

Analytic properties of loop integrands

Pascal Wasser, Johannes Gutenberg-Universität Mainz

based on

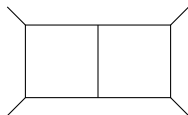
Master thesis¹ in group of Johannes Henn

Loops and Legs

3rd May 2018, St. Goar

¹https://publications.ub.uni-mainz.de/theses/frontdoor.php?source_opus=100001967

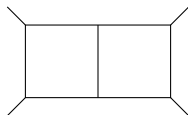
Motivation



$$\mathcal{I}_{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9} = \int \frac{d^D k_1 d^D k_2 P_8^{-a_8} P_9^{-a_9}}{P_1^{a_1} P_2^{a_2} P_3^{a_3} P_4^{a_4} P_5^{a_5} P_6^{a_6} P_7^{a_7}}$$

$$\vec{m} = (\mathcal{I}_{0,1,0,0,1,0,1,0,0}, \mathcal{I}_{1,0,0,1,0,0,1,0,0}, \mathcal{I}_{0,1,1,0,0,1,1,0,0}, \mathcal{I}_{1,0,1,1,0,1,0,0,0}, \\ \mathcal{I}_{1,1,1,0,1,0,1,0,0}, \mathcal{I}_{0,1,1,0,1,1,1,0,0}, \mathcal{I}_{1,1,1,1,1,1,1,0,0}, \mathcal{I}_{2,1,1,1,1,1,1,0,0})^T$$

Motivation



$$\mathcal{I}_{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9} = \int \frac{d^D k_1 d^D k_2 P_8^{-a_8} P_9^{-a_9}}{P_1^{a_1} P_2^{a_2} P_3^{a_3} P_4^{a_4} P_5^{a_5} P_6^{a_6} P_7^{a_7}}$$

$$\vec{m} = (\mathcal{I}_{0,1,0,0,1,0,1,0,0}, \mathcal{I}_{1,0,0,1,0,0,1,0,0}, \mathcal{I}_{0,1,1,0,0,1,1,0,0}, \mathcal{I}_{1,0,1,1,0,1,0,0,0}, \\ \mathcal{I}_{1,1,1,0,1,0,1,0,0}, \mathcal{I}_{0,1,1,0,1,1,1,0,0}, \mathcal{I}_{1,1,1,1,1,1,1,0,0}, \mathcal{I}_{2,1,1,1,1,1,1,0,0})^T$$

What is the simplest basis of master integrals for a given integral topology?

Leading Singularities

$$\int_{-\infty}^{\infty} dk_0 \int_{-\infty}^{\infty} dk_1 \int_{-\infty}^{\infty} dk_2 \int_{-\infty}^{\infty} dk_3 \frac{1}{P_1 P_2 P_3 P_4}$$

Leading Singularities

$$\int_{-\infty}^{\infty} dk_0 \int_{-\infty}^{\infty} dk_1 \int_{-\infty}^{\infty} dk_2 \int_{-\infty}^{\infty} dk_3 \frac{1}{P_1 P_2 P_3 P_4}$$



$$\frac{1}{(2\pi i)^4} \oint_{P_1=0} dk_0 \oint_{P_2=0} dk_1 \oint_{P_3=0} dk_2 \oint_{P_4=0} dk_3 \frac{1}{P_1 P_2 P_3 P_4}$$

Leading Singularities

$$\int_{-\infty}^{\infty} dk_0 \int_{-\infty}^{\infty} dk_1 \int_{-\infty}^{\infty} dk_2 \int_{-\infty}^{\infty} dk_3 \frac{1}{P_1 P_2 P_3 P_4}$$



$$\frac{1}{(2\pi i)^4} \oint_{P_1=0} dk_0 \oint_{P_2=0} dk_1 \oint_{P_3=0} dk_2 \oint_{P_4=0} dk_3 \frac{1}{P_1 P_2 P_3 P_4}$$

Idea: Leading singularities as simple as possible

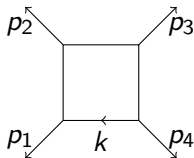
Outline

- Introduction
 - Differential equations
 - Dlog forms and leading singularities
- Method
 - Algorithmic computation of leading singularities and dlog forms
- Results
 - Planar double box family
 - Three loop four point families

Introduction

Differential equations, dlog forms and leading singularities

Differential equations



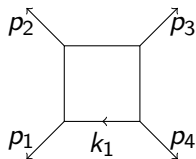
$$\mathcal{I}_{a_1, a_2, a_3, a_4}(s, t)$$

$$s = (p_1 + p_2)^2$$

$$t = (p_1 + p_4)^2$$

$$\vec{f} = (\mathcal{I}_{1,1,1,1}, \mathcal{I}_{1,0,1,0}, \mathcal{I}_{0,1,0,1})^T$$

$$\frac{\partial}{\partial s} \vec{f} = A(s, t, D) \vec{f}$$

Simplified Differential Equations [Henn, 2013](#)

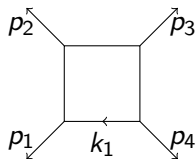
$$g_1 = c\epsilon^2(-s)^\epsilon t \mathcal{I}_{0,1,1,1,1}, \quad D = 4 - 2\epsilon$$

$$g_2 = c\epsilon^2(-s)^\epsilon s \mathcal{I}_{1,1,1,0,0}$$

$$g_3 = c\epsilon^2(-s)^\epsilon st \mathcal{I}_{1,1,1,1,1}$$

$$\frac{\partial}{\partial x} \vec{g}(x, \epsilon) = \epsilon \left(\frac{a}{x} + \frac{b}{1+x} \right) \vec{g}(x, \epsilon), \quad x = t/s$$

$$a = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & -1 \end{pmatrix}, \quad b = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -2 & -2 & 1 \end{pmatrix}$$

Simplified Differential Equations [Henn, 2013](#)

$$g_1 = c\epsilon^2(-s)^\epsilon t \mathcal{I}_{0,1,1,1,1}, \quad D = 4 - 2\epsilon$$

$$g_2 = c\epsilon^2(-s)^\epsilon s \mathcal{I}_{1,1,1,0,0}$$

$$g_3 = c\epsilon^2(-s)^\epsilon st \mathcal{I}_{1,1,1,1,1}$$

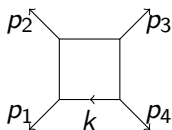
$$\frac{\partial}{\partial x} \vec{g}(x, \epsilon) = \epsilon \left(\frac{a}{x} + \frac{b}{1+x} \right) \vec{g}(x, \epsilon), \quad x = t/s$$

$$a = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & -1 \end{pmatrix}, \quad b = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -2 & -2 & 1 \end{pmatrix}$$

Uniform transcendental weight solution:

$$g_3 = 4 + \epsilon(-2 \log x) + \epsilon^2 \left(-\frac{4\pi^2}{3} \right) + \epsilon^3 \left(\frac{7\pi^2}{6} \log x + \frac{1}{3} \log^3 x + \dots \right) + \mathcal{O}(\epsilon^4)$$

Dlog form of box integrand



$$d\mathcal{I}_4 = \frac{d^4 k}{k^2(k-p_1)^2(k-p_1-p_2)^2(k+p_4)^2}$$

$$= \frac{1}{st} d\log \frac{k^2}{(k-k^*)^2} \wedge d\log \frac{(k-p_1)^2}{(k-k^*)^2} \\ \wedge d\log \frac{(k-p_1-p_2)^2}{(k-k^*)^2} \wedge d\log \frac{(k+p_4)^2}{(k-k^*)^2} \quad \text{Arkani-Hamed et al., 2012}$$

k^* : Solution to $k^2 = (k-p_1)^2 = (k-p_1-p_2)^2 = (k+p_4)^2 = 0$.

Leading singularities

$$\frac{1}{(2\pi i)^4} \oint_{P_1=0} dk_0 \oint_{P_2=0} dk_1 \oint_{P_3=0} dk_2 \oint_{P_4=0} dk_3 \frac{1}{P_1 P_2 P_3 P_4}$$

Leading singularities

$$\frac{1}{(2\pi i)^4} \oint_{P_1=0} dk_0 \oint_{P_2=0} dk_1 \oint_{P_3=0} dk_2 \oint_{P_4=0} dk_3 \frac{1}{P_1 P_2 P_3 P_4}$$
$$c \, d\log(q_1) \wedge d\log(q_2) \wedge d\log(q_3) \wedge d\log(q_4)$$

Leading singularities

$$\frac{1}{(2\pi i)^4} \oint_{P_1=0} dk_0 \oint_{P_2=0} dk_1 \oint_{P_3=0} dk_2 \oint_{P_4=0} dk_3 \frac{1}{P_1 P_2 P_3 P_4}$$
$$c \, d\log(q_1) \wedge d\log(q_2) \wedge d\log(q_3) \wedge d\log(q_4)$$
$$= c \frac{dq_1}{q_1} \wedge \frac{dq_2}{q_2} \wedge \frac{dq_3}{q_3} \wedge \frac{dq_4}{q_4}$$

Leading singularities

$$\frac{1}{(2\pi i)^4} \oint_{P_1=0} dk_0 \oint_{P_2=0} dk_1 \oint_{P_3=0} dk_2 \oint_{P_4=0} dk_3 \frac{1}{P_1 P_2 P_3 P_4}$$

$$c \, d\log(q_1) \wedge d\log(q_2) \wedge d\log(q_3) \wedge d\log(q_4)$$

$$= c \frac{dq_1}{q_1} \wedge \frac{dq_2}{q_2} \wedge \frac{dq_3}{q_3} \wedge \frac{dq_4}{q_4}$$

Leading singularity: c

Dlog forms and leading singularities

$$d\Omega = \sum_{i=1}^n c_i d\log(g_{i1}) \wedge d\log(g_{i2}) \wedge \dots \wedge d\log(g_{ik})$$

Values of c_j : Leading singularities of $d\Omega$.

Dlog forms and leading singularities

$$d\Omega = \sum_{i=1}^n c_i d\log(g_{i1}) \wedge d\log(g_{i2}) \wedge \dots \wedge d\log(g_{ik})$$

Values of c_i : Leading singularities of $d\Omega$.

Conjecture

Dlog integrands with constant leading singularities fulfill differential equations in the canonical form

Method

Algorithmic computation of leading singularities and dlog forms

Simple example

$$df(x, y) = \frac{dx \wedge dy}{xy(x + y + 1)}$$

Simple example

$$df(x, y) = \frac{dx \wedge dy}{xy(x + y + 1)} = \frac{dx \wedge dy}{xy(x + 1)} + \frac{dx \wedge dy}{x(-x - 1)(x + y + 1)}$$

Simple example

$$\begin{aligned}df(x, y) &= \frac{dx \wedge dy}{xy(x + y + 1)} = \frac{dx \wedge dy}{xy(x + 1)} + \frac{dx \wedge dy}{x(-x - 1)(x + y + 1)} \\ &= \frac{dx}{x(x + 1)} \wedge d\log(y) - \frac{dx}{x(x + 1)} \wedge d\log(x + y + 1)\end{aligned}$$

Simple example

$$\begin{aligned}df(x, y) &= \frac{dx \wedge dy}{xy(x + y + 1)} = \frac{dx \wedge dy}{xy(x + 1)} + \frac{dx \wedge dy}{x(-x - 1)(x + y + 1)} \\&= \frac{dx}{x(x + 1)} \wedge d\log(y) - \frac{dx}{x(x + 1)} \wedge d\log(x + y + 1) \\&= \left[\frac{dx}{x} - \frac{dx}{x + 1} \right] \wedge [d\log(y) - d\log(x + y + 1)]\end{aligned}$$

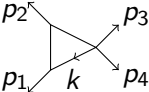
Simple example

$$\begin{aligned}df(x, y) &= \frac{dx \wedge dy}{xy(x + y + 1)} = \frac{dx \wedge dy}{xy(x + 1)} + \frac{dx \wedge dy}{x(-x - 1)(x + y + 1)} \\&= \frac{dx}{x(x + 1)} \wedge d\log(y) - \frac{dx}{x(x + 1)} \wedge d\log(x + y + 1) \\&= \left[\frac{dx}{x} - \frac{dx}{x + 1} \right] \wedge [d\log(y) - d\log(x + y + 1)] \\&= [d\log(x) - d\log(x + 1)] \wedge [d\log(y) - d\log(x + y + 1)]\end{aligned}$$

Simple example

$$\begin{aligned}
 df(x, y) &= \frac{dx \wedge dy}{xy(x+y+1)} = \frac{dx \wedge dy}{xy(x+1)} + \frac{dx \wedge dy}{x(-x-1)(x+y+1)} \\
 &= \frac{dx}{x(x+1)} \wedge d\log(y) - \frac{dx}{x(x+1)} \wedge d\log(x+y+1) \\
 &= \left[\frac{dx}{x} - \frac{dx}{x+1} \right] \wedge [d\log(y) - d\log(x+y+1)] \\
 &= [d\log(x) - d\log(x+1)] \wedge [d\log(y) - d\log(x+y+1)] \\
 &= d\log \frac{x}{x+1} \wedge d\log \frac{y}{x+y+1}
 \end{aligned}$$

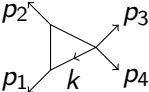
Triangle



The diagram shows a triangle loop with three internal propagator lines. The external momenta are labeled p_1 , p_2 , p_3 , and p_4 . The internal momentum is labeled k . The diagram is equated to the following integral expression:

$$= \frac{s d^4 k}{k^2 (k - p_1)^2 (k - p_1 - p_2)^2}$$

Triangle

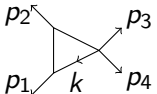


A Feynman diagram of a triangle loop. Three internal lines form a triangle, with an internal momentum k flowing clockwise. Four external lines are attached to the vertices: p_1 enters from the bottom-left, p_2 enters from the top-left, p_3 exits to the top-right, and p_4 exits to the bottom-right.

$$= \frac{s d^4 k}{k^2 (k - p_1)^2 (k - p_1 - p_2)^2}$$

$$p_1 = \lambda_1 \tilde{\lambda}_1, \quad p_2 = \lambda_2 \tilde{\lambda}_2, \quad \lambda_1 \tilde{\lambda}_2, \quad \lambda_2 \tilde{\lambda}_1$$

Triangle



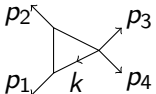
The diagram shows a triangle loop with three internal propagator lines. The external momenta are labeled p_1 , p_2 , p_3 , and p_4 . The internal momentum is labeled k . The diagram is followed by an equals sign and a fraction.

$$= \frac{s d^4 k}{k^2 (k - p_1)^2 (k - p_1 - p_2)^2}$$

$$p_1 = \lambda_1 \tilde{\lambda}_1, \quad p_2 = \lambda_2 \tilde{\lambda}_2, \quad \lambda_1 \tilde{\lambda}_2, \quad \lambda_2 \tilde{\lambda}_1$$

$$k = \alpha_1 \lambda_1 \tilde{\lambda}_1 + \alpha_2 \lambda_2 \tilde{\lambda}_2 + \frac{\langle 23 \rangle}{\langle 13 \rangle} \alpha_3 \lambda_1 \tilde{\lambda}_2 + \frac{\langle 13 \rangle}{\langle 23 \rangle} \alpha_4 \lambda_2 \tilde{\lambda}_1$$

Triangle



The diagram shows a triangle loop with three internal lines and four external lines. The external momenta are labeled p_1 , p_2 , p_3 , and p_4 . The internal momentum is labeled k . The external lines are: p_1 (bottom-left, incoming), p_2 (top-left, incoming), p_3 (top-right, outgoing), and p_4 (bottom-right, outgoing). The internal lines connect the vertices: k (bottom), k (left), and k (right).

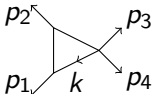
$$= \frac{s d^4 k}{k^2(k-p_1)^2(k-p_1-p_2)^2}$$

$$p_1 = \lambda_1 \tilde{\lambda}_1, \quad p_2 = \lambda_2 \tilde{\lambda}_2, \quad \lambda_1 \tilde{\lambda}_2, \quad \lambda_2 \tilde{\lambda}_1$$

$$k = \alpha_1 \lambda_1 \tilde{\lambda}_1 + \alpha_2 \lambda_2 \tilde{\lambda}_2 + \frac{\langle 23 \rangle}{\langle 13 \rangle} \alpha_3 \lambda_1 \tilde{\lambda}_2 + \frac{\langle 13 \rangle}{\langle 23 \rangle} \alpha_4 \lambda_2 \tilde{\lambda}_1$$

$$d\mathcal{I}_3 = \frac{d\alpha_1 \wedge d\alpha_2 \wedge d\alpha_3 \wedge d\alpha_4}{(\alpha_1 \alpha_2 - \alpha_3 \alpha_4)(\alpha_1 \alpha_2 - \alpha_2 - \alpha_3 \alpha_4)(1 - \alpha_1 - \alpha_2 + \alpha_1 \alpha_2 - \alpha_3 \alpha_4)}$$

Triangle



The diagram shows a triangle loop with vertices. External momenta are labeled p_1 , p_2 , p_3 , and p_4 . Internal momentum is labeled k . The loop is formed by three internal lines connecting the vertices.

$$= \frac{s d^4 k}{k^2 (k - p_1)^2 (k - p_1 - p_2)^2}$$

$$p_1 = \lambda_1 \tilde{\lambda}_1, \quad p_2 = \lambda_2 \tilde{\lambda}_2, \quad \lambda_1 \tilde{\lambda}_2, \quad \lambda_2 \tilde{\lambda}_1$$

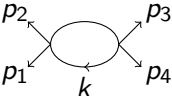
$$k = \alpha_1 \lambda_1 \tilde{\lambda}_1 + \alpha_2 \lambda_2 \tilde{\lambda}_2 + \frac{\langle 23 \rangle}{\langle 13 \rangle} \alpha_3 \lambda_1 \tilde{\lambda}_2 + \frac{\langle 13 \rangle}{\langle 23 \rangle} \alpha_4 \lambda_2 \tilde{\lambda}_1$$

$$d\mathcal{I}_3 = \frac{d\alpha_1 \wedge d\alpha_2 \wedge d\alpha_3 \wedge d\alpha_4}{(\alpha_1 \alpha_2 - \alpha_3 \alpha_4)(\alpha_1 \alpha_2 - \alpha_2 - \alpha_3 \alpha_4)(1 - \alpha_1 - \alpha_2 + \alpha_1 \alpha_2 - \alpha_3 \alpha_4)}$$

$$= d\log(\alpha_4) \wedge d\log(\alpha_2) \wedge d\log(\alpha_3) \wedge d\log(-\alpha_2 + \alpha_1 \alpha_2 - \alpha_3 \alpha_4)$$

$$+ (\dots) + (\dots) + (\dots)$$

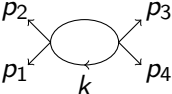
Bubble



A Feynman diagram of a bubble loop. It consists of a central circle with four external lines. The bottom-left line is labeled p_1 with an arrow pointing into the circle. The top-left line is labeled p_2 with an arrow pointing out of the circle. The top-right line is labeled p_3 with an arrow pointing out of the circle. The bottom-right line is labeled p_4 with an arrow pointing out of the circle. The bottom line of the circle is labeled k with an arrow pointing into the circle from the bottom.

$$= \frac{s^2 d^4 k}{k^2 (k - p_1 - p_2)^2}$$

Bubble

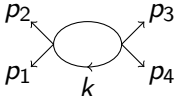


A Feynman diagram of a bubble loop. It consists of a central circle with four external lines. The top-left line is labeled p_2 with an arrow pointing away from the loop. The top-right line is labeled p_3 with an arrow pointing away from the loop. The bottom-left line is labeled p_1 with an arrow pointing towards the loop. The bottom-right line is labeled p_4 with an arrow pointing towards the loop. The bottom internal line is labeled k with an arrow pointing clockwise around the loop.

$$= \frac{s^2 d^4 k}{k^2 (k - p_1 - p_2)^2}$$

$$d\mathcal{I}_2 = \frac{d\alpha_1 \wedge d\alpha_2 \wedge d\alpha_3 \wedge d\alpha_4}{(\alpha_1 \alpha_2 - \alpha_3 \alpha_4)(1 - \alpha_1 + \alpha_1 \alpha_2 - \alpha_2 - \alpha_3 \alpha_4)}$$

Bubble



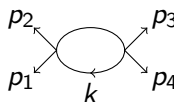
A Feynman diagram of a bubble loop. It consists of two vertices connected by two internal lines forming a loop. The internal momentum is labeled k . Four external lines are attached to the vertices, with momenta labeled p_1 , p_2 , p_3 , and p_4 . The diagram is equated to the mathematical expression $\frac{s^2 d^4 k}{k^2(k - p_1 - p_2)^2}$.

$$= \frac{s^2 d^4 k}{k^2(k - p_1 - p_2)^2}$$

$$d\mathcal{I}_2 = \frac{d\alpha_1 \wedge d\alpha_2 \wedge d\alpha_3 \wedge d\alpha_4}{(\alpha_1\alpha_2 - \alpha_3\alpha_4)(1 - \alpha_1 + \alpha_1\alpha_2 - \alpha_2 - \alpha_3\alpha_4)}$$

$$d\mathcal{I}_2 = d\alpha_1 \wedge d\log(-1 + \alpha_1 + \alpha_2) \wedge d\log(\alpha_3) \\ \wedge [d\log(\alpha_1\alpha_2 - \alpha_3\alpha_4) - d\log(1 - \alpha_1 - \alpha_2 + \alpha_1\alpha_2 - \alpha_3\alpha_4)]$$

Bubble



$$= \frac{s^2 d^4 k}{k^2 (k - p_1 - p_2)^2}$$

$$d\mathcal{I}_2 = \frac{d\alpha_1 \wedge d\alpha_2 \wedge d\alpha_3 \wedge d\alpha_4}{(\alpha_1 \alpha_2 - \alpha_3 \alpha_4)(1 - \alpha_1 + \alpha_1 \alpha_2 - \alpha_2 - \alpha_3 \alpha_4)}$$

$$d\mathcal{I}_2 = d\alpha_1 \wedge d\log(-1 + \alpha_1 + \alpha_2) \wedge d\log(\alpha_3) \\ \wedge [d\log(\alpha_1 \alpha_2 - \alpha_3 \alpha_4) - d\log(1 - \alpha_1 - \alpha_2 + \alpha_1 \alpha_2 - \alpha_3 \alpha_4)]$$

No dlog form! Double pole at $\alpha_1 \rightarrow \infty$.

$$d\alpha_1 = \frac{d\tilde{\alpha}_1}{\tilde{\alpha}_1^2}, \text{ with } \tilde{\alpha}_1 = \alpha_1^{-1}$$

Square roots

- $\frac{dx(ux + v)}{ax^2 + bx + c} \rightarrow$ Residue with square roots

Square roots

- $\frac{dx(ux + v)}{ax^2 + bx + c} \rightarrow$ Residue with square roots
- $\frac{dx}{x\sqrt{(x+a)(x+b)}} \rightarrow$ Residue: $\frac{1}{\sqrt{ab}}$

Square roots

- $\frac{dx(ux + v)}{ax^2 + bx + c} \rightarrow$ Residue with square roots
- $\frac{dx}{x\sqrt{(x+a)(x+b)}} \rightarrow$ Residue: $\frac{1}{\sqrt{ab}}$
- $\frac{dx}{\sqrt{(x+a)(x+b)(x+c)}} \rightarrow$ Elliptic Integral

Square roots

- $\frac{dx(ux + v)}{ax^2 + bx + c} \rightarrow$ Residue with square roots

- $\frac{dx}{x\sqrt{(x+a)(x+b)}} \rightarrow$ Residue: $\frac{1}{\sqrt{ab}}$

- $\frac{dx}{\sqrt{(x+a)(x+b)(x+c)}} \rightarrow$ Elliptic Integral

- $\frac{dx \wedge dy}{\sqrt{\mathcal{O}(x^3y^3)}} \rightarrow$ Rational reparametrization?

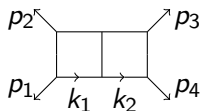
Try: $x \rightarrow x + n_1 + n_2y + n_3xy$

and solve for n_1, n_2, n_3 such that integrand simplifies

Applications

*Planar double box and three loop
four point families*

Planar double box



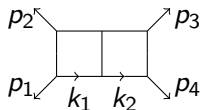
$$p_{12} = p_1 + p_2$$

$$p_{123} = p_1 + p_2 + p_3$$

$$d\mathcal{I}^P = \frac{d^D k_1 d^D k_2 N(k_1, k_2)}{k_1^2 (k_1 + p_1)^2 (k_1 + p_{12})^2 k_2^2 (k_2 + p_{12})^2 (k_2 + p_{123})^2 (k_1 - k_2)^2}$$

$$\begin{array}{cccc}
 k_1^2, & (k_1 + p_1)^2, & (k_1 + p_{12})^2, & (k_1 + p_{123})^2 \\
 k_2^2, & (k_2 + p_1)^2, & (k_2 + p_{12})^2, & (k_2 + p_{123})^2 \\
 (k_1 - k_2)^2 & & &
 \end{array}$$

Planar double box

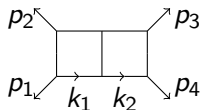


$$p_{12} = p_1 + p_2$$

$$p_{123} = p_1 + p_2 + p_3$$

$$\begin{aligned}
 N(k_1, k_2) = & n_1 + k_1^2 n_2 + (k_1 + p_1)^2 n_3 + (k_1 + p_{12})^2 n_4 \\
 & + (k_1 + p_{123})^2 n_5 + k_2^2 n_6 + (k_2 + p_1)^2 n_7 \\
 & + (k_2 + p_{12})^2 n_8 + (k_2 + p_{123})^2 n_9 + (k_1 - k_2)^2 n_{10} \\
 & + k_1^2 k_2^2 n_{11} + k_1^2 (k_2 + p_1)^2 n_{12} + k_1^2 (k_2 + p_{12})^2 n_{13} \\
 & + k_1^2 (k_2 + p_{123})^2 n_{14} + (k_1 + p_1)^2 k_2^2 n_{15} \\
 & + (k_1 + p_1)^2 (k_2 + p_1)^2 n_{16} + (k_1 + p_1)^2 (k_2 + p_{12})^2 n_{17} \\
 & + (k_1 + p_1)^2 (k_2 + p_{123})^2 n_{18} + (k_1 + p_{12})^2 k_2^2 n_{19} \\
 & + (k_1 + p_{12})^2 (k_2 + p_1)^2 n_{20} + (k_1 + p_{12})^2 (k_2 + p_{12})^2 n_{21} \\
 & + (k_1 + p_{12})^2 (k_2 + p_{123})^2 n_{22} + (k_1 + p_{123})^2 k_2^2 n_{23} \\
 & + (k_1 + p_{123})^2 (k_2 + p_1)^2 n_{24} + (k_1 + p_{123})^2 (k_2 + p_{12})^2 n_{25} \\
 & + (k_1 + p_{123})^2 (k_2 + p_{123})^2 n_{26}
 \end{aligned}$$

Planar double box



$$p_{12} = p_1 + p_2$$

$$p_{123} = p_1 + p_2 + p_3$$

$$\begin{aligned}
 N(k_1, k_2) = & n_1 + k_1^2 n_2 + (k_1 + p_1)^2 n_3 + (k_1 + p_{12})^2 n_4 \\
 & + (k_1 + p_{123})^2 n_5 + k_2^2 n_6 + (k_2 + p_1)^2 n_7 \\
 & + (k_2 + p_{12})^2 n_8 + (k_2 + p_{123})^2 n_9 + (k_1 - k_2)^2 n_{10} \\
 & + k_1^2 k_2^2 n_{11} + k_1^2 (k_2 + p_1)^2 n_{12} + k_1^2 (k_2 + p_{12})^2 n_{13} \\
 & + k_1^2 (k_2 + p_{123})^2 n_{14} + (k_1 + p_1)^2 k_2^2 n_{15} \\
 & + (k_1 + p_1)^2 (k_2 + p_1)^2 n_{16} + (k_1 + p_1)^2 (k_2 + p_{12})^2 n_{17} \\
 & + (k_1 + p_1)^2 (k_2 + p_{123})^2 n_{18} + (k_1 + p_{12})^2 k_2^2 n_{19} \\
 & + (k_1 + p_{12})^2 (k_2 + p_1)^2 n_{20} + (k_1 + p_{12})^2 (k_2 + p_{12})^2 n_{21} \\
 & + (k_1 + p_{12})^2 (k_2 + p_{123})^2 n_{22} + (k_1 + p_{123})^2 k_2^2 n_{23} \\
 & + (k_1 + p_{123})^2 (k_2 + p_1)^2 n_{24} + (k_1 + p_{123})^2 (k_2 + p_{12})^2 n_{25} \\
 & + (k_1 + p_{123})^2 (k_2 + p_{123})^2 n_{26}
 \end{aligned}$$

Absence of
double poles:

$$n_{16} = 0$$

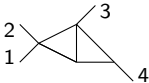
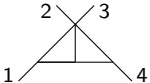
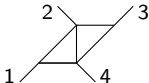
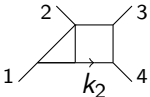
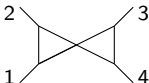
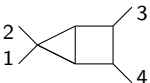
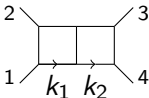
$$n_{26} = 0$$

$$n_{18} = -n_{24}$$

Leading singularities of planar double box

$$\begin{aligned}
 & -\frac{n_5 t + n_9 t - n_1}{s^2 t}, \quad \frac{n_5 t - n_1}{s^2 t}, \quad \frac{n_5 - n_{18} s}{s^2}, \quad \frac{n_7 t - n_1}{s^2 t}, \\
 & -\frac{n_8 s - n_1}{s^2 t}, \quad -\frac{n_3 - n_{17} s}{s^2}, \quad -\frac{n_{18} t^2 + n_5 t + n_7 t + n_{10} t - n_1}{s^2 t}, \\
 & \frac{n_1}{s^2 t}, \quad -\frac{n_{15} s - n_3}{s^2}, \quad \frac{-n_3 t - n_7 t + n_1}{s^2 t}, \quad \frac{-n_{21} s^2 + n_4 s + n_8 s - n_1}{s^2 t}, \\
 & \frac{-n_{22} s t + n_4 s + n_9 t - n_1}{s^2 t}, \quad -\frac{n_4}{s t}, \quad -\frac{n_{20} t - n_4}{s t}, \\
 & \frac{n_{19} s^2 t + n_{20} s^2 t + n_{18} s t^2 - n_{23} s t^2 - n_4 s(s+t) + n_5 t(s+t)}{s^2 t(s+t)}, \\
 & \frac{n_2 s - n_1}{s^2 t}, \quad -\frac{n_{12} s^2(-t) - n_{13} s^2 t - n_{18} s t^2 + n_{25} s t^2 + n_2 s(s+t) - n_5 t(s+t)}{s^2 t(s+t)}, \\
 & \frac{n_{25} s t - n_8 s - n_5 t + n_1}{s^2 t}, \quad \frac{-n_{12} s t + n_2 s + n_7 t - n_1}{s^2 t}, \\
 & -\frac{-n_{14} s t + n_2 s + n_9 t - n_1}{s^2 t}, \quad \frac{-n_{11} s^2 + n_2 s + n_6 s - n_1}{s^2 t}, \quad \frac{n_{23} s - n_5}{s^2}, \quad -\frac{n_6 s - n_1}{s^2 t}
 \end{aligned}$$

Dlog basis of planar double box

	$j_1) s$		$j_2) t$
	$j_3) s+t$		$j_4) st$ $j_5) s(k_2+p_1)^2$ $-sk_2^2$
	$j_6) s^2$		$j_7) s$
	$j_8) s^2 t$ $j_9) s^2(k_2+p_1)^2$ $j_{10}) s(k_2+p_1)^2(k_1+p_1+p_2+p_3)^2$ $-s(k_1+p_1)^2(k_2+p_1+p_2+p_3)^2 - (k_1-k_2)^2 st$ $-k_2^2 s(k_1+p_1+p_2)^2 - k_1^2 s(k_2+p_1+p_2)^2$		

Simple IBP-relations

$$\begin{aligned}j_1 + j_2 - j_3 - \frac{1}{3}j_4 - j_5 &= 0 \\ -3j_1 - 5j_2 + 5j_3 - \frac{4}{3}j_4 + j_6 + \frac{1}{2}j_7 - \frac{1}{2}j_8 - \frac{3}{2}j_9 + j_{10} &= 0\end{aligned}$$

Uniform transcendental weight

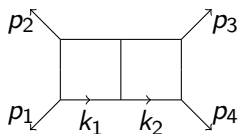
$$\mathbf{a} = \begin{pmatrix} -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{3}{2} & 0 & 0 & 0 & -2 & 0 & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & -2 & 0 & 0 \\ -3 & -3 & 0 & 0 & 4 & 12 & -2 & 0 \\ \frac{9}{2} & 3 & -3 & -1 & -4 & -18 & 1 & 1 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{3}{2} & 0 & 3 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 3 & 6 & 6 & 2 & -4 & -12 & 2 & 2 \\ -\frac{9}{2} & -3 & 3 & -1 & 4 & 18 & -1 & -1 \end{pmatrix}$$

Derive

$$\frac{\partial}{\partial x} \vec{f} = \epsilon \left(\frac{\mathbf{a}}{x} + \frac{\mathbf{b}}{1+x} \right) \vec{f}, \quad x = \frac{t}{s},$$

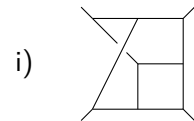
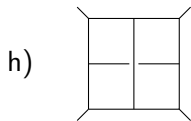
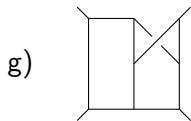
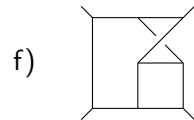
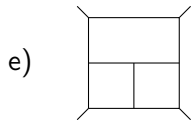
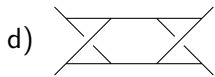
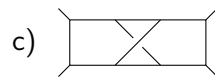
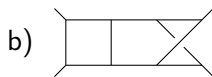
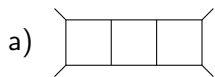
where a and b are constant matrices, to prove uniform transcendental weight property.

Planar double box summary



- Ansatz with 26 integrands
- Integrand basis with 23 integrands (10 non equivalent)
- All integrals have uniform transcendental weight
- Integral basis with 8 integrals (complete)

Three-loop four-point



Vanishing Gram Determinants

$$G \begin{pmatrix} p_1, p_2, p_3, p_4, p_5 \\ q_1, q_2, q_3, q_4, q_5 \end{pmatrix} \equiv \det_{i,j \in \{1,2,3,4,5\}}(2p_i \cdot q_j)$$

always vanishes in four dimensions.

Vanishing gram determinants

$$G \begin{pmatrix} k_1, k_2, p_1, p_2, p_3 \\ k_1, k_2, p_1, p_2, p_3 \end{pmatrix} = 0, \quad \text{in } D = 4$$

$$d\mathcal{I} = \frac{d^4 k_1 d^4 k_2 G(\dots)}{\text{denominator}} = 0$$

- Extend the solution space of dlog integrals
- In general no uniform transcendental weight
- Extended analysis in $d = 6$ or $d = 4 - 2\epsilon$

Summary

- Conjecture: Integrands with constant leading singularities fulfill differential equations in the canonical form
- Leading singularities can be computed in an algorithmic way
- Construct complete dlog bases for integral family
- Applications to two loop and three loop families confirms conjecture

Thank you for listening