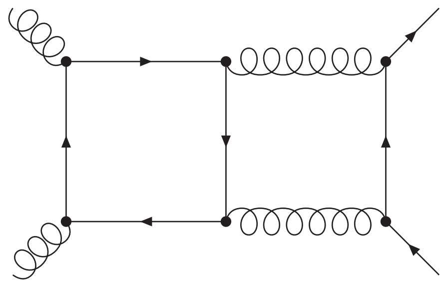
From elliptic curves to Feynman integrals

Stefan Weinzierl

Institut für Physik, Universität Mainz

in collaboration with Luise Adams and Ekta Chaubey



L. Adams, E. Chaubey, S.W., arXiv:1804.11144

Motivation

- The planar double box integral for top-pair production with a closed top loop enters the NNLO $pp \to t\bar{t}$ cross section.

- In the existing NNLO calculation for $pp \to t\bar{t}$ this integral has been treated numerically.

  Czakon, Fiedler, Mitov, ’13,

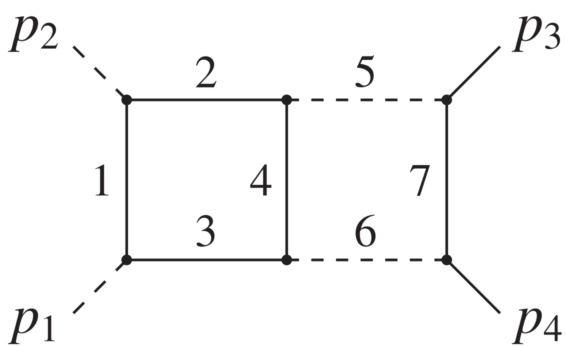
  Bärnreuther, Czakon, Fiedler, ’13

- This integral is up to now not known analytically. Impedes further progress on the analytical side.

- Contains the massive sunrise integral as sub-topology. Expect elliptic stuff.

Kinematics

$$I_{\nu_1\nu_2\nu_3\nu_4\nu_5\nu_6\nu_7}\left(D,\frac{s}{m^2},\frac{t}{m^2}\right) = e^{2\gamma_E\varepsilon}\left(m^2\right)^{\sum\limits_{j=1}^{7}\nu_j - D}\int\frac{d^Dk_1}{i\pi^{\frac{D}{2}}}\frac{d^Dk_2}{i\pi^{\frac{D}{2}}}\prod_{j=1}^{7}\frac{1}{P_j^{\nu_j}},$$



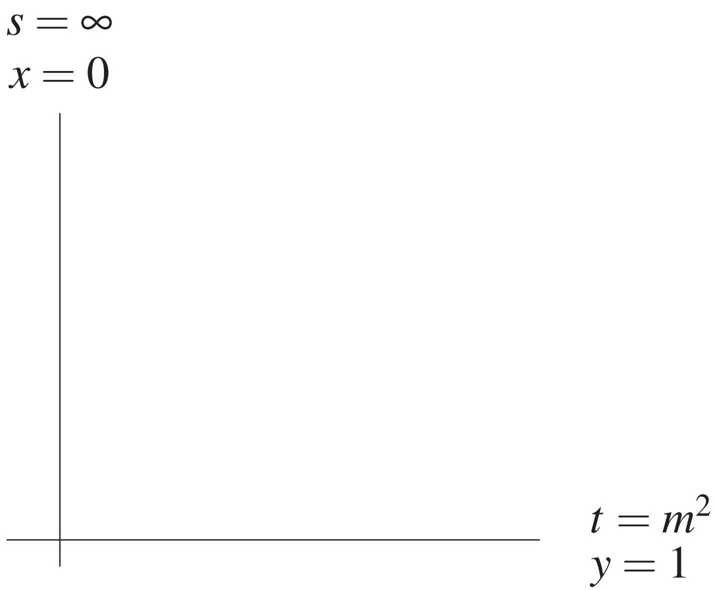
$$p_1^2 = p_2^2 = 0, \qquad p_3^2 = p_4^2 = m^2,$$
$$s = (p_1+p_2)^2, \qquad t = (p_2+p_3)^2.$$

Executive summary

The Laurent expansion in ε of $I_{1111111}$ (and all its sub-topologies) can be computed systematically to all orders in ε in terms of iterated integrals.

The solution

- reduces to multiple polylogarithms for $t = m^2$ and
- reduces to iterated integrals of modular forms of $\Gamma_1(6)$ for $s = \infty$.



Iterated integrals

For ω_1, ..., ω_k differential 1-forms on a manifold M and $\gamma : [0,1] \rightarrow M$ a path, write for the pull-back of ω_j to the interval $[0,1]$

$$f_j(\lambda)\, d\lambda = \gamma^* \omega_j.$$

The iterated integral is defined by

$$I_\gamma(\omega_1, ..., \omega_k; \lambda) = \int\limits_0^{\lambda} d\lambda_1 f_1(\lambda_1) \int\limits_0^{\lambda_1} d\lambda_2 f_2(\lambda_2) \dots \int\limits_0^{\lambda_{k-1}} d\lambda_k f_k(\lambda_k).$$

Multiple polylogarithms:

$$\omega_j = \frac{d\lambda}{\lambda - c_j}.$$

Iterated integrals of modular forms: $f(\tau)$ a modular form, simply write f instead of $2\pi i f d\tau$ in the arguments of iterated integrals.

Outline of the calculation

- Derive the differential equation in a pre-canonical basis (standard, but ...)
- Transform the differential equation (essential step)
- Solve the differential equation (easy)

The differential equation in the pre-canonical basis

In a pre-canonical basis with higher powers of the propagators but no numerators:

$$\begin{aligned} \frac{d}{ds} I_{\nu_1\nu_2\nu_3\nu_4\nu_5\nu_6\nu_7}(D) &= \mathbf{D}^+ \mathcal{F}_s\left(\nu_1 \mathbf{1}^+, ..., \nu_7 \mathbf{7}^+\right) I_{\nu_1\nu_2\nu_3\nu_4\nu_5\nu_6\nu_7}(D) \\ \frac{d}{dt} I_{\nu_1\nu_2\nu_3\nu_4\nu_5\nu_6\nu_7}(D) &= \mathbf{D}^+ \mathcal{F}_t\left(\nu_1 \mathbf{1}^+, ..., \nu_7 \mathbf{7}^+\right) I_{\nu_1\nu_2\nu_3\nu_4\nu_5\nu_6\nu_7}(D) \end{aligned}$$

Dimensional shift relation:

$$\mathbf{D}^- I_{\nu_1\nu_2\nu_3\nu_4\nu_5\nu_6\nu_7}(D) = \mathcal{U}\left(\nu_1 \mathbf{1}^+, ..., \nu_7 \mathbf{7}^+\right) I_{\nu_1\nu_2\nu_3\nu_4\nu_5\nu_6\nu_7}(D)$$

Not necessarily the smartest way to compute the differential equations, but foolproof.

$$d\vec{I} = A(x, y, \varepsilon)\vec{I}$$

Integral reduction

We used `Reduze`, `Kira` and `Fire` for the integral reductions. Taking trivial symmetry relations into account, all programs give 45 master integrals.

Observations:

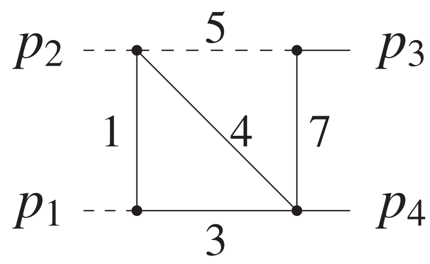
- The reductions disagree for the three most complicated topologies.

- The results of two of the three programs fail the integrability check

$$dA = A \wedge A$$

What shall we conclude from these observations?

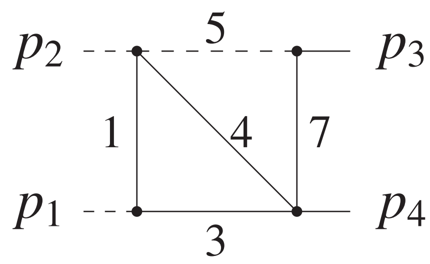
Integral reduction



- All three programs correctly implement the Laporta algorithm.
- The algorithm does not guarantee that all relations among the Feynman integrals are found.
- There exists an additional relation.
- Taking this relation into account, the results from `Reduze`, `Kira` and `Fire` agree and the integrability condition is satisfied for all of them.
- In addition, we verified numerically the first few terms in the ε-expansion of this relation.

The additional relation

$$
\begin{aligned}
0 \quad = \quad & 3\,(D-4)\left(m^2-s-t\right)\left(m^2-t\right) I_{1011101} \\
& +2\left(m^2-s-t\right)\left(m^2-t\right) m^2 I_{2011101} \\
& +2\left(m^2-s-t\right)\left(m^2-t\right) m^2 I_{1021101} \\
& +\left(2m^4-3m^2s-2m^2t+st\right)\left(m^2-t\right) I_{1012101} \\
& +4\,s^2 m^2 I_{1011201} \\
& +\text{sub-topologies}.
\end{aligned}
$$



Reduces the number of master integrals in the sector $I_{\nu_1 0 \nu_3 \nu_4 \nu_5 0 \nu_7}$ from 5 to 4.

Basis transformation

We seek a transformation (Henn, ’13)

$$\vec{J} = U\vec{I}$$

such that the transformed differential equation is linear in ε

$$d\vec{J} = \left(A^{(0)}(x,y) + \varepsilon A^{(1)}(x,y)\right)\vec{J}$$

and

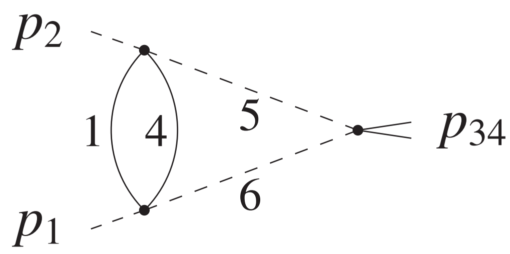
- $A^{(0)}$ is strictly lower triangular

- $A^{(1)}$ is block triangular

This is equivalent to an ε-form, if one introduces primitives for the entries of $A^{(0)}$.

Kinematic variables

Introduce dimensionless variables x and y through



$$\frac{s}{m^2} = -\frac{\left(1+x^2\right)^2}{x\left(1-x^2\right)}, \qquad \frac{t}{m^2} = y.$$

The definition of x simultaneously rationalises the square roots

$$\sqrt{-s(4m^2-s)} \quad \text{and} \quad \sqrt{-s(-4m^2-s)}.$$

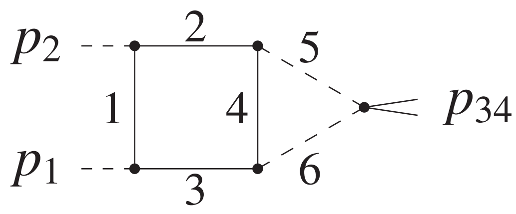
The alphabet for the polylogarithms

All sub-topologies, which depend only on s/m^2 (and all integrals in the limit $y = 1$) can be expressed as multiple polylogarithms / iterated integrals with letters given by

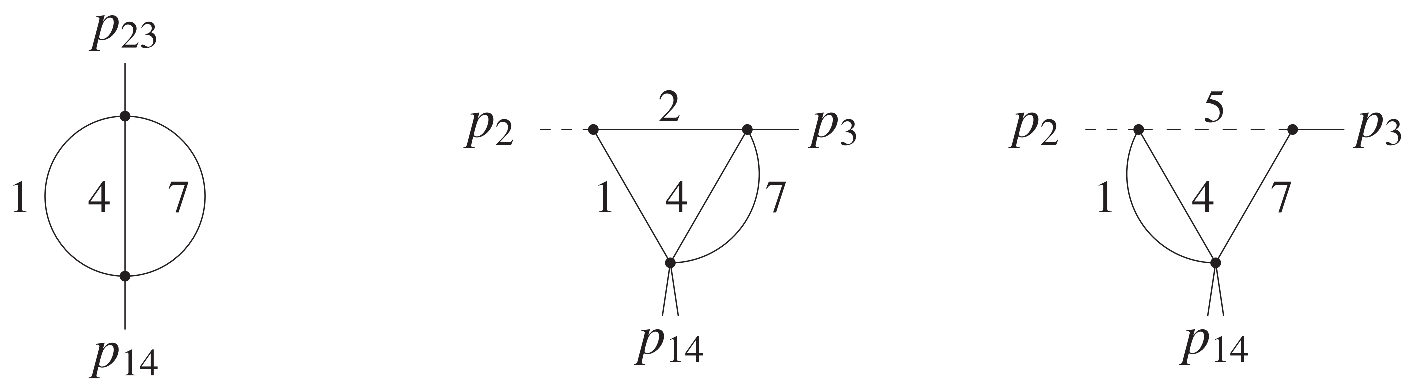
$$
\begin{aligned}
\omega_0 &= \frac{ds}{s} &&= \frac{2dx}{x-i} + \frac{2dx}{x+i} - \frac{dx}{x-1} - \frac{dx}{x+1} - \frac{dx}{x}, \\
\omega_4 &= \frac{ds}{s-4m^2} &&= \frac{2dx}{x-\left(1+\sqrt{2}\right)} + \frac{2dx}{x-\left(1-\sqrt{2}\right)} - \frac{dx}{x-1} - \frac{dx}{x+1} - \frac{dx}{x}, \\
\omega_{-4} &= \frac{ds}{s+4m^2} &&= \frac{2dx}{x-\left(-1+\sqrt{2}\right)} + \frac{2dx}{x-\left(-1-\sqrt{2}\right)} - \frac{dx}{x-1} - \frac{dx}{x+1} - \frac{dx}{x}, \\
\omega_{0,4} &= \frac{ds}{\sqrt{-s\left(4m^2-s\right)}} &&= \frac{dx}{x-1} - \frac{dx}{x+1} + \frac{dx}{x}, \\
\omega_{-4,0} &= \frac{ds}{\sqrt{-s\left(-4m^2-s\right)}} &&= -\frac{dx}{x-1} + \frac{dx}{x+1} + \frac{dx}{x}.
\end{aligned}
$$

Example

$$
\begin{aligned}
J_{36} &= \varepsilon^4 \frac{\left(x^2+1\right)^3\left(x^2-2x-1\right)}{\left(x-1\right)^2\left(x+1\right)^2 x^2} I_{1111110} \\
&= \Big[2I\left(\omega_{0,4},\omega_{0,4},\omega_{0},\omega_{0,4};x\right)+2I\left(\omega_{0,4},\omega_{0,4},\omega_{0,4},\omega_{0};x\right) \\
&\quad -7I\left(\omega_{0,4},\omega_{0},\omega_{0,4},\omega_{0,4};x\right)+4I\left(\omega_{0,4},\omega_{-4,0},\omega_{-4,0},\omega_{0};x\right) \\
&\quad -4\zeta_2 I\left(\omega_{0,4},\omega_{-4,0};x\right)-10\zeta_3 I\left(\omega_{0,4};x\right)-\frac{39}{2}\zeta_4\Big]\varepsilon^4 \\
&\quad +O\left(\varepsilon^5\right)
\end{aligned}
$$



Integrals, which only depend on t



These integrals (and all integrals in the limit $s = \infty$) are expressed in terms of iterated integrals of modular forms

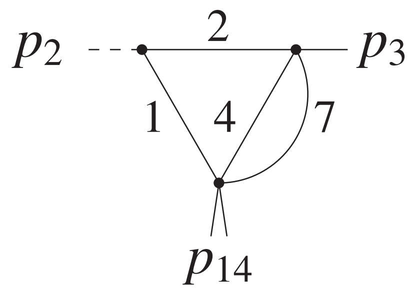
$$\{1, g_{2,0}, g_{2,1}, g_{2,9}, g_{3,1}, p_{3,0}, g_{4,0}, g_{4,1}, g_{4,9}, p_{4,0}, p_{4,1}\}$$

where

$$g_{n,r} = -\frac{1}{2}\frac{y\,(y-1)\,(y-9)}{y-r}\left(\frac{\psi_1^{(a)}}{\pi}\right)^n, \qquad p_{n,s} = -\frac{1}{2}y\,(y-1)^{1+s}\,(y-9)\left(\frac{\psi_1^{(a)}}{\pi}\right)^n.$$

Example

$$
\begin{aligned}
J_{14} &= \varepsilon^3 (1-y) I_{1102001} \\
&= \left[-I\left(p_{3,0}, 1, p_{3,0}; \tau_6^{(a)}\right) - 2\zeta_2 I\left(p_{3,0}; \tau_6^{(a)}\right)\right] \varepsilon^3 \\
&\quad + \left[I\left(p_{3,0}, 1, f_2, p_{3,0}; \tau_6^{(a)}\right) + I\left(p_{3,0}, f_2, 1, p_{3,0}; \tau_6^{(a)}\right) \right. \\
&\quad + 2\zeta_2 I\left(p_{3,0}, f_2; \tau_6^{(a)}\right) - 2\zeta_2 I\left(p_{3,0}, 1; \tau_6^{(a)}\right) \\
&\quad \left. - (7\zeta_3 - 12\zeta_2 \ln(2)) I\left(p_{3,0}; \tau_6^{(a)}\right)\right] \varepsilon^4 \\
&\quad + O\left(\varepsilon^5\right)
\end{aligned}
$$

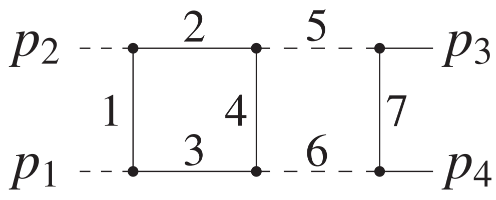


with

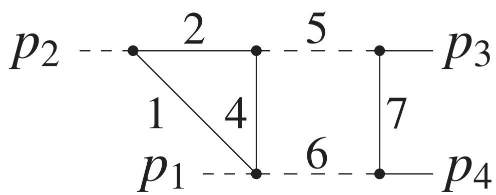
$$
f_2 = -\frac{1}{2} g_{2,0} + g_{2,1} + g_{2,9}, \qquad \tau_6^{(a)} = \frac{\psi_2^{(a)}}{6\psi_1^{(a)}}
$$

Integrals, which depend on s and t

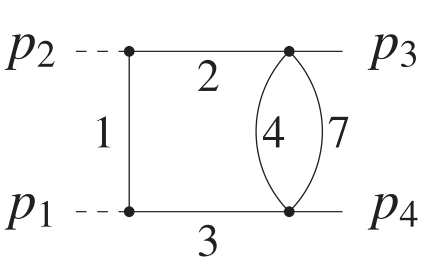
7 propagators:

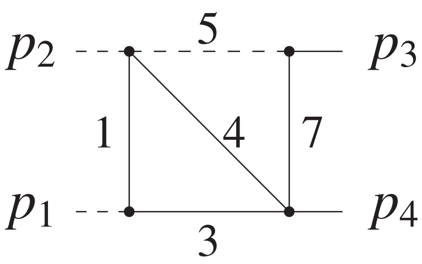


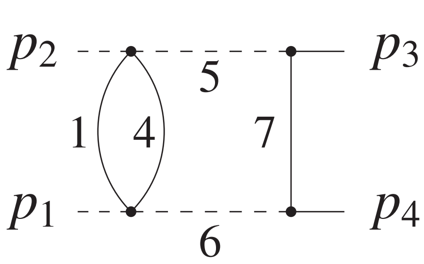
6 propagators:



5 propagators:







Construction of the basis J

- For the diagonal blocks:
  - Exploit the factorisation properties of Picard-Fuchs operators

    Adams, Chaubey, S.W., '17
  - Study the maximal cut

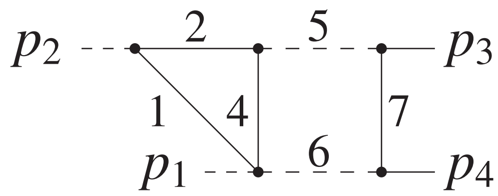
    Baikov '96; Lee '10; Kosower, Larsen, '11; Caron-Huot, Larsen, '12; Frellesvig, Papadopoulos, '17; Bosma, Sogaard, Zhang, '17; Harley, Moriello, Schabinger, '17

- For the non-diagonal blocks:
  - Modified version of the algorithm of Meyer

    (Meyer '16)

Factorisation properties of the Picard-Fuchs operator

Picard-Fuchs operator of sector 123 factorises into two first-order operators:



$$
\begin{aligned}
J_{38} &= 2\varepsilon^4 \frac{\left(x^2+1\right)\left(x^2-2x-1\right)}{(x-1)(x+1)x}\left[I_{11011110(-1)} - (y-2)\, I_{1101111}\right] \\
&\quad -4\frac{x+1}{x^2+1} J_{22} \\
J_{39} &= \varepsilon^4 (1-y) \frac{\left(x^2+1\right)^2}{(x-1)(x+1)x} I_{1101111}
\end{aligned}
$$

In all other sectors: Maximal second-order irreducible differential operators.

Picard-Fuchs operator of elliptic curves

- Sunrise integral: An elliptic curve can be obtained either from
  - Feynman graph polynomial
  - maximal cut

  The periods ψ_1, ψ_2 are the solutions of the homogeneous differential equations.

  Adams, Bogner, S.W., '13, '14

- In general: The maximal cuts are solutions of the homogeneous differential equations.

  Primo, Tancredi, '16

Search for Feynman integrals, whose maximal cuts are periods of an elliptic curve.

Maximal cuts

Sunrise :

$$\mathrm{MaxCut}_{\mathcal{C}}\, I_{1001001}\,(2-2\varepsilon) =$$

$$\frac{um^2}{\pi^2}\int_{\mathcal{C}} \frac{dP}{(P-t)^{\frac{1}{2}}\,(P-t+4m^2)^{\frac{1}{2}}\,(P^2+2m^2P-4m^2t+m^4)^{\frac{1}{2}}} + O(\varepsilon)\,.$$

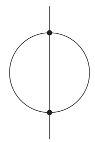
Double box :

$$\mathrm{MaxCut}_{\mathcal{C}}\, I_{1111111}\,(4-2\varepsilon) =$$

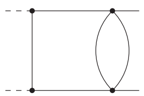
$$\frac{um^6}{4\pi^4 s^2}\int_{\mathcal{C}} \frac{dP}{(P-t)^{\frac{1}{2}}\,(P-t+4m^2)^{\frac{1}{2}}\left(P^2+2m^2P-4m^2t+m^4-\frac{4m^2\left(m^2-t\right)^2}{s}\right)^{\frac{1}{2}}} + O(\varepsilon)\,.$$

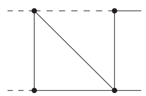
Three elliptic curves

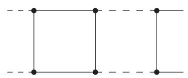
$$E^{(a)} \quad : \quad w^2 = (z-t)\left(z-t+4m^2\right)\left(z^2+2m^2z-4m^2t+m^4\right)$$



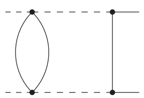
$$E^{(b)} \quad : \quad w^2 = (z-t)\left(z-t+4m^2\right)\left(z^2+2m^2z-4m^2t+m^4-\frac{4m^2\left(m^2-t\right)^2}{s}\right)$$







$$E^{(c)} \quad : \quad w^2 = (z-t)\left(z-t+4m^2\right)\left(z^2+\frac{2m^2\left(s+4t\right)}{\left(s-4m^2\right)}z+\frac{sm^2\left(m^2-4t\right)-4m^2t^2}{s-4m^2}\right)$$



Remarks

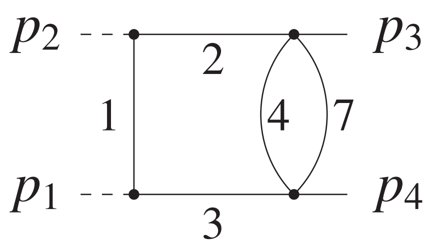
- $E^{(a)}$ gives rise to iterated integrals of modular forms of $\Gamma_1(6)$.
- For $s \to \infty$ the curves $E^{(b)}$ and $E^{(c)}$ degenerate to $E^{(a)}$.
- If we would have only one curve, we expect that the result can be written in elliptic polylogarithms.
- We have three elliptic curves.

Master integrals in the elliptic sector

$$J_{24} = \varepsilon^3 \frac{\left(1+x^2\right)^2}{x\left(1-x^2\right)} \frac{\pi}{\psi_1^{(b)}} I_{1112001},$$

$$J_{25} = \varepsilon^3 \left(1-2\varepsilon\right) \frac{\left(1+x^2\right)^2}{x\left(1-x^2\right)} I_{1111001} + R_{25,24} \frac{\psi_1^{(b)}}{\pi} J_{24},$$

$$J_{26} = \frac{6}{\varepsilon} \frac{\left(\psi_1^{(b)}\right)^2}{2\pi i W_y^{(b)}} \frac{d}{dy} J_{24} + R_{26,24} \left(\frac{\psi_1^{(b)}}{\pi}\right)^2 J_{24}$$
$$- \frac{\varepsilon^2}{24} \left(y^2 - 30y - 27\right) \frac{\psi_1^{(b)}}{\pi} \mathbf{D}^- I_{1001001},$$



This pattern applies to all elliptic sectors.

Summary on the differential equation

$$d\vec{J} \;=\; \left(A^{(0)}(x,y)+\varepsilon A^{(1)}(x,y)\right)\vec{J}$$

- $A^{(0)}$ is strictly lower triangular

- $A^{(1)}$ is block triangular

- $A^{(0)}$ vanishes for $t=m^2$ or $s=\infty$.

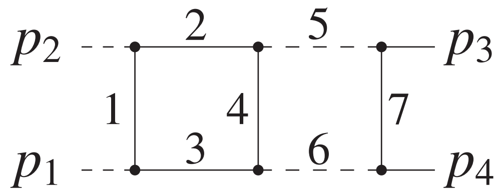
- $A^{(1)}$ reduces to one-forms associated with polylogarithms for $t=m^2$ and to modular forms for $s=\infty$.

- $A^{(0)}$ and $A^{(1)}$ are rational in

$$\left\{ x, y,\ \psi_1^{(a)}, \psi_1^{(b)}, \psi_1^{(c)},\ \partial_y\psi_1^{(a)}, \partial_y\psi_1^{(b)}, \partial_y\psi_1^{(c)} \right\}$$

The double box integral

$$J_{41} = \varepsilon^4 (1+2\varepsilon) \frac{\left(x^2+1\right)^4}{x^2 (x-1)^2 (x+1)^2} \frac{\pi}{\psi_1^{(b)}} I_{1111111}$$



- Starts at ε^4.

- Currently roughly $O(200)$-terms at $O(\varepsilon^4)$.

- Can still be massaged.

Conclusions

- Loop integrals with masses important for top, W/Z- and H-physics.
- May involve elliptic sectors from two loops onwards.
- The planar double box integral relavant to $t\bar{t}$-production with a closed top loop depends on two variables and involves several elliptic sub-sectors.
- More than one elliptic curve occurs.
- We may expect more results in the near future.