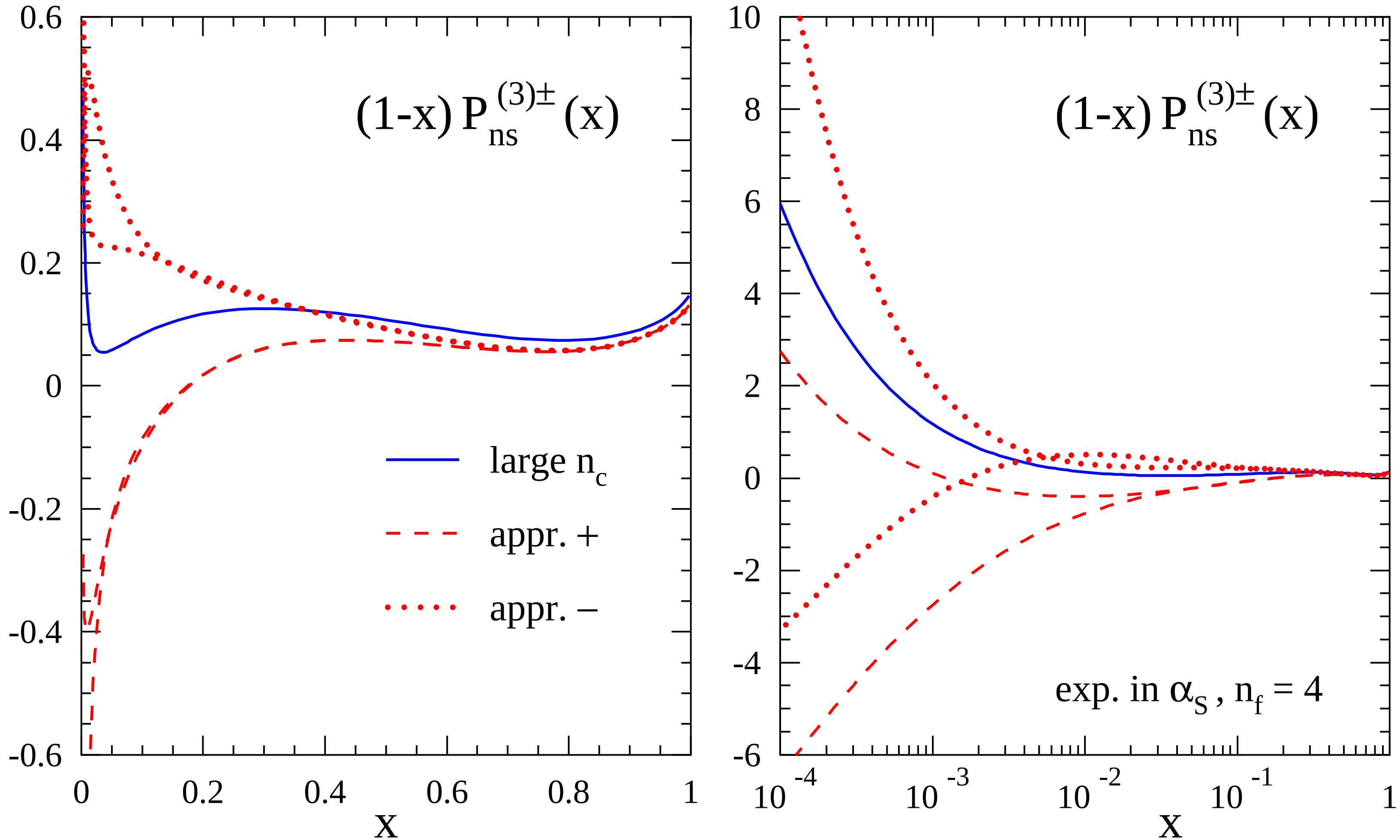
Example: approximations of the n_f^0 component



Factor $(1-x)$ $\rightarrow$ value at $x=1$: contribution to cusp anom. dimension A_4

Korchemsky ('89); Albino, R. Ball ('00), . . .

4-loop large-n_c limit vs $P_{\rm ns}^{\pm}(x)$ in QCD

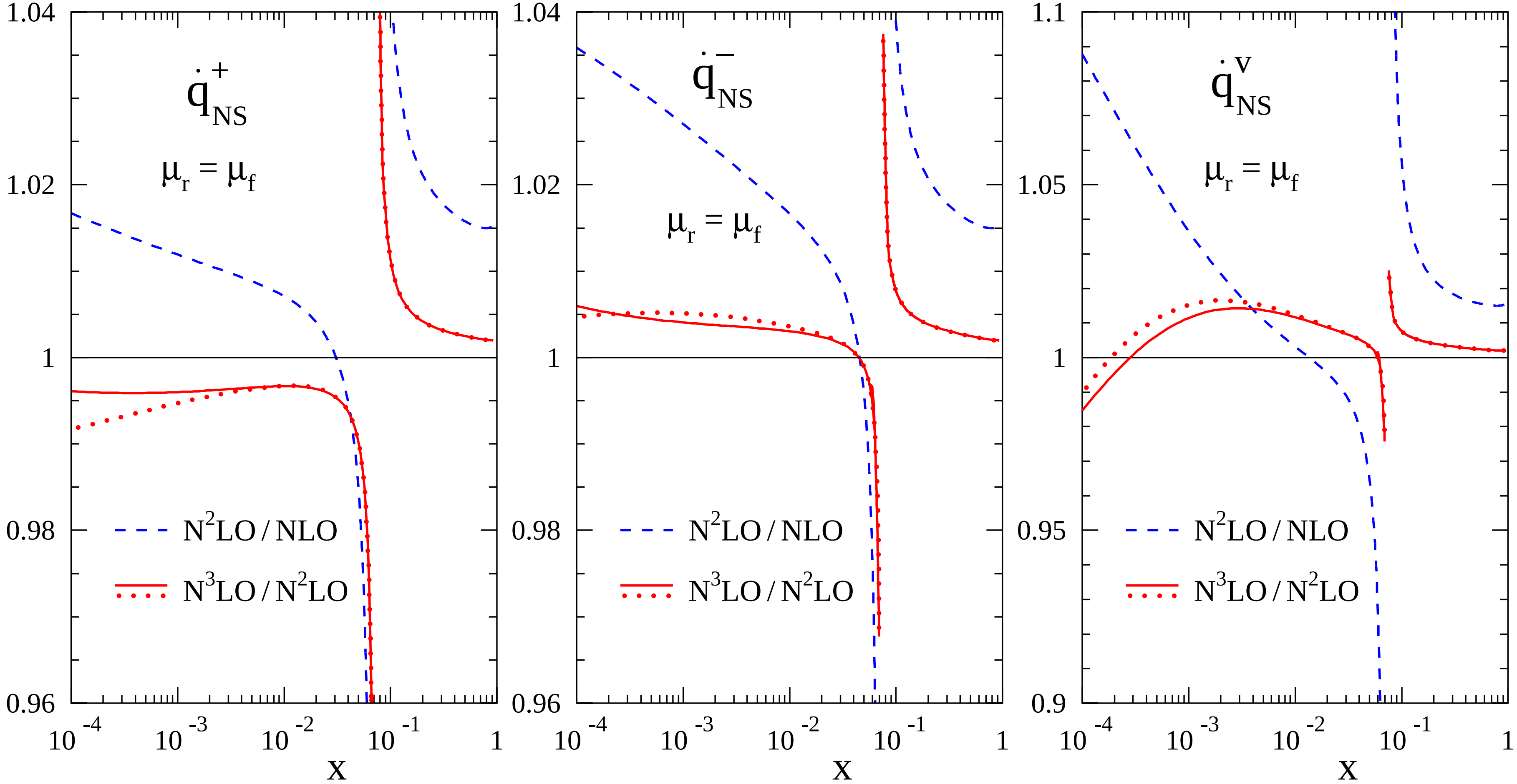


Pattern similar to 3-loop: $+ < -$ at $x < 0.6$, $+$ below Ln_c at small x

$w = 6$ HPLs: Gehrmann, Remiddi (ext. by T.G.); Ablinger, Blümlein, Round, Schneider

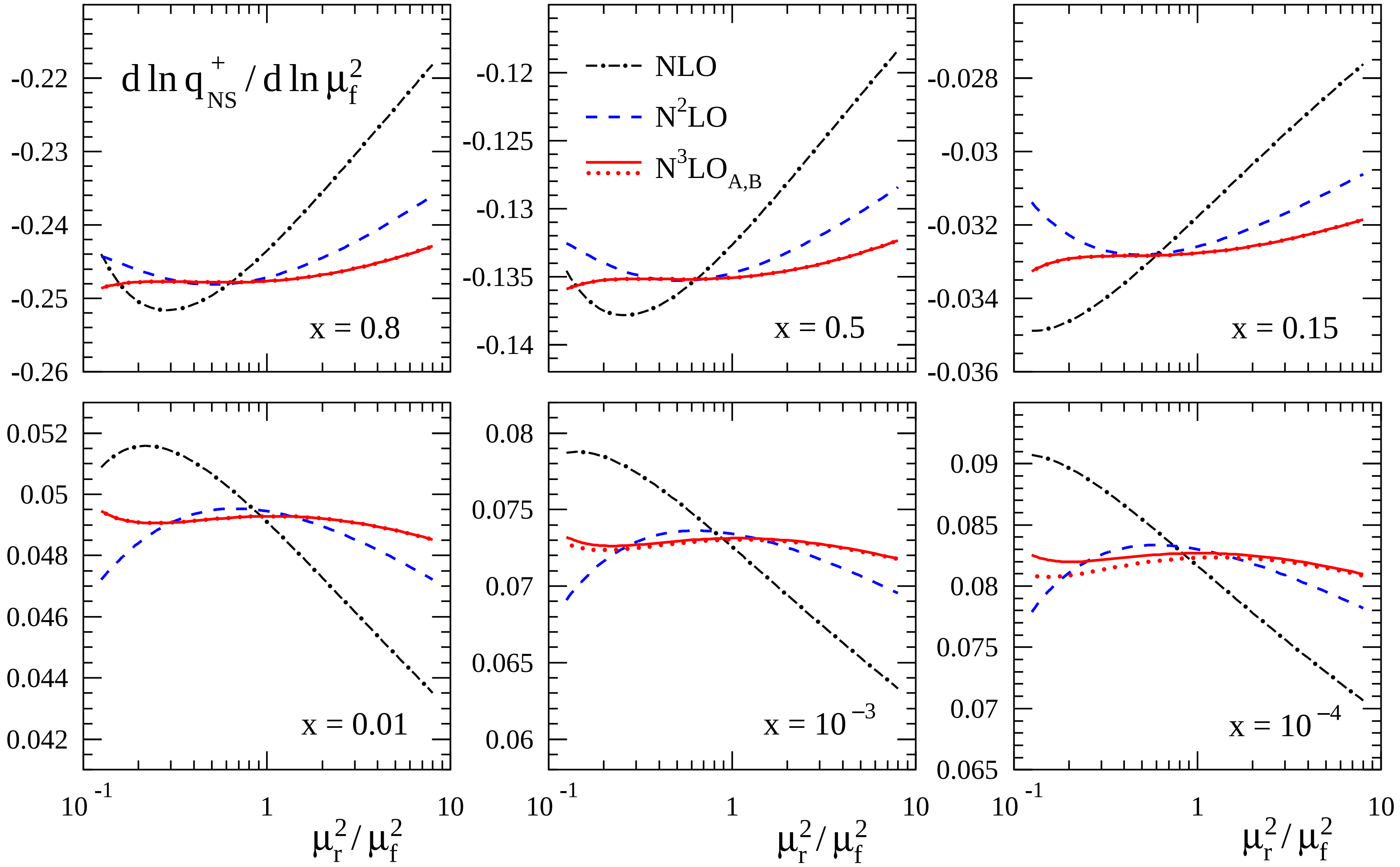
Higher-order corr's to the non-singlet evolutions

Logarithmic derivatives w.r.t. the factorization scale, $\dot{q}^{\,i}_{\rm ns} \equiv d \ln q^{\,i}_{\rm ns} / d \ln \mu_f^2$



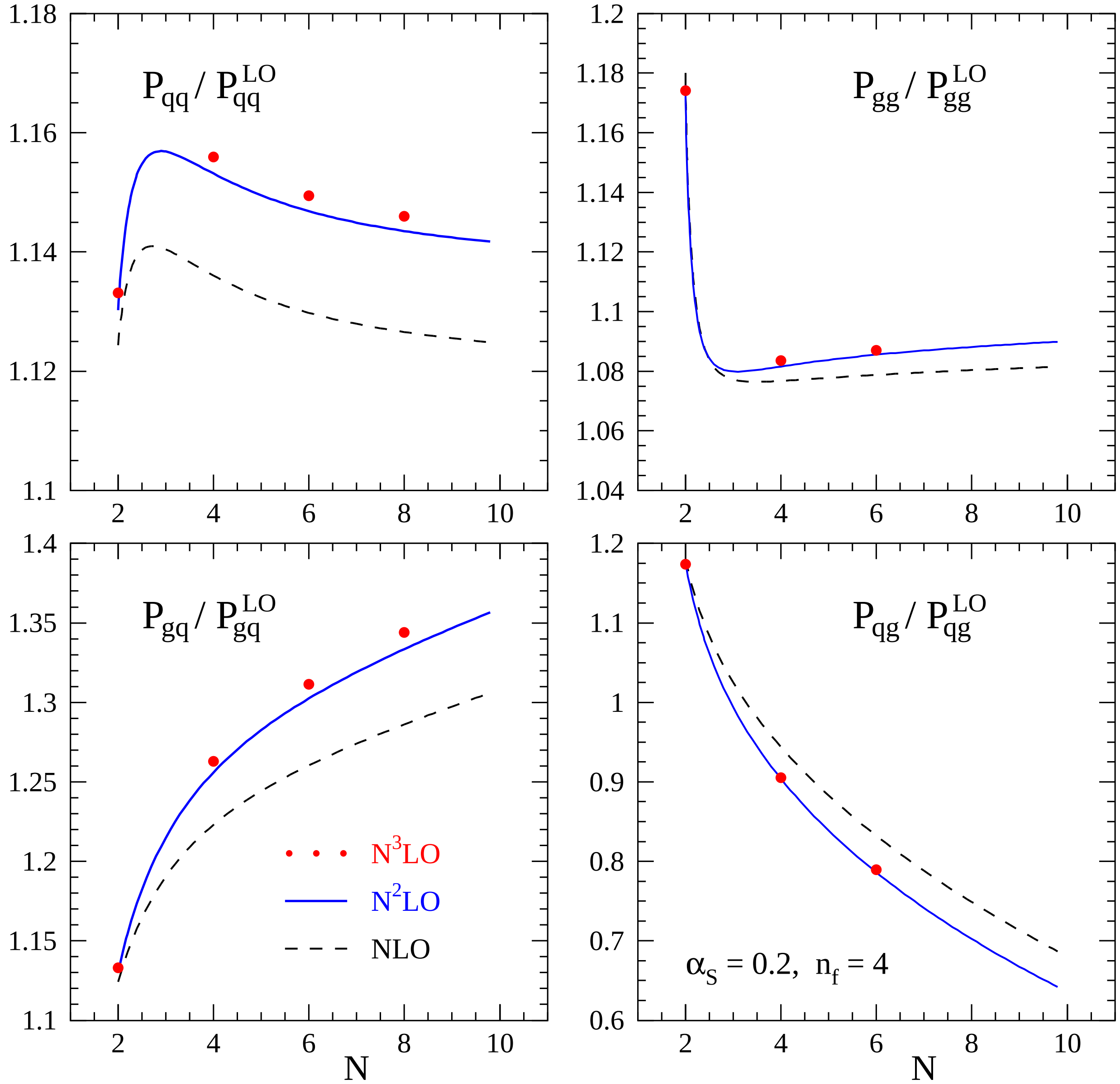
Initial condition $xq^{\pm,\rm v}_{\rm ns}(x,\mu_0^2) = x^{\,0.5}(1-x)^3$ **at** $\alpha_{\rm s}(\mu_0^2) = 0.2\,,\ n_f = 4$

NS$^+$ evolution: renormalization scale dependence



$\frac{1}{2}\mu_f \leq \mu_r \leq 2\mu_f$: **scale uncertainty below 1% except close to sign change**

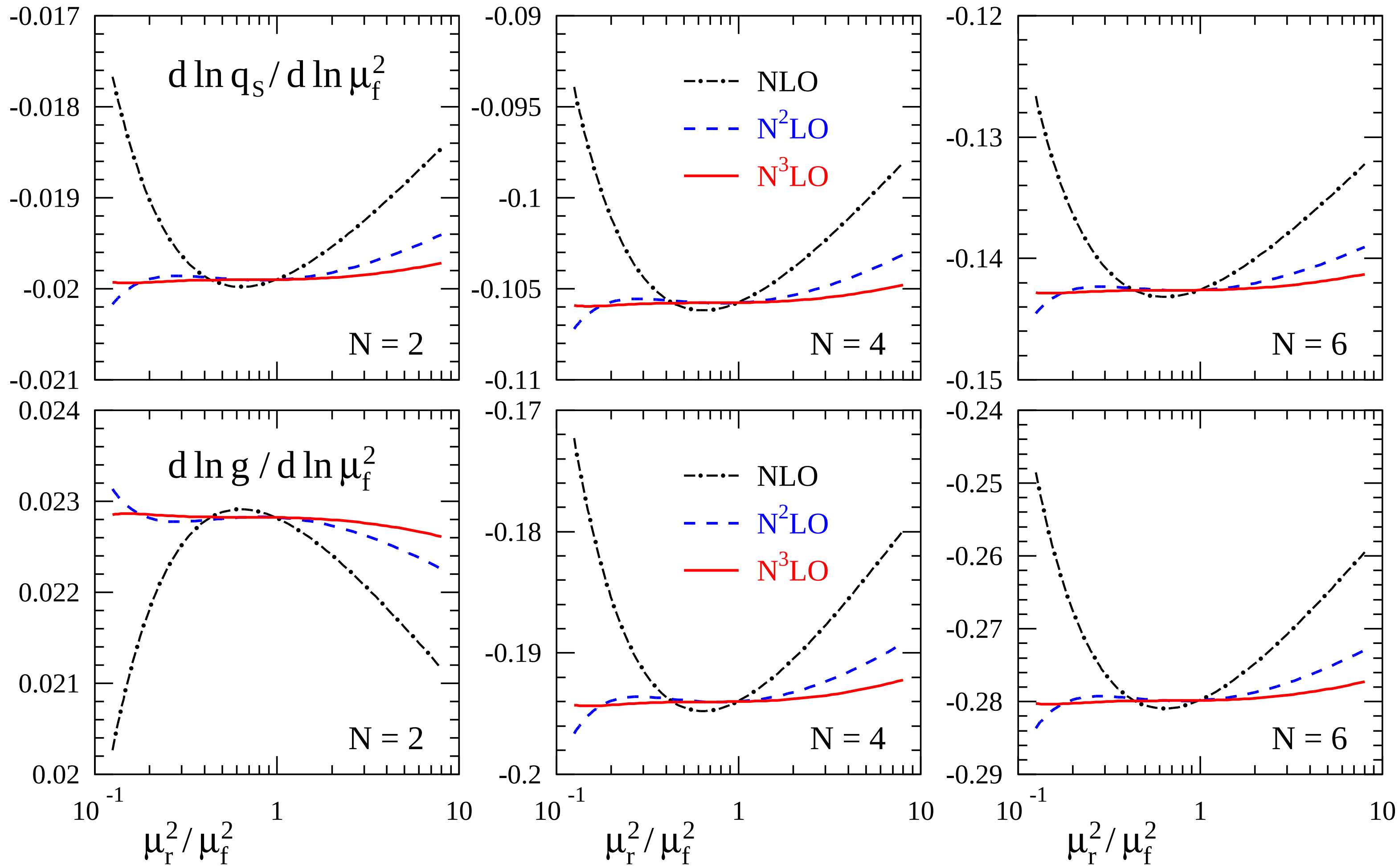
Singlet: N³LO corrections in N-space relative to LO



Absolute sizes

$$|P_{qq}| < |P_{gg}|$$
$$\gg$$
$$P_{gq} \sim P_{qg}$$

Singlet evolution: scale stability in N-space



at $xq_s(x,\mu_0^2) \;=\; 0.6\,x^{-0.3}(1-x)^{3.5}\,(1+5.0\,x^{0.8})\,,$

$xg(x,\mu_0^2) \;=\; 1.6\,x^{-0.3}(1-x)^{4.5}\,(1-0.6\,x^{0.3})\,, \quad \alpha_s(\mu_0^2)=0.2\,,\; n_f=4$

Singlet splitting functions: quartic group invariants

4-loop: 'leading order' for terms with $d_{\rm xy}^{(4)} \equiv d_x^{abcd} d_y^{abcd}$ $\rightarrow$ SUSY relation

$$\gamma_{\rm qq}^{(3)}(N) + \gamma_{\rm gq}^{(3)}(N) - \gamma_{\rm qg}^{(3)}(N) - \gamma_{\rm gg}^{(3)}(N) \;=\; 0$$

for

$$(2n_f)^2 \frac{d_{FF}^{(4)}}{n_a} = 2n_f \frac{d_{FA}^{(4)}}{n_a} = 2n_f \frac{d_{FF}^{(4)}}{n_c} = \frac{d_{FA}^{(4)}}{n_c} = \frac{d_{AA}^{(4)}}{n_a}$$

New off-diagonal relation: $\gamma_{\rm qg}^{(0)}\,\gamma_{\rm gq}^{(3)}|_{d_{Fx}^{(4)}} \;=\; \gamma_{\rm gq}^{(0)}\,\gamma_{\rm qg}^{(3)}|_{d_{Fx}^{(4)}}$ (RR quantities)

Consistent with but going beyond Basso, Korchemsky ('06)

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Consistent with but going beyond Basso, Korchemsky ('06)

All-N expressions for all $\zeta_5 d_{\rm xy}^{(4)}$ parts of the anomalous dimensions $\gamma_{\rm ik}^{(3)}$

- New analytic structure: $\zeta_5 N(N+1)$ in $\gamma_{\rm gg}^{(3)}$ – cancels in SUSY limit
- $\gamma_{\rm gg,\,gq}^{(3)}$ include $1/(N-1)^2 \;\leftrightarrow\; x^{-1}\ln x$ NNLL small-x contributions

No complete (NNLL) BFKL predictions from 3-loop results

$N \le 16$: $d_{\rm xy}^{(4)}$ parts of gluon cusp anomalous dimension $A_{4,\rm g}$, numerically

$\rightarrow$ generalized Casimir scaling: $A_{\rm q}$ coeff's $\Leftrightarrow$ $A_{\rm g}$ coeff's cf. Dixon ('17)

Cusp anomalous dimensions, general group

quark	gluon	$A_{4,\mathrm{q}}$	$A_{4,\mathrm{g}}$
C_F^4	—	0	—
$C_F^3\, C_A$	—	0	—
$C_F^2 C_A^2$	—	0	—
$C_F C_A^3$	C_A^4	610.3 ± 0.3	
$d_{FA}^{(4)}/N_F$	$d_{AA}^{(4)}/N_A$	-507.5 ± 6.0	$-507.0 \pm 5.0^*$
$n_f\, C_F^3$	$n_f\, C_F^2 C_A$	-31.00 ± 0.4	
$n_f\, C_F^2 C_A$	$n_f\, C_F C_A^2$	38.75 ± 0.2	
$n_f\, C_F C_A^2$	$n_f C_A^3$	-440.65 ± 0.2	
$n_f\, d_{FF}^{(4)}/N_F$	$n_f\, d_{FA}^{(4)}/N_A$	-123.90 ± 0.2	-124.0 ± 0.6
$n_f^2\, C_F^2$	$n_f^2\, C_F C_A$	-21.31439	
$n_f^2\, C_F C_A$	$n_f^2\, C_A^2$	58.36737	
—	$n_f^2\, d_{FF}^{(4)}/N_A$	—	0.0 ± 0.1
$n_f^3\, C_F$	$n_f^3\, C_A$	2.454258	2.454258

Large-n_c of $A_{4,\mathrm{q}}$ known: errors correlated. Non-Ln_c pieces done separately in QCD

*** Preliminary results, from $N = 2,\ 4,\ \ldots,\ 14$**

Cusp anomalous dimensions in QCD

4-loop quark cusp anom. dim. for QCD with n_f flavours, expanded in $\alpha_s/4\pi$

$$A_{4,q} = 20702(2) - 5171.9(2)\, n_f + 195.5772\, n_f^2 + 3.272344\, n_f^3$$

In brackets: uncertainty of the preceeding digit(s)

$$
\begin{aligned}
A_q(\alpha_s, n_f = 3) &= 0.42441\, \alpha_s\, [1 + 0.72657\, \alpha_s + 0.73405\, \alpha_s^2 + 0.6647(2)\, \alpha_s^3 + \ldots] \\
A_q(\alpha_s, n_f = 4) &= 0.42441\, \alpha_s\, [1 + 0.63815\, \alpha_s + 0.50998\, \alpha_s^2 + 0.3168(2)\, \alpha_s^3 + \ldots] \\
A_q(\alpha_s, n_f = 5) &= 0.42441\, \alpha_s\, [1 + 0.54973\, \alpha_s + 0.28403\, \alpha_s^2 + 0.0133(2)\, \alpha_s^3 + \ldots]
\end{aligned}
$$

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A_q(\alpha_s, n_f\!=\!5) &= 0.42441\,\alpha_s\,[1 + 0.54973\,\alpha_s + 0.28403\,\alpha_s^2 + 0.0133(2)\,\alpha_s^3 + \ldots]
\end{aligned}$$

Corresponding gluon result, for now

$$A_{4,g} \;=\; 40875(70) - 11714(1)\, n_f + 440.0488\, n_f^2 + 7.362774\, n_f^3$$

$$\begin{aligned}
A_q(\alpha_s, n_f\!=\!3) &= 0.95493\,\alpha_s\,[1 + 0.72657\,\alpha_s + 0.73405\,\alpha_s^2 + 0.415(3)\,\alpha_s^3 + \ldots] \\
A_q(\alpha_s, n_f\!=\!4) &= 0.95493\,\alpha_s\,[1 + 0.63815\,\alpha_s + 0.50998\,\alpha_s^2 + 0.064(3)\,\alpha_s^3 + \ldots] \\
A_q(\alpha_s, n_f\!=\!5) &= 0.95493\,\alpha_s\,[1 + 0.54973\,\alpha_s + 0.28403\,\alpha_s^2 - 0.243(3)\,\alpha_s^3 + \ldots]
\end{aligned}$$

q/g diff. in square brackets: large-n_c suppressed parts of $d_{xy}^{(4)}$. E.g., at n_f^0

$$d_{RA}^{(4)}/N_A = n_c(n_c^2+6)/48 \quad \text{vs} \quad d_{AA}^{(4)}/N_A = n_c^2(n_c^2+36)/24 \quad \text{in} \quad SU(n_c)$$

Numerical Casimir scaling restored in large-n_c limit (of the $d_{xy}^{(4)}$)

5-loop non-singlet results at $N = 2$ and $N = 3$

Lowest N possible via FORCER + R* operation

Chetyrkin, Tkachov ('82); Chetyrkin, Smirnov ('84); ... ; Herzog, Ruijl ('17), see Ben's talk (Thu)

Check of the setup, calculation with gauge parameter: $\gamma_{\rm ns}^{(4)-}(N=1) = 0$.

$N = 2$, numerically for $n_f = 3, 4, 5$:

$$
\begin{aligned}
\gamma_{\rm ns}^{+} &= 0.2829\alpha_s(1 + 0.8695\alpha_s + 0.7980\alpha_s^2 + 0.9258\alpha_s^3 + 1.781\alpha_s^4 + \ldots)\\
\gamma_{\rm ns}^{+} &= 0.2829\alpha_s(1 + 0.7987\alpha_s + 0.5451\alpha_s^2 + 0.5215\alpha_s^3 + 1.223\alpha_s^4 + \ldots)\\
\gamma_{\rm ns}^{+} &= 0.2829\alpha_s(1 + 0.7280\alpha_s + 0.2877\alpha_s^2 + 0.1512\alpha_s^3 + 0.849\alpha_s^4 + \ldots)
\end{aligned}
$$

$N = 3$, numerically for $n_f = 3, 4, 5$:

$$
\begin{aligned}
\gamma_{\rm ns}^{-} &= 0.4421\alpha_s(1 + 0.7952\alpha_s + 0.7183\alpha_s^2 + 0.7605\alpha_s^3 + 1.508\alpha_s^4 + \ldots)\\
\gamma_{\rm ns}^{-} &= 0.4421\alpha_s(1 + 0.7218\alpha_s + 0.4767\alpha_s^2 + 0.3921\alpha_s^3 + 1.031\alpha_s^4 + \ldots)\\
\gamma_{\rm ns}^{-} &= 0.4421\alpha_s(1 + 0.6484\alpha_s + 0.2310\alpha_s^2 + 0.0645\alpha_s^3 + 0.727\alpha_s^4 + \ldots)
\end{aligned}
$$

Together with 4-loop coefficient functions: N^{4}LO physical evolution kernels

π^2 terms (ζ_4, ζ_6): predicted by/support the 'no-π^2 conjecture/theorem'

Jamin, Miravitllas ('17); Davies, A.V. ('17); Baikov, Chetyrkin ('18); see Kostja's talk (Mon)

Summary and Outlook

FORCER: parametric red'n of 4-loop self-energy integrals Ruijl, Ueda, Vermaseren

$\Rightarrow$ More moments of 4-loop splitting functions & coefficient fct's accessible

Non-singlet: exact $P_{\rm ns}^{(3)}(x)$ in the large n_c limit + approximations of the rest

Ln_c: used all known/conjectured features & endpoint results + Diophantine equations

Ok for (almost) all phenomenological purposes, incl. quark cusp anom. dim.

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Singlet $P_{ik}^{(3)}(N)$ in progress: now at $N \leq 6(8)$, quartic invariants to $N = 16$

Mid-/large-x pert. stability, structures (ζ_5 all-N: form, small-x), gluon cusp

Realistic short-term aim: $N = 14$ or 16 $\rightarrow$ x-space approx. $\rightarrow$ N^{3}LO PDFs

Longer term: all-N/all-x calc'n [3-loop: MVV ('04)]: next generation, anyone?

5-loop anomalous dim's at $N = 2, 3$ via FORCER + R^* operation ..., Herzog, Ruijl

'Large-ish' numerical mid-/large-x corr's; structural features, e.g., for ζ_4 & ζ_6