Regge limit of scattering amplitudes from an anomalous dimension

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in collaboration with:

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based on arXiv:18021.02524

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Contents

Massive scattering amplitudes

- Study amplitude in different kinematic limits
- Go beyond leading term in the limits
  $\longrightarrow$ concentrate on Regge limit

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Massive scattering amplitudes

- Study amplitude in different kinematic limits
- Go beyond leading term in the limits
  $\longrightarrow$ concentrate on Regge limit

Wilson loops

- Understand Regge limit from an effective field theory point of view
- Leads to a description involving Wilson lines

Massive scattering amplitudes

Setup

Model

- Consider $\mathcal{N} = 4$ SYM in the planar limit
- Introduce masses via Higgs mechanism (all masses equal):

$$SU(N_c) \xrightarrow{\text{SSB}} SU(N_c - 4) \times SU(4) \times U(1)$$

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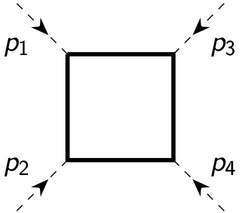
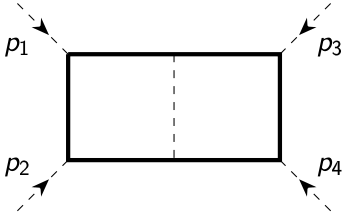
Scattering amplitude





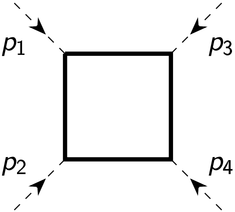
$$A = A_{\text{tree}}\, \mathcal{M}\left(\frac{4m^2}{-s}, \frac{4m^2}{-t}\right)$$

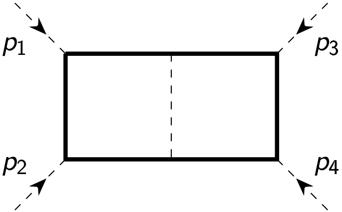
Variables:

$$s = (p_1 + p_2)^2\,, \qquad t = (p_2 + p_3)^2\,, \qquad m = \text{mass of } W \text{ bosons}$$

Perturbative expansion in $g^2 = (g_{\text{YM}}^2 N_c)/(16\pi^2)$:

$$\mathcal{M} = 1 + g^2\, \mathcal{M}^{(1)} + g^4\, \mathcal{M}^{(2)} + g^6\, \mathcal{M}^{(3)} + \mathcal{O}\left(g^8\right)$$





Amplitude

Remarks on the amplitude / integrals

- Amplitudes has dual conformal symmetry!
  [Alday, Henn, Plefka, Schuster, '10]
- UV-finite
- IR-finite: mass of W bosons regulates soft and collinear divergences

$\Rightarrow$ Can calculate the integrals in $D = 4$ dimension

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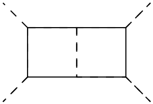
Remarks on the calculation

- Integrand derived using unitarity cuts [Bern, Carrasco, Dennen, Huang, Ita, '10]
- Loop integrals evaluated up to three loops analytically in $D = 4$ dimensions using the differential equation method [Caron-Huot, Henn, '14]

Amplitude $A(s, t, m^2)$



Kinematic limits

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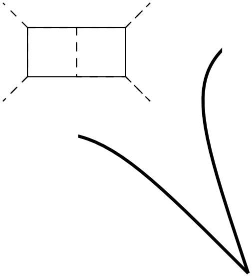
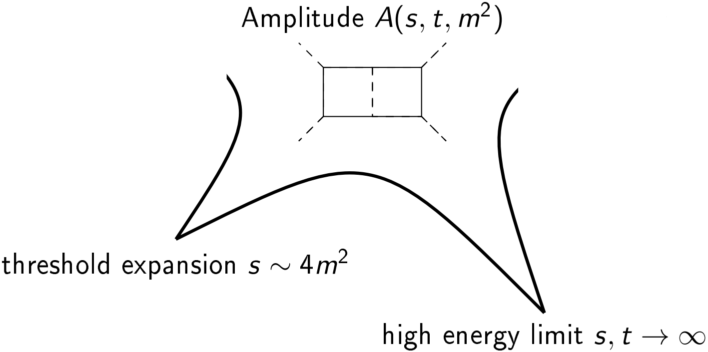


high energy limit $s, t \to \infty$

Kinematic limits



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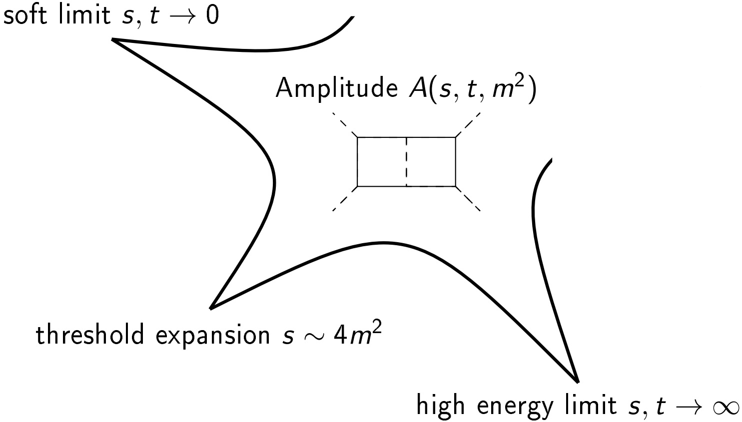

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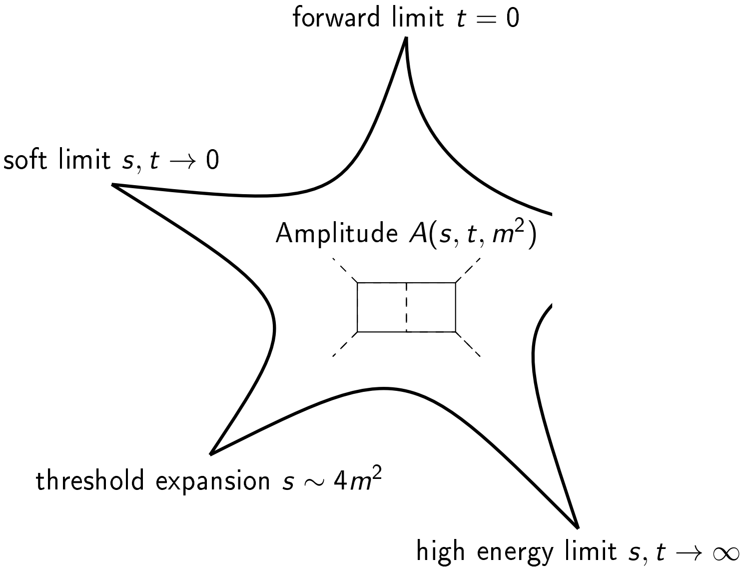

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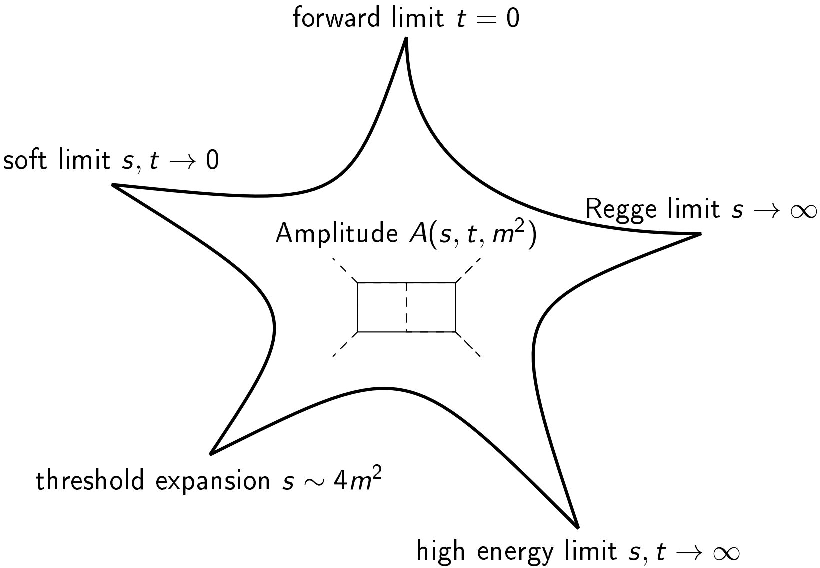

Regge limit

- Regge limit described by $s \gg t, m^2$
- Leading term is given by a simple power law [Henn, Naculich, Schnitzer, Spradlin, '10]

$$\lim_{s\to\infty} \mathcal{M}\left(\frac{4m^2}{-s}, \frac{4m^2}{-t}\right) = r_0(t) \left(\frac{-s}{4m^2}\right)^{\tilde{j}_0(t)} + \mathcal{O}(1/s)$$

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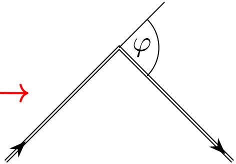
with:

$$\tilde{j}_0(t) = -\Gamma_{\text{cusp}}(\varphi)\,, \qquad t = 4m^2\sin^2(\varphi/2)$$

anomalous dim. of a cusped Wilson loop



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- Single power law does not work for $\mathcal{O}(1/s)$ term

Can improve situation by using better variables

Dual conformal partial waves

- First improvement by a partial wave expansion

$$A = \sum_j a_j P_j\big(\cos\theta_{SO(3)}\big)\,, \qquad \cos\theta_{SO(3)} = 1 + \frac{2s}{t}$$

$\longrightarrow$ this uses the rotational symmetry $SO(3)$

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- In our setup we have a larger $SO(4)$ symmetry, due to dual conformal symmetry

- The corresponding angle is given by

$$\cos\theta_{SO(4)} = 1 + \frac{2s}{t} - \frac{s}{2m^2}$$

- Calculate the Regge limit in $SO(4)$ variable $Y = e^{i\theta_{SO(4)}} \sim 1/s$

Regge limit in $SO(4)$ variable

Leading and subleading terms are described by single power laws:

$$\lim_{Y\to 0} \frac{1+Y}{1-Y}\mathcal{M} = r_0(t)Y^{\tilde{j}_0(t)} + r_1(t)Y^{\tilde{j}_1(t)+1} + \mathcal{O}\left(Y^2\right)$$

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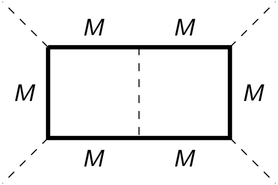
with $(4m^2)/(-t) = 4x/(1-x^2)$ we have

$$r_0 = 1 + \mathcal{O}\left(g^2\right) , \qquad r_1 = 2 + \mathcal{O}\left(g^2\right) , \qquad \tilde{j}_0 = -\Gamma_{\text{cusp}} ,$$

$$\tilde{j}_1 = -4g^2 + g^4\left[\frac{1+x}{1-x}\left(\frac{4\log^3(x)}{3} + \frac{16}{3}\pi^2\log(x)\right) + 8\log^2(x) + 16\frac{1-x}{1+x}\log(x) + \frac{8\pi^2}{3} + 16\right] + \mathcal{O}\left(g^6\right)$$

(calculated up to three loops)

From the Regge limit to Wilson lines



Amplitude:

$$u = \frac{4m^2}{-s}\,,\ v = \frac{4m^2}{-t}$$

[Henn, Naculich, Schnitzer, Spradlin, '10]

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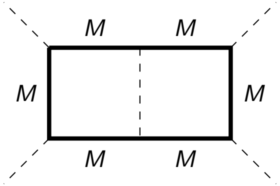

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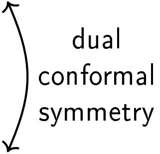

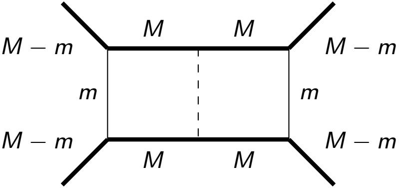

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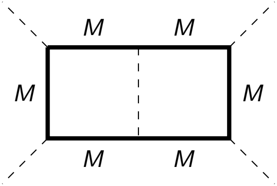

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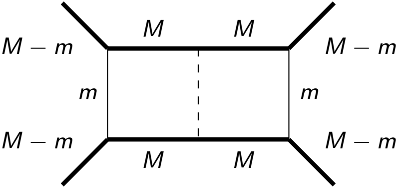
dual conformal symmetry



Amplitude:

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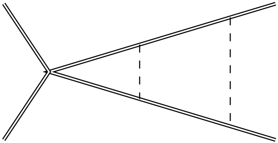
$m \ll M \ll s$



Wilson line:
cusp angle φ

Wilson loops

Cusp anomalous dimension in $\mathcal{N} = 4$ SYM

We consider the Maldacena Wilson loop operator

$$W[C] = P \exp\left[i g_{\text{YM}} \int_C dx_\mu A^\mu + g_{\text{YM}} \int_C |dx| \sqrt{x^2} \phi \right] ,$$

where ϕ is one of the six scalars.

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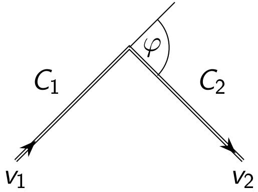
In particular interested in a Wilson loop with a cusp:

$$\boxed{W_{\text{cusp}} = W[C_1]\, W[C_2]}$$



$$v_1^2 = v_2^2 = 1$$

$$\cos(\varphi) = v_1 \cdot v_2 = \frac{1}{2}\left(x + \frac{1}{x}\right)$$

$$x = e^{i\varphi}$$

Cusp anomalous dimension in $\mathcal{N} = 4$ SYM

- In the case of a straight line $\varphi = 0$, the operator W_{cusp} is protected
- Known in planar $\mathcal{N} = 4$ SYM up to 4 loops [Drukker, Forini, '06] [Correa, Henn, Maldacena, Sever '12] [Henn, Huber, '13]
- Planar case governed by integrability [Drukker, '12] [Correa, Maldacena, Sever, '12]

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Result for the anomalous dimension of W_{cusp}:

$$\Gamma_{\text{cusp}} = -2g^2 \frac{1-x}{1+x} \log(x) + \mathcal{O}\left(g^4\right), \qquad x = e^{i\varphi}, \qquad g^2 = \frac{g_{\text{YM}}^2 N_c}{16\pi^2}$$

checked up to three loops

$$\lim_{Y \to 0} \frac{1+Y}{1-Y} \mathcal{M} = r_0(t) Y^{\tilde{j}_0(t)} + r_1(t) Y^{\tilde{j}_1(t)+1} + \mathcal{O}\left(Y^2\right)$$

Effective field theory approach

- We want to describe power corrections $\longrightarrow \tilde{j}_1$
- Power corrections are described by higher dimensional operators in an effective field theory

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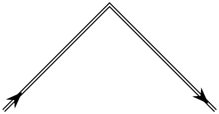
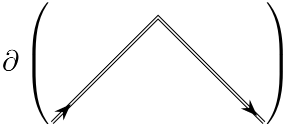
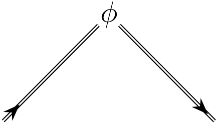
leading term $\tilde{j}_0$	subleading term $\tilde{j}_1$	
$W_{\text{cusp}} =$	$\partial W_{\text{cusp}} =$	$W_{\text{cusp},\phi} =$
$W[C_1]\,W[C_2]$	$\frac{i}{g_{\text{YM}}}(v_2 - v_1)_\mu \partial^\mu\, W_{\text{cusp}}$	$W[C_1]\,\phi(x)\,W[C_2]$

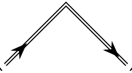






Renormalization of Wilson line operators

Consider:

$$\vec{W} = \{\partial W_{\text{cusp}}, W_{\text{cusp},\phi}\} = \Big\{ \partial \Big($$

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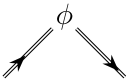

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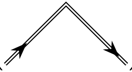
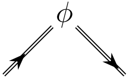
$$\vec{W} = \{\partial W_{\text{cusp}}, W_{\text{cusp},\phi}\} = \Big\{ \partial \Big($$  $$\Big), $$ 

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UV renormalization: $\vec{W}^{\text{ren}} = \mathbf{Z}^{-1}\vec{W}$

$$\mathbf{Z} = \begin{pmatrix} Z_{\text{cusp}} & 0 \\ Z_{\text{mix}} & Z_{\text{cusp},\phi} \end{pmatrix},$$

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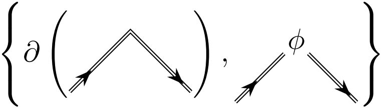

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Straight line case: [Alday, Maldacena, '07]

$\longrightarrow$ non-vanishing anomalous dim.

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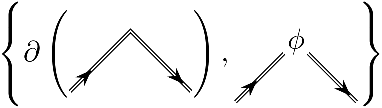

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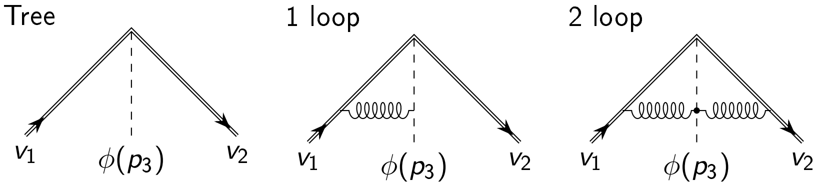
Choose suitable correlator to calculate **Z**: $\langle 0|\vec{W}|\phi(p_3)\rangle$

Correlators

Typical Feynman diagrams for $\langle 0 | W_{\mathrm{cusp},\phi} | \phi(p_3) \rangle$:



We have three invariants:

$$\cos(\varphi) = v_1 \cdot v_2 = \frac{1}{2}\left(x + \frac{1}{x}\right), \quad s_1 = -2 v_1 \cdot p_3 \,, \quad s_2 = 2 v_2 \cdot p_3$$

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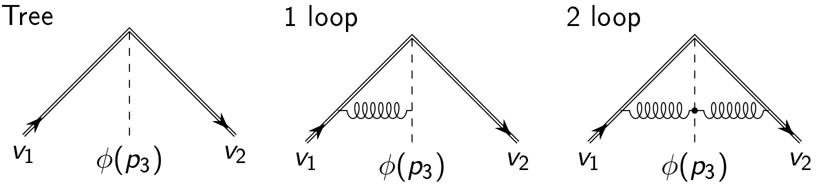

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Divergences:

- UV divergences from the cusp: $\to \mathbf{Z}(x)$
- IR and collinear divergences from on-shell scalar: $\to Z_{\text{IR}}(s_1, s_2)$

Renormalization and anomalous dimensions

Calculation

Renormalization condition + known Z_{cusp}:

$$Z_{\text{IR}}^{-1}\,\mathbf{Z}^{-1}\langle 0|\,\vec{W}|\phi(p_3)\rangle = \text{finite} \quad\Longrightarrow\quad Z_{\text{IR}},\ Z_{\text{cusp},\phi},\ Z_{\text{mix}}$$

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no mixing: $Z_{\text{mix}} = 0$

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$Z_{\text{cusp},\phi} \rightarrow \Gamma_{\text{cusp},\phi}$

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Compare with amplitude

$\Gamma_{\text{cusp},\phi}$

checked up to two loops

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Γ_{cusp} checked up to three loops

Summary and Outlook

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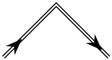
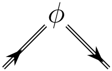
- Massive scattering amplitude in $\mathcal{N} = 4$ SYM
- Regge limit described by Wilson line operators:

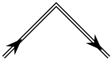
leading term $\longrightarrow$ Γ_{cusp} 

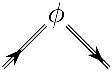
subleading term $\longrightarrow$ $\Gamma_{\text{cusp},\phi}$ 

Summary and Outlook

Summary

- Massive scattering amplitude in $\mathcal{N} = 4$ SYM
- Regge limit described by Wilson line operators:

leading term $\longrightarrow$ Γ_{cusp} 

subleading term $\longrightarrow$ $\Gamma_{\text{cusp},\phi}$ 

Outlook

- Study subsubleading term in Regge limit
- $\Gamma_{\text{cusp},\phi}$ from integrability:
  $\rightarrow$ Inserted scalar does not couple to the Wilson line [Gromov, Levkovich-Maslyuk, '15]
  $\rightarrow$ Inserted scalar does couple to the Wilson line (ladder limit, $\theta = i\infty$) [Cavaglià , Gromov, Levkovich-Maslyuk, '18]

Backup slides

Kinematic limits

- High energy limit $s, t \to \infty$
  soft and collinear divergences regulated by mass
- Threshold expansion $s \sim 4m^2$
  relation to hydrogen-like system
- Soft limit $s, t \to 0$
  effective action
- Forward limit $t = 0$
  total cross section, conjectured exact formula for $\lim_{s\to\infty} \sigma(s)$
- Regge limit $s \to \infty$
  Wilson line description, integrability

Asymptotic expansion of loop integrals

We want to compute the loop integrals in different kinematic limits:

- Use the known differential equation (DE) in $D = 4$ dimensions
- Solve the DE in an asymptotic expansion [Wasow, '65]

  DE:

$$x \frac{\partial \vec{f}}{\partial x} = \frac{\partial A}{\partial x} \vec{f}, \qquad \frac{\partial A}{\partial x} = A_0 + A_1 x + \mathcal{O}\left(x^2\right)$$

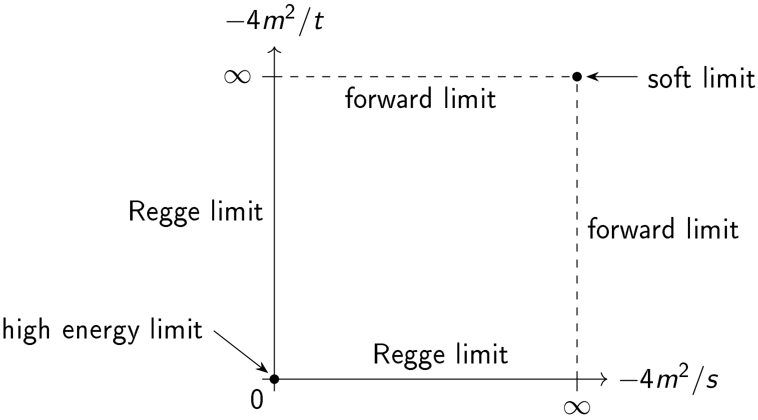
  solution:

$$\vec{f}(x) = P(x) e^{A_0 \log(x)} \vec{f_0}, \qquad P(x) = \mathbb{1} + P_1 x + \mathcal{O}\left(x^2\right)$$

- Complicated part: Calculation of the (regularized) boundary value $\vec{f_0}$

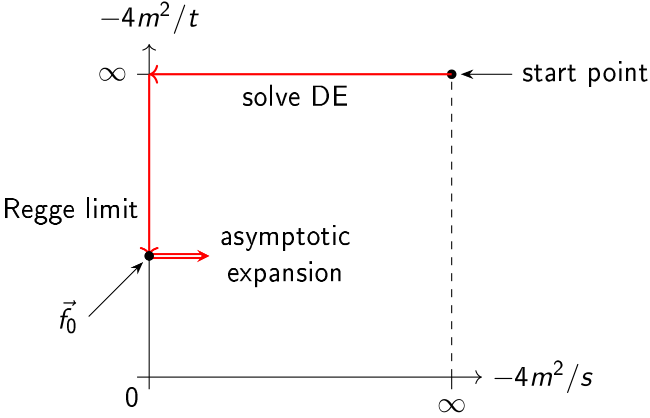
Asymptotic expansion of loop integrals

Transportation of the boundary value $\vec{f}_0$



Asymptotic expansion of loop integrals

Transportation of the boundary value $\vec{f}_0$



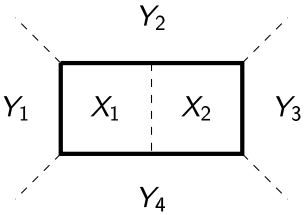
Three loop result for $\tilde{j}_1$

$$
\begin{aligned}
\tilde{j}_1 = &-4g^2 + g^4\left[\frac{-H_{1,1,1} - \frac{2}{3}\pi^2 H_1}{\xi} + 4H_{1,1} - 8H_1\xi + \frac{8\pi^2}{3} + 16\right] \\
&+ g^6\Bigg[-16\xi^2 H_{1,1} + \xi\Big(-32H_{1,1} + 28H_{1,1,1} + 8\pi^2 H_1 + 96H_1 - 64H_2\Big) \\
&\quad + \Big(-8\pi^2 H_{1,1} - 24H_{1,1} + 32H_{1,2} + 16H_{1,1,1} - 20H_{1,1,1,1} \\
&\quad + 96\zeta_3 - \frac{8\pi^4}{3} - \frac{32\pi^2}{3} - 128\Big) \\
&\quad + \frac{1}{\xi}\Big(\frac{8}{3}\pi^2 H_{1,1,1} + 4H_{1,1,1} - 8H_{1,1,2} - 4H_{1,1,1,1} \\
&\quad + 6H_{1,1,1,1,1} - 24\zeta_3 H_1 + \frac{44\pi^4 H_1}{45} + \frac{8\pi^2 H_1}{3}\Big)\Bigg]
\end{aligned}
$$

$H_{\vec{a}} = H_{\vec{a}}(1-x^2)$ [Harmonic Polylogarithm], $\xi = (1-x)/(1+x)$.

Dual coordinates

Introduce 6-vectors for internal and external regions with $X_i^2 = 0$ and $Y_i^2 = -m^2$



Dual conformal symmetry $SO(4,2)$ acts linearly on these vectors

Propagators:

$$-2X_i \cdot X_j = (x_i - x_j)^2\,, \qquad -2X_i \cdot Y_j = (x_i - y_j)^2 + m^2\,.$$

$SO(4)$ angle

Choose a frame:

$$Y_1 = \begin{pmatrix} -\vec{p}_1 \\ \frac{t}{2\alpha} \\ \hline 0 \\ \frac{2m^2}{\alpha} \end{pmatrix}, \quad Y_2 = \begin{pmatrix} \vec{0} \\ 0 \\ \hline \frac{1}{2}\sqrt{t} \\ \frac{\alpha}{2} \end{pmatrix}, \quad Y_3 = \begin{pmatrix} \vec{p}_2 \\ \frac{t}{2\alpha} \\ \hline 0 \\ \frac{2m^2}{\alpha} \end{pmatrix}, \quad Y_4 = \begin{pmatrix} \vec{0} \\ 0 \\ \hline -\frac{1}{2}\sqrt{t} \\ \frac{\alpha}{2} \end{pmatrix}$$

with $\alpha = \sqrt{4m^2 - t}$

Then we have:

$$\cos\theta_{SO(4)} = \sphericalangle(Y_1, Y_3) = 1 + \frac{2s}{t} - \frac{s}{2m^2}$$

Note: Rotations of the spacelike components ($\to SO(4)$) leaves t invariant: $Y_2 \cdot Y_4 = \frac{1}{2}t - m^2$

Renormalization

Renormalization matrix:

$$\mathbf{Z} = \begin{pmatrix} Z_{\text{cusp}} & 0 \\ Z_{\text{mix}} & Z_{\text{cusp},\phi} \end{pmatrix} ,$$

IR Z-factor:

$$\log\left(Z_{\text{IR}}^{-1}\right) = \sum_{L\geq 1} \left(g^2\right)^L \left(\frac{\gamma^{(L)}}{8} \frac{1}{(\epsilon L)^2} \left(1 - \epsilon L \log\left(\frac{s_1 s_2}{\mu^2}\right)\right) - \frac{\gamma_{HgH}^{(L)}}{2\epsilon L} \right)$$

Renormalization condition:

$$Z_{\text{IR}}^{-1}\, \mathbf{Z}^{-1} \langle 0 |\, \vec{W} | \phi(p_3) \rangle = \text{finite}$$

1. row:

$$Z_{\text{IR}}^{-1}\, Z_{\text{cusp}}^{-1} \langle 0 |\, \partial W_{\text{cusp}} | \phi(p_3) \rangle = \text{finite} \qquad \xRightarrow{Z_{\text{cusp}} \text{ known}} \qquad Z_{\text{IR}}$$

Renormalization

2. row:

$$\frac{Z_{\text{mix}}^{-1}}{\det(\mathbf{Z})}\left[Z_{\text{IR}}^{-1}\langle 0|\partial W_{\text{cusp}}|\phi(p_3)\rangle\right] + Z_{\text{cusp},\phi}^{-1}\left[Z_{\text{IR}}^{-1}\langle 0|W_{\text{cusp},\phi}|\phi(p_3)\rangle\right] = \text{finite}$$

We have at one-loop

$$Z_{\text{IR}}^{-1}\langle 0|\partial W_{\text{cusp}}|\phi(p_3)\rangle = \frac{(s_1+s_2)^2}{s_1 s_2}\left(1+\frac{g^2}{\epsilon}\frac{1-x}{1+x}\log(x)\right)$$

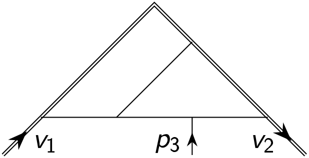
$$Z_{\text{IR}}^{-1}\langle 0|W_{\text{cusp},\phi}|\phi(p_3)\rangle = 1-\frac{2g^2}{\epsilon}$$

The factor $(s_1+s_2)^2/(s_1 s_2)$ allows us to fix the missing pieces:

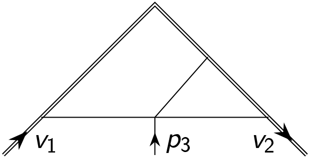
$$Z_{\text{mix}} = 0 \qquad \text{and} \qquad Z_{\text{cusp},\phi} = 1-\frac{2g^2}{\epsilon}$$

Soft current integrals

Two loop integral family



Elliptic sector



- Alphabet in canonical form (excluding elliptic sector)

$$\{s_1, s_2, x, 1-x, 1+x, s_1+s_2x, s_2+s_1x\}$$

- All integrals at one and two loops up to the order in ϵ needed for the renormalization can be written in terms of harmonic polylogarithms with arguments: $x, s_1/s_2, xs_1/s_2, xs_2/s_1$