Conformal Symmetry and Feynman Integrals

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Conformal Symmetry and Feynman Integrals

Anomalous conformal symmetry

Bootstrapping symbols

6D penta-box

Outlook

Conformal symmetry

- In massless theories the Poincaré group can be extended to the conformal group with the addition of
  - dilatations

$$x^\mu \to \lambda x^\mu$$

  - conformal boosts

$$x^\mu \to \frac{x^\mu - b^\mu x^2}{1 - 2(b \cdot x) + b^2 x^2}$$

- Euclidean spacetime: transformations preserving angles
- Minkowski spacetime: transformations preserving causality

timelike points	$\longrightarrow$	timelike points
lightlike points	$\longrightarrow$	lightlike points
spacelike points	$\longrightarrow$	spacelike points

Conformal symmetry in momentum space

- Standard methods to compute correlation functions using conformal symmetry in position space date back to the '80s
- Goal: application to scattering amplitudes
  - momentum space
  - on-shell configuration $p_i^2 = 0$
- The generator of conformal boosts becomes 2nd order

$$K_{\mu;\Delta} = \sum_{i=1}^{n} \left[-p_{i\mu} \Box_{p_i} + 2 p_i^\nu \frac{\partial}{\partial p_i^\nu} \frac{\partial}{\partial p_i^\mu} + 2(D - \Delta_i) \frac{\partial}{\partial p_i^\mu} \right]$$

- Integrals which are exactly conformal in position space undergo subtle symmetry breaking in on-shell momentum space [Chicherin, Sokatchev 2017]

Conformal anomaly

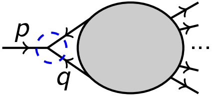
[Chicherin, Sokatchev 2017]

$D = 6$ scalar Φ^3 theory

- Contact anomaly

$$K_\mu \frac{1}{q^2(q+p)^2} = 4i\pi^3 p_\mu \int_0^1 d\xi \xi(1-\xi)\delta^{(6)}(q+\xi p)$$

on-shell corner



generator of conformal boosts

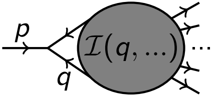
- Localized on the collinear configuration

$$q = -\xi p, \qquad \xi \in [0,1]$$

Anomalous conformal Ward identities

- The contact anomaly localizes loop-integrations

$$\int d^6k$$



$$\longrightarrow \int_0^1 d\xi \xi(1-\xi)\mathcal{I}(q=-\xi p, \dots)$$

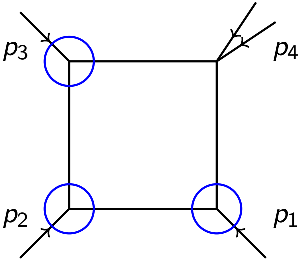
- System of linear non-homogeneous 2nd-order DEs

$$K_\mu \delta^{(6)}(P)\mathcal{I}^{(\ell)} = \delta^{(6)}(P) A_\mu^{(\ell-1)}$$

$\mathcal{I}^{(\ell)}$ ℓ-loop Feynman integral
$A_\mu^{(\ell-1)}$ anomaly ($\ell - 1$-loop integrals)

? Assuming we can calculate the anomaly, can we solve these equations and find $\mathcal{I}^{(\ell)}$?

A toy example: the 6D 1-mass box



$p_i^2 = 0 \quad \forall i = 1, 2, 3$

$p_4^2 \neq 0$

$s_{ij} = 2 p_i \cdot p_j$

- anomalous conformal Ward identities
- symmetry under exchange $p_1 \Leftrightarrow p_3$
- absence of u-channel singularities

$$\mathcal{I}_{1m} = \frac{1}{s_{13}} \left[\mathrm{Li}_2 \left(1 - \frac{p_4^2}{s_{12}^2} \right) + \mathrm{Li}_2 \left(1 - \frac{p_4^2}{s_{23}^2} \right) + \frac{1}{2} \log^2 \left(\frac{s_{12}^2}{s_{23}^2} \right) + \frac{\pi^2}{6} \right]$$

A toy example: the 6D 1-mass box

$$\mathcal{I}_{1m}(x,z,s_{123}) = \frac{1}{s_{123}} g(x,z)$$

$$\begin{cases} s_{123} = s_{12} + s_{13} + s_{23} \\ x = s_{12}/s_{123} \\ z = s_{23}/s_{123} \end{cases}$$

Anomalous conformal Ward identities:

$$\left\{ z \left[x^2 \frac{d^2}{dx^2} + (z-1)^2 \frac{d^2}{dz^2} + 2x(z-1) \frac{d}{dx}\frac{d}{dz} \right] + 2x(2z-1)\frac{d}{dx} + 2(z-1)(2z-1)\frac{d}{dz} + 2(z-1) \right\} g(x,z) =$$
$$= -\frac{z(x-1) + (x+z-1)\log\left(\frac{1}{x}\right)}{(x-1)^2 z},$$

$$\left\{ (1-x-z) \left[x^2 \frac{d^2}{dx^2} + z^2 \frac{d^2}{dz^2} + 2xz \frac{d}{dx}\frac{d}{dz} \right] + 2x(1-2x-2z)\frac{d}{dx} + 2z(1-2x-2z)\frac{d}{dz} - 2(x+z) \right\} g(x,z) =$$
$$= \frac{(z-1)^2 \log\left(\frac{1}{x}\right) + (x-1)(z-1)(x+z-2) + (x-1)^2 \log\left(\frac{1}{z}\right)}{(x-1)^2 (z-1)^2},$$

$$\left\{ x \left[(x-1)^2 \frac{d^2}{dx^2} + z^2 \frac{d^2}{dz^2} + 2z(x-1) \frac{d}{dx}\frac{d}{dz} \right] + 2(2x-1)(x-1)\frac{d}{dx} + 2z(2x-1)\frac{d}{dz} + 2(x-1) \right\} g(x,z) =$$
$$= -\frac{x(z-1) + (x+z-1)\log\left(\frac{1}{z}\right)}{x(z-1)^2}$$

What do we learn?

- Observation
  - $\mathcal{I}_{1m}$ is weight-2 (Li_2)
  - $\mathcal{A}^\mu$ contains weight-1 (log) and weight-0 functions

  A great simplification is achieved by projecting the Ward identities along q^μ such that

  $$(q \cdot K)\mathcal{I}_{1m} = q \cdot A \equiv \text{weight-0}$$

  $\Rightarrow$ maximum weight drop

- The Ward identities are quite complicated already at 4-point $\Rightarrow$ different approach needed for more interesting applications

Bootstrap strategy

[talk by Henn]

- Write down an ansatz ($s \equiv$ kinematic variables)

$$\mathcal{I}(s) = \sum_{i,j} c_{ij} r_i(s) f_j(s)$$

c_{ij} finite number of coefficients
$\Rightarrow$ to be fixed by imposing constraints

- graph symmetry
- anomalous conformal Ward identities...

$r_i(s)$ algebraic (ideally rational) functions
$\Rightarrow$ leading singularities

[Cachazo 2008; Arkani-Ahmed, Bourjaily, Cachazo, Trnka]

$f_j(s)$ special functions
$\Rightarrow$ polylogarithms and multivariate generalizations

Iterated integrals and symbols

- Chen iterated integral

$$\int_\gamma \omega_1\omega_2...\omega_k = \int_0^1 dt_k f_k(t_k)\left(\int_0^{t_k}\omega_1...\omega_{k-1}\right)$$

$$\gamma^*\omega_i = f_i(t)dt \to \text{pull-back on } [0,1]$$

- Alphabet: set of differential forms $\omega_i = d\log\alpha_i$

$$\Omega = \{\alpha_1, ..., \alpha_n\}$$

  Letters α_i are algebraic functions of the kinematic variables

- Symbol map associates a "word" to each iterated integral

$$\mathcal{S}\left[\int_\gamma d\log\alpha_1 \circ ... \circ d\log\alpha_k\right] = \alpha_1 \otimes ... \otimes \alpha_k$$

Integrable symbols

- Given an alphabet $\Omega = \{\alpha_1, ..., \alpha_k\}$, is any word made of the letters $\alpha_i \in \Omega$ allowed?

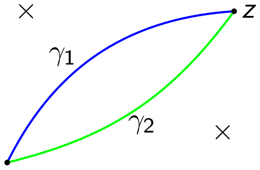
Integrable symbols

- Given an alphabet $\Omega = \{\alpha_1, ..., \alpha_k\}$, is any word made of the letters $\alpha_i \in \Omega$ allowed? No!
- Local path independence

$$\mathcal{S}(f) = \alpha_1 \otimes ... \otimes \alpha_k$$

$$\omega = d\log\alpha_1 \circ ... \circ d\log\alpha_k$$

$$f(z) = \int_{\gamma_1}\omega \stackrel{!}{=} \int_{\gamma_2}\omega$$



$\gamma_1 \sim \gamma_2$ same homotopy class

- Integrability conditions for $\mathcal{S} = \sum_{i_1,\ldots,i_n} c_{i_1\ldots i_n}(\alpha_{i_1}\otimes ... \otimes \alpha_{i_n})$

$$\sum_{i_1,\ldots,i_n} c_{i_1\ldots i_n}(d\log\alpha_{i_a} \wedge d\log\alpha_{i_{a+1}})\,\alpha_{i_1}\otimes ... \otimes \hat{\alpha}_{i_a} \otimes \hat{\alpha}_{i_{a+1}} \otimes ... \otimes \alpha_{i_n} = 0$$

$$\forall a = 1, ..., n-1$$

Symbols vs functions

Integrand $\longrightarrow$ Symbol $\longrightarrow$ Function (polylogs...)

- Symbols fix the leading functional transcendentality piece

$$\mathrm{Li}_2(1-x) + \log(x)\log(1-x) = -\mathrm{Li}_2(x) + \frac{\pi^2}{6}$$

$$-x\otimes(1-x) + x\otimes(1-x) + (1-x)\otimes x + (1-x)\otimes x = (1-x)\otimes x$$

$\Rightarrow$ they capture the most complicated part of the function

- The lower functional transcendentality pieces can be fixed as well $\Rightarrow$ Bootstrap strategy: accommodate them in the ansatz
- Additional work required to fully upgrade a symbol to function (Goncharov polylogarithms)

An example: 4-point integrals

- On-shell massless
- Up to three loops

$\Rightarrow$ Alphabet

$$\Omega = \{x, 1+x\} \qquad x = \frac{s}{t}$$

[Remiddi, Gehrmann, Henn, Smirnov, Mistlberger...]

$\Rightarrow$ Harmonic polylogarithms

$$H(\vec{m}_w; x) = \int_0^x dx' f(a; x') H(\vec{m}_{w-1}; x') \qquad \vec{m}_w = (a, \vec{m}_{w-1})$$

$$f(0; x) = \frac{1}{x} \qquad f(1; x) = \frac{1}{1-x} \qquad f(-1; x) = \frac{1}{1+x}$$

Pentagon functions: the alphabet

[talk by Henn]

- Five independent kinematic variables $v_i = 2p_i \cdot p_{i+1}$
- 2-loop on-shell 5-particle scattering amplitudes

Planar
[Gehrmann, Henn, Lo Presti 2015]

$$\alpha_i = v_i$$
$$\alpha_{5+i} = v_{i+2} + v_{i+3}$$
$$\alpha_{10+i} = v_i - v_{i+3}$$
$$\alpha_{15+i} = v_i + v_{i+1} - v_{i+3}$$

Non-planar
[Chicherin, Henn, Mitev 2017]

$$\alpha_{20+i} = v_{2+i} + v_{3+i} - v_i - v_{i+1}$$
$$\alpha_{25+i} = \frac{a_i - \sqrt{\Delta}}{a_i + \sqrt{\Delta}}$$
$$\alpha_{31} = \sqrt{\Delta}$$

$$a_i = v_i v_{i+1} - v_{i+1} v_{i+2} + v_{i+2} v_{i+3} - v_{i+3} v_{i+4} - v_{i+4} v_i$$
$$\Delta = \det\left(2p_i \cdot p_j\right)$$

Pentagon functions: constraints

[Chicherin, Henn, Mitev 2017]

- First entry condition
  - planar symbols → $\{\alpha_i\}_{i=1}^{5} = \{s_{12}$ and cyclic$\}$
  - non-planar symbols → $\{\alpha_i\}_{i=1}^{5} \cup \{\alpha_i\}_{i=16}^{20} = \{s_{ij}\}_{i<j=1}^{5}$
- Conjectured second entry condition

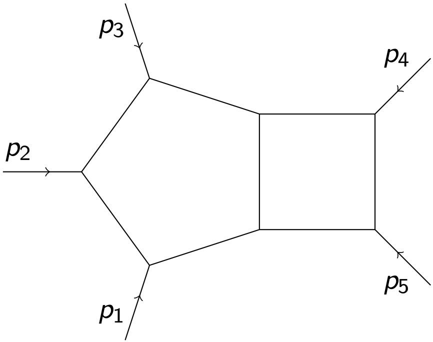
  Experimental observation that certain pairs $\alpha_i \otimes \alpha_j \otimes \ldots$ do not appear in Feynman integrals

How many functions? E.g. for the planar alphabet (26 letters)

Weight	1	2	3	4	5	6
1^{st} entry cond.	5	25	126	651	3436	18426
2^{nd} entry cond.	5	20	81	346	1551	7201

[Chicherin, Henn, Mitev]

6D penta-box



Basic facts:

1. planar alphabet (1st entry condition)
2. symmetric under exchange $\{1 \leftrightarrow 3, 4 \leftrightarrow 5\}$
3. even under complex conjugation
4. leading singularity $1/\sqrt{\Delta}$

Last entry condition from conformal symmetry

- Conformal symmetry constrains the ansatz even before knowing the explicit expression of the anomaly

$$(q \cdot K)\left[\frac{\alpha_1 \otimes \ldots \otimes \alpha_k}{\sqrt{\Delta}}\right] = (q \cdot K)\left[\frac{\log \alpha_k}{\sqrt{\Delta}}\right](\alpha_1 \otimes \ldots \otimes \alpha_{k-1}) + \text{weight-}(k-2)$$

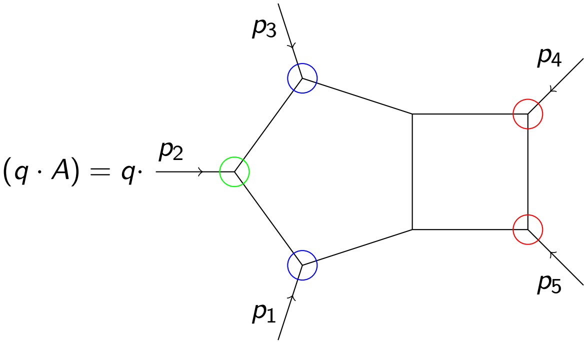
- 1-mass box example: Consider q^μ such that

$$(q \cdot K)\mathcal{I}_k = (q \cdot A) \equiv \text{weight-}(k-2)$$

$\Rightarrow$ Constraint on the allowed last entries

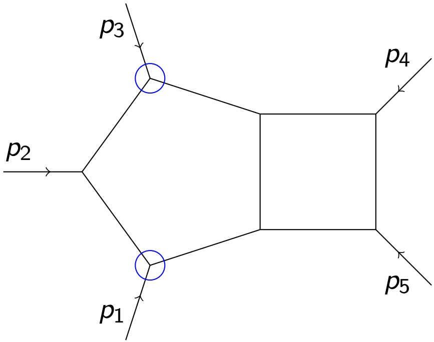
$$(q \cdot K)\left[\sum_i c_i \frac{\log \alpha_i}{\sqrt{\Delta}}\right] = 0$$

6D penta-box: anomaly



6D penta-box: anomaly

$(q \cdot A) = q \cdot$



$$q \perp p_2, p_4, p_5$$

$$(q \cdot A) = \frac{\text{(weight-3)}}{s_{12} s_{14} (s_{23} - s_{45})} + (1 \leftrightarrow 3, 4 \leftrightarrow 5)$$

6D penta-box: ansatz

- Info from anomalous conformal symmetry:
  5. weight 5
  6. last entry condition (11 allowed letters out of 26)
- Ansatz (leading functional transcendentality)

$$\underbrace{\overbrace{\mathcal{S}\left[\mathcal{I}_5\right]}^{\text{even}}}_{\text{symm.}} = \underbrace{\overbrace{\frac{1}{\sqrt{\Delta}}}^{\text{odd}}}_{\text{symm.}} \sum_{I=(i_1,\ldots,i_5)} c_I \underbrace{\overbrace{\left(\alpha_{i_1} \otimes \ldots \otimes \alpha_{i_5}\right)}^{\text{odd}}}_{\text{symm.}}, \qquad \alpha_i \in \mathbb{A}_{\mathrm{P}} \; \forall i$$

Constraints	weight-5 integrable symbols ($\mathbb{A}_{\mathrm{P}}$)
1st entry condition	3436
odd	161
last entry condition	59
exchange symmetry	33

6D penta-box: symbol

- Plug the ansatz

$$\mathcal{S}\left[\mathcal{I}_5\right] = \frac{1}{\sqrt{\Delta}} \sum_{i=1}^{33} c_i \left(\alpha_{i_1} \otimes \ldots \otimes \alpha_{i_5}\right)$$

into the anomalous conformal Ward identity

$$\left(q \cdot K\right) \mathcal{S}\left[\mathcal{I}_5\right] = q \cdot \mathcal{S}\left[A\right]$$

$\Rightarrow$ All coefficients are uniquely fixed by one anomaly!

✓ Agreement with the known result

✓ Agreement with conjectured 2nd entry condition

$$\mathcal{S}\left[\mathcal{I}_5\right] \sim 10000 \text{ terms}$$

6D penta-box: work in progress

- To do list:
  - ☑ Algorithmically upgrade to function (Goncharov polylogs)

$$\mathcal{S}\left[\mathcal{I}_5\right] \to \mathcal{I}_5^* \qquad \mathcal{S}\left[\mathcal{I}_5\right] = \mathcal{S}\left[\mathcal{I}_5^*\right]$$

  - ☐ Write ansatz for the lower functional transcendentality piece

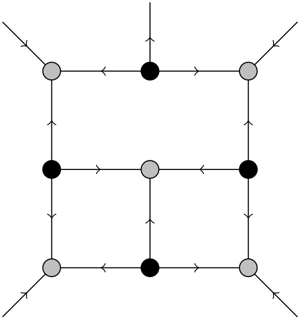
$$\mathcal{I}_5 - \mathcal{I}_5^* = \pi W^{(4)} + \pi^2 W^{(3)} + \ldots \equiv \Delta\mathcal{I}_5$$

  - ☐ Fix the ansatz using anomalous conformal Ward identities

$$K^\mu \Delta\mathcal{I}_5 = A^\mu - K^\mu \mathcal{I}_5^* = \pi W^{(2)} + \pi^2 W^{(1)} + \pi^3$$

Outlook

- Many improvements possible
  - Upgrade bootstrap to function level (Goncharov polylogs)
  - Study structure of the DEs (characteristics, asymptotic behaviours, singularities)
- Super-conformal symmetry [Chicherin, Henn, Sokatchev 2018]
  - Wess-Zumino model of $\mathcal{N} = 1$ massless supersymmetric matter
  - 1st order Ward identities $\Rightarrow$ more powerful!



Iterated integrals and symbols

- Differential equations in the canonical form

$$d\vec{f}(s;\epsilon) = \epsilon\ d\left(\sum_k A_k d\log\alpha_k(s)\right)\vec{f}(s;\epsilon) \qquad \text{[Henn]}$$

$\alpha_k(s)$ rational functions of the kinematic variables $s \equiv$ letters
A_k constant matrices

- Alphabet $\Omega = \{\alpha_1, ..., \alpha_n\}$ set of all possible letters
$\to$ specifies which class of functions is required

- General solution

$$\vec{f}(s;\epsilon) = \mathcal{P}\exp\left[\epsilon\int_\gamma d\left(\sum_k A_k d\log\alpha_k(s)\right)\right]\vec{f}_0(\epsilon)$$

Iterated integrals and symbols: a few examples

- Polylogarithms

$$\mathrm{Li}_s(z) = \int_0^z d\log(t)\mathrm{Li}_{s-1}(z), \qquad \mathrm{Li}_1(z) = -\log(1-z)$$

$$\mathrm{Li}_s(z) = -\int_0^z d\log(t_1)\int_0^{t_1} d\log(t_2)\ldots\int_0^{t_{s-1}} d\log(1-t_s)$$

$$\mathcal{S}\Big[\mathrm{Li}_s(z)\Big] = -(1-z)\otimes\overbrace{z\otimes\ldots\otimes z}^{s-1\ \text{times}} \quad \Rightarrow \Omega = \{z, 1-z\}$$

- Harmonic polylogarithms

$$\text{e.g. } H(0,1;z) = \int_0^z \frac{dt}{t}H(1,t) = -\int_0^z \frac{dt}{t}\int_0^t \frac{dt'}{1-t'}$$

$$\mathcal{S}\Big[H(0,1;z)\Big] = -(1-z)\otimes z \equiv \mathcal{S}\Big[\mathrm{Li}_2(z)\Big]$$

From symbols to functions

- Differentiation acts on the last entry of the symbol

$$d\left(\alpha_1 \otimes \ldots \otimes \alpha_n\right) = d \ln \alpha_n \left(\alpha_1 \otimes \ldots \otimes \alpha_{n-1}\right)$$

- The differential of a weight-n symbol is written in terms of weight-$(n-1)$ symbols

$$d\mathcal{S}\left[f^{(n)}\right] = \sum_i c_i d \ln \alpha_i \; \mathcal{S}\left[f_i^{(n-1)}\right]$$

- Differential equation with a natural grading

$$d \begin{pmatrix} f^{(n)} \\ \vec{f}^{(n-1)} \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & A & & & \\ & 0 & B & & \mathbf{0} \\ & & 0 & \ddots & \\ & \mathbf{0} & & \ddots & C \\ & & & & 0 \end{pmatrix} \begin{pmatrix} f^{(n)} \\ \vec{f}^{(n-1)} \\ \vdots \\ 1 \end{pmatrix}$$

[Caron-Huot, Henn]

From symbols to functions

- Algorithm: given a symbol $\mathcal{S}^n$ of weight-n
  - by differentiating from weight-n down to weight-0 we determine the block triangular matrix M in

$$d\vec{f} = dM\ \vec{f} \qquad \vec{f} = \left(f^{(n)}, \vec{f}^{(n-1)}, ..., 1\right)^T$$

  - we integrate iteratively from weight-0 up to weight-n

✓ Function whose symbol matches $\mathcal{S}^n$

X The lower functional transcendentality pieces are still missing!

⇒ Bootstrap strategy: accommodate them into the ansatz