The corolla polynomial: a graph polynomial on half-edges

Dirk Kreimer, Humboldt University, Berlin

based on collaborations with Karen Yeats, Matthias Sars, Walter van Suijlekom and David Prinz

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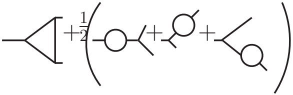
- New graph polynomial: the corolla polynomial
- Some literature:
  Dirk Kreimer, Karen Yeats: *Properties of the corolla polynomial of a 3-regular graph*, arXiv:1207.5460 [math.CO], El.J.Comb.**20**, Issue 1 (2013) Paper no.P41.
  Dirk Kreimer, Matthias Sars, Walter D. van Suijlekom: *Quantization of gauge fields, graph polynomials and graph homology*, [arXiv:1208.6477 [hep-th]], Annals Phys. 336 (2013) 180-222.
  David Prinz: *The Corolla Polynomial for spontaneously broken Gauge Theories*, [arXiv:arXiv:1603.03321 [math-ph]], Math.Phys.Anal.Geom.**19** (2016) no.3, 18.
  James Conan, Karen Vogtmann: *On a theorem of Kontsevich*, [arXiv:math/0208169v2 [math.QA]] Algebr.Geom.Topol.**3** (2003) 1167-1224.

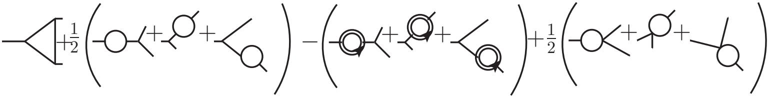
A comparison

There exists a corolla differential D_Γ such that the Feynman integrand I_Γ for connected 3-regular graphs Γ,

$$I_\Gamma = \frac{e^{-\frac{|N_\Gamma|_{\mathrm{Pf}}}{\psi_\Gamma}}}{\psi_\Gamma^2},$$

gives rise, when summed over connected graphs Γ, to the total gauge theory amplitude, using $D_\Gamma I_\Gamma$.





Graph Homology: shrinking edges

Let e be an edge connecting two 3-gluon vertices in a graph Γ, χ_+^e be the operator which shrinks edge e, extend χ_+^e to zero when acting between any other two vertices. Let S, with $S^2 = 0$, be the corresponding graph homology operator. Then, for a gauge theory amplitude r:

Theorem

Let $X^{r,n}_{0\mathrm{x};j\mathrm{gl}}$ be the sum of all 3-regular connected graphs, with j ghost loops, and with external legs determined by r and loop number n, weighted by colour and symmetry, let $X^{r,n}_{/\mathrm{x},j\mathrm{gl}}$ be the same allowing for 3- and 4-valent vertices. We have

$$i) : \ e^{\chi_+} X^{r,n}_{0\ \mathrm{x};j\mathrm{gl}} = X^{r,n}_{/\mathrm{x};j\mathrm{gl}},$$

$$ii) : \ S e^{\chi_+} X^{r,n}_{0\mathrm{x};j\mathrm{gl}} = 0.$$

Cycle Homology: marking loops

Let δ_+^C be the operator which marks a cycle C through 3-valent vertices and unmarked edges, extend δ_+^C to zero on any other cycle. Let T, with $T^2 = 0$, be the corresponding cycle homology operator. Then:

Theorem

Let $X^{r,n}_{j\mathrm{x};0\mathrm{gl}}$ be the sum of all connected graphs with j 4-vertices contributing to amplitude r and loop number n and no ghost loops, weighted by colour and symmetry, $X^{r,n}_{j\mathrm{x};/\mathrm{gl}}$ be the same allowing for any possible number of ghost loops. We have

$$i): \; e^{\delta_+} X^{r,n}_{j\mathrm{x};0\mathrm{gl}} = X^{r,n}_{j\mathrm{x};/\mathrm{gl}},$$

$$ii): \; T e^{\delta_+} X^{r,n}_{j\mathrm{x};0\mathrm{gl}} = 0.$$

Total Homology: gauge theory amplitudes

These two operations are compatible:

Theorem

i) *We have* $[S, T] = 0 \Leftrightarrow (S+T)^2 = 0$ *and*

$$Te^{\delta_+ + \chi_+} X^{r,n}_{0\mathrm{x};0\mathrm{gl}} = 0, \qquad Se^{\delta_+ + \chi_+} X^{r,n}_{0\mathrm{x};0\mathrm{gl}} = 0.$$

ii) *Together, they generate the whole gauge theory amplitude from 3-regular graphs:*

$$e^{\delta_+ + \chi_+} X^{r,n}_{0\mathrm{x};0\mathrm{gl}} = X^{r,n}_{/\mathrm{x};/\mathrm{gl}} =: X^{r,n}.$$

$X^{r,n}$ is the only non-trivial element in the bicomplex of cycle- and graph-homology. This is BRST homology graph-theoretically.

The corolla polynomial

It is a polynomial based on half-edge variables $a_{v,j}$ assigned to any half-edge (v, j) determined by a vertex v and an edge j. We need the following definitions:

- For a vertex $v \in V$ let $n(v)$ be the set of edges incident to v (internal or external).
- For a vertex $v \in V$ let $D_v = \sum_{j \in n(v)} a_{v,j}$.
- Let $\mathcal{C}$ be the set of all cycles of Γ (cycles, not circuits). This is a finite set.
- For C a cycle and v a vertex in V, since Γ is 3-regular, there is a unique edge of Γ incident to v and not in C, let v_C be this edge.
- For $i \geq 0$ let

$$C^i = \sum_{\substack{C_1, C_2, \dots C_i \in \mathcal{C} \\ C_j \text{pairwise disjoint}}} \left(\left(\prod_{j=1}^{i} \prod_{v \in C_j} a_{v,v_C} \right) \prod_{v \notin C_1 \cup C_2 \cup \dots \cup C_i} D_v \right)$$

- Let

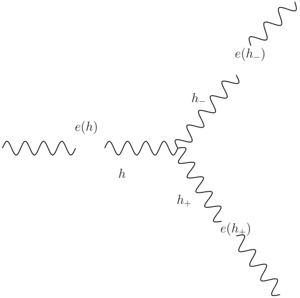
$$C = \sum_{j \geq 0} (-1)^j C^j$$

The corolla differential

Acting on the parametric integrand:

$$D_g(h) := -\tfrac{1}{2} g^{\mu_{h_+}\mu_{h_-}} \left(\epsilon_{h_+} \frac{1}{A_{e(h_+)}} \frac{\partial}{\partial \xi(h_+)_{\mu_h}} - \epsilon_{h_-} \frac{1}{A_{e(h_-)}} \frac{\partial}{\partial \xi(h_-)_{\mu_h}} \right),$$

for any half-edge h.



Note: double differentials wrt the same half edge generate the Feynman rules for a 4-valent vertex via Cauchy's residue formula.

From scalar theory to gauge theory

Finally, we get the Feynman integrands in the unrenormalized and renormalized case for a gauge theory amplitude r from 3-regular connected graphs of scalar fields.

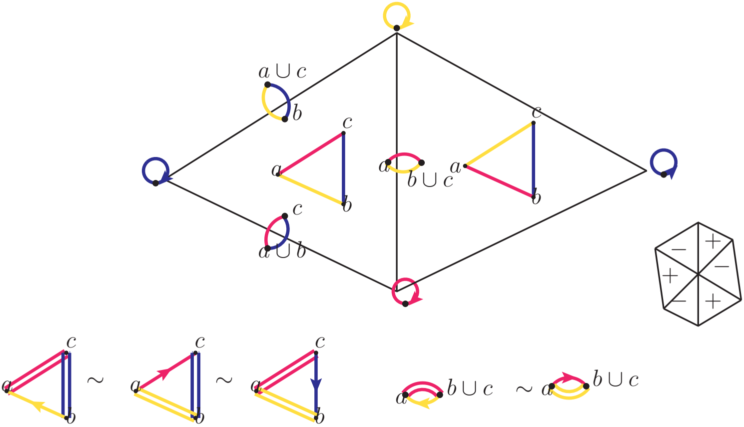
Theorem

The full Yang–Mills amplitude $\bar{U}_\Gamma$ *for a graph* Γ *can be obtained by acting with a corolla differential operator on the scalar integrand* $U_\Gamma(\{\xi_e\})$ *for* Γ*, setting the edge momenta* $\xi_e = 0$ *afterwards. Moreover,* $\bar{U}_\Gamma$ *gives rise to a differential form* $J_\Gamma^{\bar{U}_\Gamma}$ *and there exists a vector* H_Γ *such that the unrenormalized Feynman integrand for the sum of all Feynman graphs contributing to the connected* k*-loop amplitude* r *is*

$$\Phi(X^{r,k}) = \sum_{|\Gamma|=k, \mathbf{res}(\Gamma)=r} \frac{\mathrm{colour}(\gamma)}{\mathrm{sym}(\Gamma)} \int (H_\Gamma \cdot J_\Gamma^{\bar{U}_\Gamma}),$$

The renormalized analogue is given by writing $\bar{U}_\Gamma^R$ *instead of* $\bar{U}_\Gamma$*.*

4-valent gauge couplings as co-dimension one hypersurfaces



The Green Function is an integral over Outer Space - a sum of integrals over the volume of all cells of any codimension free of tadpoles.

Remarks and Outlook

- The two Kirchhoff polynomials are distinguished as unique polynomials on edge variables having recusive contraction deletion properties.
- The corolla polynomial is similarly distinguished amongst half-edge polynomials having recursive half-edge deletion properties.
- David Prinz has generalized this to the full SM.
- Marcel Golz is turning this into a very efficient algorithm for QED amplitudes.
- What is the corolla polynomial for spin 2 bosons?
- Can we combine the reduction to master integrals (Laporta's algorithm) with this transition to gauge theory?
- Thanks for your attention.