Analytic properties of loop integrands

Pascal Wasser, Johannes Gutenberg-Universität Mainz

based on

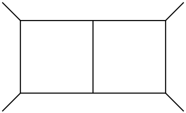
Master thesis[1] in group of Johannes Henn

Loops and Legs

3rd May 2018, St. Goar

[1] https://publications.ub.uni-mainz.de/theses/frontdoor.php?source_opus=100001967

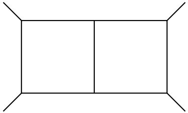
Motivation



$$\mathcal{I}_{a_1,a_2,a_3,a_4,a_5,a_6,a_7,a_8,a_9} = \int \frac{d^D k_1 d^D k_2 P_8^{-a_8} P_9^{-a_9}}{P_1^{a_1} P_2^{a_2} P_3^{a_3} P_4^{a_4} P_5^{a_5} P_6^{a_6} P_7^{a_7}}$$

$$\vec{m} = (\mathcal{I}_{0,1,0,0,1,0,1,0,0}, \mathcal{I}_{1,0,0,1,0,0,1,0,0}, \mathcal{I}_{0,1,1,0,0,1,1,0,0}, \mathcal{I}_{1,0,1,1,0,1,0,0,0}, \mathcal{I}_{1,1,1,0,1,0,1,0,0}, \mathcal{I}_{0,1,1,0,1,1,1,0,0}, \mathcal{I}_{1,1,1,1,1,1,1,0,0}, \mathcal{I}_{2,1,1,1,1,1,1,0,0})^T$$

Motivation



$$\mathcal{I}_{a_1,a_2,a_3,a_4,a_5,a_6,a_7,a_8,a_9} = \int \frac{d^D k_1 d^D k_2 P_8^{-a_8} P_9^{-a_9}}{P_1^{a_1} P_2^{a_2} P_3^{a_3} P_4^{a_4} P_5^{a_5} P_6^{a_6} P_7^{a_7}}$$

$$\vec{m} = (\mathcal{I}_{0,1,0,0,1,0,1,0,0}, \mathcal{I}_{1,0,0,1,0,0,1,0,0}, \mathcal{I}_{0,1,1,0,0,1,1,0,0}, \mathcal{I}_{1,0,1,1,0,1,0,0,0}, \mathcal{I}_{1,1,1,0,1,0,1,0,0}, \mathcal{I}_{0,1,1,0,1,1,1,0,0}, \mathcal{I}_{1,1,1,1,1,1,1,0,0}, \mathcal{I}_{2,1,1,1,1,1,1,0,0})^T$$

What is the simplest basis of master integrals for a given integral topology?

Leading Singularities

$$\int_{-\infty}^{\infty} dk_0 \int_{-\infty}^{\infty} dk_1 \int_{-\infty}^{\infty} dk_2 \int_{-\infty}^{\infty} dk_3 \frac{1}{P_1 P_2 P_3 P_4}$$

Leading Singularities

$$\int_{-\infty}^{\infty} dk_0 \int_{-\infty}^{\infty} dk_1 \int_{-\infty}^{\infty} dk_2 \int_{-\infty}^{\infty} dk_3 \frac{1}{P_1 P_2 P_3 P_4}$$

$$\Downarrow$$

$$\frac{1}{(2\pi i)^4} \oint_{P_1=0} dk_0 \oint_{P_2=0} dk_1 \oint_{P_3=0} dk_2 \oint_{P_4=0} dk_3 \frac{1}{P_1 P_2 P_3 P_4}$$

Leading Singularities

$$\int_{-\infty}^{\infty} dk_0 \int_{-\infty}^{\infty} dk_1 \int_{-\infty}^{\infty} dk_2 \int_{-\infty}^{\infty} dk_3 \frac{1}{P_1 P_2 P_3 P_4}$$

$$\Big\downarrow$$

$$\frac{1}{(2\pi i)^4} \oint_{P_1=0} dk_0 \oint_{P_2=0} dk_1 \oint_{P_3=0} dk_2 \oint_{P_4=0} dk_3 \frac{1}{P_1 P_2 P_3 P_4}$$

Idea: Leading singularities as simple as possible

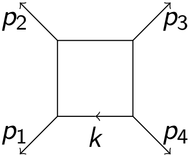
Outline

- Introduction
  - Differential equations
  - Dlog forms and leading singularities
- Method
  - Algorithmic computation of leading singularities and dlog forms
- Results
  - Planar double box family
  - Three loop four point families

Introduction

Differential equations, dlog forms and leading singularities

Differential equations



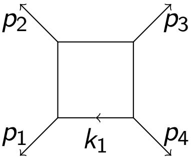
$$\mathcal{I}_{a_1,a_2,a_3,a_4}(s,t)$$

$$s = (p_1 + p_2)^2$$
$$t = (p_1 + p_4)^2$$

$$\vec{f} = \left(\mathcal{I}_{1,1,1,1}, \mathcal{I}_{1,0,1,0}, \mathcal{I}_{0,1,0,1}\right)^T$$

$$\frac{\partial}{\partial s}\vec{f} = A(s,t,D)\vec{f}$$

Simplified Differential Equations Henn, 2013

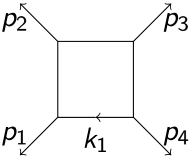


$$g_1 = c\epsilon^2(-s)^\epsilon t\mathcal{I}_{0,1,1,1}, \qquad D = 4 - 2\epsilon$$
$$g_2 = c\epsilon^2(-s)^\epsilon s\mathcal{I}_{1,1,1,0},$$
$$g_3 = c\epsilon^2(-s)^\epsilon st\mathcal{I}_{1,1,1,1}$$

$$\frac{\partial}{\partial x}\vec{g}(x,\epsilon) = \epsilon\left(\frac{a}{x} + \frac{b}{1+x}\right)\vec{g}(x,\epsilon), \quad x = t/s$$

$$a = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & -1 \end{pmatrix}, \quad b = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -2 & -2 & 1 \end{pmatrix}$$

Simplified Differential Equations Henn, 2013



$$g_1 = c\epsilon^2(-s)^\epsilon t\mathcal{I}_{0,1,1,1}, \qquad D = 4 - 2\epsilon$$
$$g_2 = c\epsilon^2(-s)^\epsilon s\mathcal{I}_{1,1,1,0},$$
$$g_3 = c\epsilon^2(-s)^\epsilon st\mathcal{I}_{1,1,1,1}$$

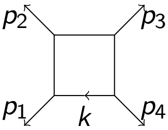
$$\frac{\partial}{\partial x}\vec{g}(x,\epsilon) = \epsilon\left(\frac{a}{x} + \frac{b}{1+x}\right)\vec{g}(x,\epsilon), \quad x = t/s$$

$$a = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & -1 \end{pmatrix}, \quad b = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -2 & -2 & 1 \end{pmatrix}$$

Uniform transcendental weight solution:

$$g_3 = 4 + \epsilon(-2\log x) + \epsilon^2\left(-\frac{4\pi^2}{3}\right) + \epsilon^3\left(\frac{7\pi^2}{6}\log x + \frac{1}{3}\log^3 x + \ldots\right) + \mathcal{O}(\epsilon^4)$$

Dlog form of box integrand



$$d\mathcal{I}_4 = \frac{d^4 k}{k^2 (k-p_1)^2 (k-p_1-p_2)^2 (k+p_4)^2}$$

$$= \frac{1}{st} d\log \frac{k^2}{(k-k^*)^2} \wedge d\log \frac{(k-p_1)^2}{(k-k^*)^2} \wedge d\log \frac{(k-p_1-p_2)^2}{(k-k^*)^2} \wedge d\log \frac{(k+p_4)^2}{(k-k^*)^2}$$

Arkani-Hamed et al., 2012

k^*: Solution to $k^2 = (k-p_1)^2 = (k-p_1-p_2)^2 = (k+p_4)^2 = 0$.

Leading singularities

$$\frac{1}{(2\pi i)^4} \oint_{P_1=0} dk_0 \oint_{P_2=0} dk_1 \oint_{P_3=0} dk_2 \oint_{P_4=0} dk_3 \frac{1}{P_1 P_2 P_3 P_4}$$

Leading singularities

$$\frac{1}{(2\pi i)^4} \oint_{P_1=0} dk_0 \oint_{P_2=0} dk_1 \oint_{P_3=0} dk_2 \oint_{P_4=0} dk_3 \frac{1}{P_1 P_2 P_3 P_4}$$

$$c \; d\log(q_1) \wedge d\log(q_2) \wedge d\log(q_3) \wedge d\log(q_4)$$

Leading singularities

$$\frac{1}{(2\pi i)^4} \oint_{P_1=0} dk_0 \oint_{P_2=0} dk_1 \oint_{P_3=0} dk_2 \oint_{P_4=0} dk_3 \frac{1}{P_1 P_2 P_3 P_4}$$

$$c \; d\log(q_1) \wedge d\log(q_2) \wedge d\log(q_3) \wedge d\log(q_4)$$

$$= c \, \frac{dq_1}{q_1} \wedge \frac{dq_2}{q_2} \wedge \frac{dq_3}{q_3} \wedge \frac{dq_4}{q_4}$$

Leading singularities

$$\frac{1}{(2\pi i)^4} \oint_{P_1=0} dk_0 \oint_{P_2=0} dk_1 \oint_{P_3=0} dk_2 \oint_{P_4=0} dk_3 \frac{1}{P_1 P_2 P_3 P_4}$$

$$c \; d\log(q_1) \wedge d\log(q_2) \wedge d\log(q_3) \wedge d\log(q_4)$$

$$= c \, \frac{dq_1}{q_1} \wedge \frac{dq_2}{q_2} \wedge \frac{dq_3}{q_3} \wedge \frac{dq_4}{q_4}$$

Leading singularity: c

Dlog forms and leading singularities

$$d\Omega = \sum_{i=1}^{n} c_i d\log(g_{i1}) \wedge d\log(g_{i2}) \wedge \ldots \wedge d\log(g_{ik})$$

Values of c_i: Leading singularities of $d\Omega$.

Dlog forms and leading singularities

$$d\Omega = \sum_{i=1}^{n} c_i d\log(g_{i1}) \wedge d\log(g_{i2}) \wedge \ldots \wedge d\log(g_{ik})$$

Values of c_i: Leading singularities of $d\Omega$.

Conjecture

Dlog integrands with constant leading singularities fulfill differential equations in the canonical form

Method

Algorithmic computation of leading singularities and dlog forms

Simple example

$$df(x,y) = \frac{dx \wedge dy}{xy(x+y+1)}$$

Simple example

$$df(x,y) = \frac{dx \wedge dy}{xy(x+y+1)} = \frac{dx \wedge dy}{xy(x+1)} + \frac{dx \wedge dy}{x(-x-1)(x+y+1)}$$

Simple example

$$df(x,y) = \frac{dx \wedge dy}{xy(x+y+1)} = \frac{dx \wedge dy}{xy(x+1)} + \frac{dx \wedge dy}{x(-x-1)(x+y+1)}$$

$$= \frac{dx}{x(x+1)} \wedge d\log(y) - \frac{dx}{x(x+1)} \wedge d\log(x+y+1)$$

Simple example

$$df(x,y) = \frac{dx \wedge dy}{xy(x+y+1)} = \frac{dx \wedge dy}{xy(x+1)} + \frac{dx \wedge dy}{x(-x-1)(x+y+1)}$$

$$= \frac{dx}{x(x+1)} \wedge d\log(y) - \frac{dx}{x(x+1)} \wedge d\log(x+y+1)$$

$$= \left[\frac{dx}{x} - \frac{dx}{x+1}\right] \wedge [d\log(y) - d\log(x+y+1)]$$

Simple example

$$df(x,y) = \frac{dx \wedge dy}{xy(x+y+1)} = \frac{dx \wedge dy}{xy(x+1)} + \frac{dx \wedge dy}{x(-x-1)(x+y+1)}$$

$$= \frac{dx}{x(x+1)} \wedge d\log(y) - \frac{dx}{x(x+1)} \wedge d\log(x+y+1)$$

$$= \left[\frac{dx}{x} - \frac{dx}{x+1}\right] \wedge [d\log(y) - d\log(x+y+1)]$$

$$= [d\log(x) - d\log(x+1)] \wedge [d\log(y) - d\log(x+y+1)]$$

Simple example

$$df(x,y) = \frac{dx \wedge dy}{xy(x+y+1)} = \frac{dx \wedge dy}{xy(x+1)} + \frac{dx \wedge dy}{x(-x-1)(x+y+1)}$$

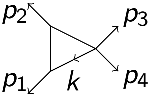
$$= \frac{dx}{x(x+1)} \wedge d\log(y) - \frac{dx}{x(x+1)} \wedge d\log(x+y+1)$$

$$= \left[\frac{dx}{x} - \frac{dx}{x+1}\right] \wedge [d\log(y) - d\log(x+y+1)]$$

$$= [d\log(x) - d\log(x+1)] \wedge [d\log(y) - d\log(x+y+1)]$$

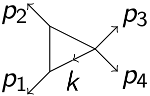
$$= d\log\frac{x}{x+1} \wedge d\log\frac{y}{x+y+1}$$

Triangle



$$= \frac{s\, d^4 k}{k^2 (k - p_1)^2 (k - p_1 - p_2)^2}$$

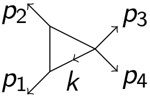
Triangle



$$= \frac{s\, d^4 k}{k^2 (k-p_1)^2 (k-p_1-p_2)^2}$$

$$p_1 = \lambda_1 \tilde{\lambda}_1, \quad p_2 = \lambda_2 \tilde{\lambda}_2, \quad \lambda_1 \tilde{\lambda}_2, \quad \lambda_2 \tilde{\lambda}_1$$

Triangle

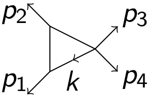


$$= \frac{s\, d^4 k}{k^2 (k-p_1)^2 (k-p_1-p_2)^2}$$

$$p_1 = \lambda_1 \tilde{\lambda}_1, \quad p_2 = \lambda_2 \tilde{\lambda}_2, \quad \lambda_1 \tilde{\lambda}_2, \quad \lambda_2 \tilde{\lambda}_1$$

$$k = \alpha_1 \lambda_1 \tilde{\lambda}_1 + \alpha_2 \lambda_2 \tilde{\lambda}_2 + \frac{\langle 23 \rangle}{\langle 13 \rangle} \alpha_3 \lambda_1 \tilde{\lambda}_2 + \frac{\langle 13 \rangle}{\langle 23 \rangle} \alpha_4 \lambda_2 \tilde{\lambda}_1$$

Triangle



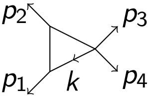
$$= \frac{s\, d^4 k}{k^2 (k - p_1)^2 (k - p_1 - p_2)^2}$$

$$p_1 = \lambda_1 \tilde{\lambda}_1, \quad p_2 = \lambda_2 \tilde{\lambda}_2, \quad \lambda_1 \tilde{\lambda}_2, \quad \lambda_2 \tilde{\lambda}_1$$

$$k = \alpha_1 \lambda_1 \tilde{\lambda}_1 + \alpha_2 \lambda_2 \tilde{\lambda}_2 + \frac{\langle 23 \rangle}{\langle 13 \rangle} \alpha_3 \lambda_1 \tilde{\lambda}_2 + \frac{\langle 13 \rangle}{\langle 23 \rangle} \alpha_4 \lambda_2 \tilde{\lambda}_1$$

$$d\mathcal{I}_3 = \frac{d\alpha_1 \wedge d\alpha_2 \wedge d\alpha_3 \wedge d\alpha_4}{(\alpha_1\alpha_2 - \alpha_3\alpha_4)(\alpha_1\alpha_2 - \alpha_2 - \alpha_3\alpha_4)(1 - \alpha_1 - \alpha_2 + \alpha_1\alpha_2 - \alpha_3\alpha_4)}$$

Triangle



$$= \frac{s\, d^4 k}{k^2 (k-p_1)^2 (k-p_1-p_2)^2}$$

$$p_1 = \lambda_1 \tilde{\lambda}_1, \quad p_2 = \lambda_2 \tilde{\lambda}_2, \quad \lambda_1 \tilde{\lambda}_2, \quad \lambda_2 \tilde{\lambda}_1$$

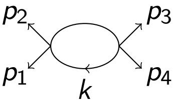
$$k = \alpha_1 \lambda_1 \tilde{\lambda}_1 + \alpha_2 \lambda_2 \tilde{\lambda}_2 + \frac{\langle 23 \rangle}{\langle 13 \rangle} \alpha_3 \lambda_1 \tilde{\lambda}_2 + \frac{\langle 13 \rangle}{\langle 23 \rangle} \alpha_4 \lambda_2 \tilde{\lambda}_1$$

$$d\mathcal{I}_3 = \frac{d\alpha_1 \wedge d\alpha_2 \wedge d\alpha_3 \wedge d\alpha_4}{(\alpha_1\alpha_2 - \alpha_3\alpha_4)(\alpha_1\alpha_2 - \alpha_2 - \alpha_3\alpha_4)(1 - \alpha_1 - \alpha_2 + \alpha_1\alpha_2 - \alpha_3\alpha_4)}$$

$$= d\log(\alpha_4) \wedge d\log(\alpha_2) \wedge d\log(\alpha_3) \wedge d\log(-\alpha_2 + \alpha_1\alpha_2 - \alpha_3\alpha_4)$$

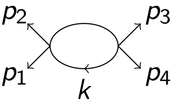
$$+(\ldots) + (\ldots) + (\ldots)$$

Bubble



$$= \frac{s^2\, d^4 k}{k^2 (k - p_1 - p_2)^2}$$

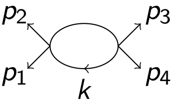
Bubble



$$= \frac{s^2\, d^4 k}{k^2 (k - p_1 - p_2)^2}$$

$$d\mathcal{I}_2 = \frac{d\alpha_1 \wedge d\alpha_2 \wedge d\alpha_3 \wedge d\alpha_4}{(\alpha_1\alpha_2 - \alpha_3\alpha_4)(1 - \alpha_1 + \alpha_1\alpha_2 - \alpha_2 - \alpha_3\alpha_4)}$$

Bubble

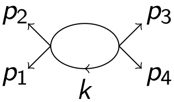


$$= \frac{s^2\, d^4 k}{k^2 (k - p_1 - p_2)^2}$$

$$d\mathcal{I}_2 = \frac{d\alpha_1 \wedge d\alpha_2 \wedge d\alpha_3 \wedge d\alpha_4}{(\alpha_1\alpha_2 - \alpha_3\alpha_4)(1 - \alpha_1 + \alpha_1\alpha_2 - \alpha_2 - \alpha_3\alpha_4)}$$

$$\begin{aligned} d\mathcal{I}_2 = {} & d\alpha_1 \wedge d\log(-1 + \alpha_1 + \alpha_2) \wedge d\log(\alpha_3) \\ & \wedge [d\log(\alpha_1\alpha_2 - \alpha_3\alpha_4) - d\log(1 - \alpha_1 - \alpha_2 + \alpha_1\alpha_2 - \alpha_3\alpha_4)] \end{aligned}$$

Bubble



$$= \frac{s^2\, d^4 k}{k^2 (k - p_1 - p_2)^2}$$

$$d\mathcal{I}_2 = \frac{d\alpha_1 \wedge d\alpha_2 \wedge d\alpha_3 \wedge d\alpha_4}{(\alpha_1\alpha_2 - \alpha_3\alpha_4)(1 - \alpha_1 + \alpha_1\alpha_2 - \alpha_2 - \alpha_3\alpha_4)}$$

$$d\mathcal{I}_2 = d\alpha_1 \wedge d\log(-1 + \alpha_1 + \alpha_2) \wedge d\log(\alpha_3)$$
$$\wedge \left[d\log(\alpha_1\alpha_2 - \alpha_3\alpha_4) - d\log(1 - \alpha_1 - \alpha_2 + \alpha_1\alpha_2 - \alpha_3\alpha_4)\right]$$

No dlog form! Double pole at $\alpha_1 \to \infty$.
$d\alpha_1 = \frac{d\tilde{\alpha}_1}{\tilde{\alpha}_1^2}$, with $\tilde{\alpha}_1 = \alpha_1^{-1}$

Square roots

- $\frac{dx(ux+v)}{ax^2+bx+c} \rightarrow$ Residue with square roots

Square roots

- $\frac{dx(ux+v)}{ax^2+bx+c} \to$ Residue with square roots
- $\frac{dx}{x\sqrt{(x+a)(x+b)}} \to$ Residue: $\frac{1}{\sqrt{ab}}$

Square roots

- $\frac{dx(ux+v)}{ax^2+bx+c} \to$ Residue with square roots
- $\frac{dx}{x\sqrt{(x+a)(x+b)}} \to$ Residue: $\frac{1}{\sqrt{ab}}$
- $\frac{dx}{\sqrt{(x+a)(x+b)(x+c)}} \to$ Elliptic Integral

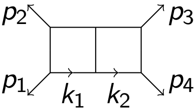
Square roots

- $\frac{dx(ux+v)}{ax^2+bx+c} \to$ Residue with square roots
- $\frac{dx}{x\sqrt{(x+a)(x+b)}} \to$ Residue: $\frac{1}{\sqrt{ab}}$
- $\frac{dx}{\sqrt{(x+a)(x+b)(x+c)}} \to$ Elliptic Integral
- $\frac{dx \wedge dy}{\sqrt{\mathcal{O}(x^3y^3)}} \to$ Rational reparametrization?
  Try: $x \to x + n_1 + n_2 y + n_3 xy$
  and solve for n_1, n_2, n_3 such that integrand simplifies

Applications

Planar double box and three loop four point families

Planar double box

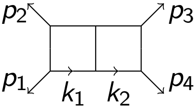


$$p_{12} = p_1 + p_2$$
$$p_{123} = p_1 + p_2 + p_3$$

$$d\mathcal{I}^{\mathrm{P}} = \frac{d^D k_1 d^D k_2 N(k_1, k_2)}{k_1^2 (k_1 + p_1)^2 (k_1 + p_{12})^2 k_2^2 (k_2 + p_{12})^2 (k_2 + p_{123})^2 (k_1 - k_2)^2}$$

$$\begin{array}{llll} k_1^2, & (k_1 + p_1)^2, & (k_1 + p_{12})^2, & {\color{red}(k_1 + p_{123})^2} \\ k_2^2, & {\color{red}(k_2 + p_1)^2}, & (k_2 + p_{12})^2, & (k_2 + p_{123})^2 \\ (k_1 - k_2)^2 & & & \end{array}$$

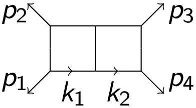
Planar double box



$$p_{12} = p_1 + p_2$$
$$p_{123} = p_1 + p_2 + p_3$$

$$\begin{aligned}
N(k_1, k_2) = {} & n_1 + k_1^2 n_2 + (k_1+p_1)^2 n_3 + (k_1+p_{12})^2 n_4 \\
& + (k_1+p_{123})^2 n_5 + k_2^2 n_6 + (k_2+p_1)^2 n_7 \\
& + (k_2+p_{12})^2 n_8 + (k_2+p_{123})^2 n_9 + (k_1-k_2)^2 n_{10} \\
& + k_1^2 k_2^2 n_{11} + k_1^2 (k_2+p_1)^2 n_{12} + k_1^2 (k_2+p_{12})^2 n_{13} \\
& + k_1^2 (k_2+p_{123})^2 n_{14} + (k_1+p_1)^2 k_2^2 n_{15} \\
& + (k_1+p_1)^2 (k_2+p_1)^2 n_{16} + (k_1+p_1)^2 (k_2+p_{12})^2 n_{17} \\
& + (k_1+p_1)^2 (k_2+p_{123})^2 n_{18} + (k_1+p_{12})^2 k_2^2 n_{19} \\
& + (k_1+p_{12})^2 (k_2+p_1)^2 n_{20} + (k_1+p_{12})^2 (k_2+p_{12})^2 n_{21} \\
& + (k_1+p_{12})^2 (k_2+p_{123})^2 n_{22} + (k_1+p_{123})^2 k_2^2 n_{23} \\
& + (k_1+p_{123})^2 (k_2+p_1)^2 n_{24} + (k_1+p_{123})^2 (k_2+p_{12})^2 n_{25} \\
& + (k_1+p_{123})^2 (k_2+p_{123})^2 n_{26}
\end{aligned}$$

Planar double box



$$p_{12} = p_1 + p_2$$
$$p_{123} = p_1 + p_2 + p_3$$

$$\begin{aligned}
N(k_1, k_2) = {} & n_1 + k_1^2 n_2 + (k_1 + p_1)^2 n_3 + (k_1 + p_{12})^2 n_4 \\
& + (k_1 + p_{123})^2 n_5 + k_2^2 n_6 + (k_2 + p_1)^2 n_7 \\
& + (k_2 + p_{12})^2 n_8 + (k_2 + p_{123})^2 n_9 + (k_1 - k_2)^2 n_{10} \\
& + k_1^2 k_2^2 n_{11} + k_1^2 (k_2 + p_1)^2 n_{12} + k_1^2 (k_2 + p_{12})^2 n_{13} \\
& + k_1^2 (k_2 + p_{123})^2 n_{14} + (k_1 + p_1)^2 k_2^2 n_{15} \\
& + (k_1 + p_1)^2 (k_2 + p_1)^2 n_{16} + (k_1 + p_1)^2 (k_2 + p_{12})^2 n_{17} \\
& + (k_1 + p_1)^2 (k_2 + p_{123})^2 n_{18} + (k_1 + p_{12})^2 k_2^2 n_{19} \\
& + (k_1 + p_{12})^2 (k_2 + p_1)^2 n_{20} + (k_1 + p_{12})^2 (k_2 + p_{12})^2 n_{21} \\
& + (k_1 + p_{12})^2 (k_2 + p_{123})^2 n_{22} + (k_1 + p_{123})^2 k_2^2 n_{23} \\
& + (k_1 + p_{123})^2 (k_2 + p_1)^2 n_{24} + (k_1 + p_{123})^2 (k_2 + p_{12})^2 n_{25} \\
& + (k_1 + p_{123})^2 (k_2 + p_{123})^2 n_{26}
\end{aligned}$$

Absence of double poles:

$$n_{16} = 0$$
$$n_{26} = 0$$
$$n_{18} = -n_{24}$$

Leading singularities of planar double box

$$-\frac{n_5 t+n_9 t-n_1}{s^2 t}, \quad \frac{n_5 t-n_1}{s^2 t}, \quad \frac{n_5-n_{18} s}{s^2}, \quad \frac{n_7 t-n_1}{s^2 t},$$

$$-\frac{n_8 s-n_1}{s^2 t}, \quad -\frac{n_3-n_{17} s}{s^2}, \quad -\frac{n_{18} t^2+n_5 t+n_7 t+n_{10} t-n_1}{s^2 t},$$

$$\frac{n_1}{s^2 t}, -\frac{n_{15} s-n_3}{s^2}, \frac{-n_3 t-n_7 t+n_1}{s^2 t}, \frac{-n_{21} s^2+n_4 s+n_8 s-n_1}{s^2 t},$$

$$\frac{-n_{22} st+n_4 s+n_9 t-n_1}{s^2 t}, \quad -\frac{n_4}{st}, \quad -\frac{n_{20} t-n_4}{st},$$

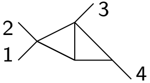
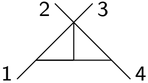
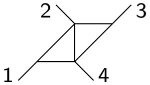
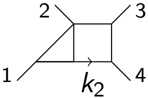
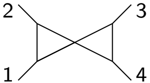
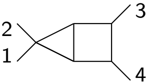
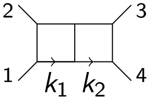
$$\frac{n_{19} s^2 t+n_{20} s^2 t+n_{18} st^2-n_{23} st^2-n_4 s(s+t)+n_5 t(s+t)}{s^2 t(s+t)},$$

$$\frac{n_2 s-n_1}{s^2 t}, \quad -\frac{n_{12} s^2(-t)-n_{13} s^2 t-n_{18} st^2+n_{25} st^2+n_2 s(s+t)-n_5 t(s+t)}{s^2 t(s+t)},$$

$$\frac{n_{25} st-n_8 s-n_5 t+n_1}{s^2 t}, \quad \frac{-n_{12} st+n_2 s+n_7 t-n_1}{s^2 t},$$

$$-\frac{-n_{14} st+n_2 s+n_9 t-n_1}{s^2 t}, \quad \frac{-n_{11} s^2+n_2 s+n_6 s-n_1}{s^2 t}, \quad \frac{n_{23} s-n_5}{s^2}, -\frac{n_6 s-n_1}{s^2 t}$$

Dlog basis of planar double box

	$j_1)\ s$		$j_2)\ t$
	$j_3)\ s+t$		$j_4)\ st$ $j_5)\ s(k_2+p_1)^2 - sk_2^2$
	$j_6)\ s^2$		$j_7)\ s$
	$j_8)\ s^2 t$ $j_9)\ s^2(k_2+p_1)^2$ $j_{10})\ s\,(k_2+p_1)^2\,(k_1+p_1+p_2+p_3)^2 - s\,(k_1+p_1)^2\,(k_2+p_1+p_2+p_3)^2 - (k_1-k_2)^2 st - k_2^2 s\,(k_1+p_1+p_2)^2 - k_1^2 s\,(k_2+p_1+p_2)^2$		

Simple IBP-relations

$$j_1 + j_2 - j_3 - \frac{1}{3} j_4 - j_5 = 0$$

$$-3j_1 - 5j_2 + 5j_3 - \frac{4}{3} j_4 + j_6 + \frac{1}{2} j_7 - \frac{1}{2} j_8 - \frac{3}{2} j_9 + j_{10} = 0$$

Uniform transcendental weight

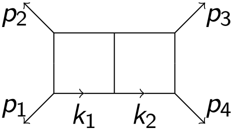
$$a = \begin{pmatrix} -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{3}{2} & 0 & 0 & 0 & -2 & 0 & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & -2 & 0 & 0 \\ -3 & -3 & 0 & 0 & 4 & 12 & -2 & 0 \\ \frac{9}{2} & 3 & -3 & -1 & -4 & -18 & 1 & 1 \end{pmatrix}, \qquad b = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{3}{2} & 0 & 3 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 3 & 6 & 6 & 2 & -4 & -12 & 2 & 2 \\ -\frac{9}{2} & -3 & 3 & -1 & 4 & 18 & -1 & -1 \end{pmatrix}$$

Derive

$$\frac{\partial}{\partial x}\vec{f} = \epsilon\left(\frac{a}{x} + \frac{b}{1+x}\right)\vec{f}, \quad x = \frac{t}{s},$$

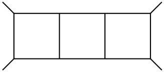
where a and b are constant matrices, to prove uniform transcendental weight property.

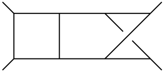
Planar double box summary

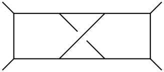


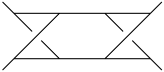
- Ansatz with 26 integrands
- Integrand basis with 23 integrands (10 non equivalent)
- All integrals have uniform transcendental weight
- Integral basis with 8 integrals (complete)

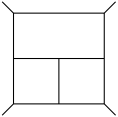
Three-loop four-point

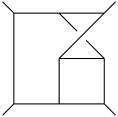
a) 

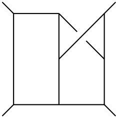
b) 

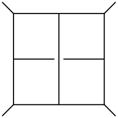
c) 

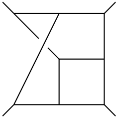
d) 

e) 

f) 

g) 

h) 

i) 

Vanishing Gram Determinants

$$G\begin{pmatrix} p_1, p_2, p_3, p_4, p_5 \\ q_1, q_2, q_3, q_4, q_5 \end{pmatrix} \equiv \det(2p_i \cdot q_j)_{i,j\in\{1,2,3,4,5\}}$$

always vanishes in four dimensions.

Vanishing gram determinants

$$G\left(\begin{array}{c} k_1, k_2, p_1, p_2, p_3 \\ k_1, k_2, p_1, p_2, p_3 \end{array}\right) = 0, \quad \text{in } D = 4$$

$$d\mathcal{I} = \frac{d^4 k_1 d^4 k_2 G(\ldots)}{\text{denominator}} = 0$$

- Extend the solution space of dlog integrals
- In general no uniform transcendental weight
- Extended analysis in $d = 6$ or $d = 4 - 2\epsilon$

Summary

- Conjecture: Integrands with constant leading singularities fulfill differential equations in the canonical form
- Leading singularities can be computed in an algorithmic way
- Construct complete dlog bases for integral family
- Applications to two loop and three loop families confirms conjecture

Thank you for listening