3-loop gauge coupling in hot QCD

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recent work with Mikko Laine, Philipp Schicho

and earlier work with I. Ghişoiu, J. Möller

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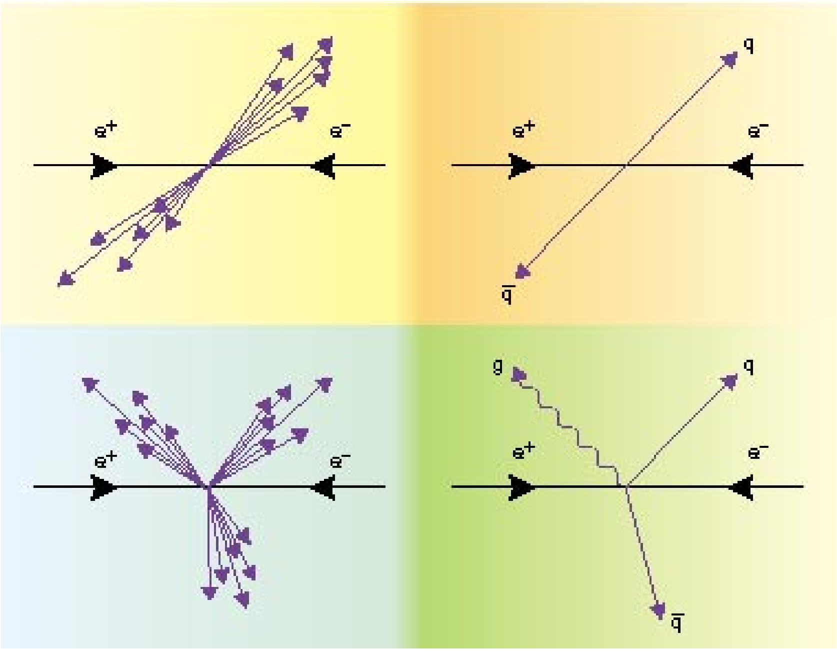
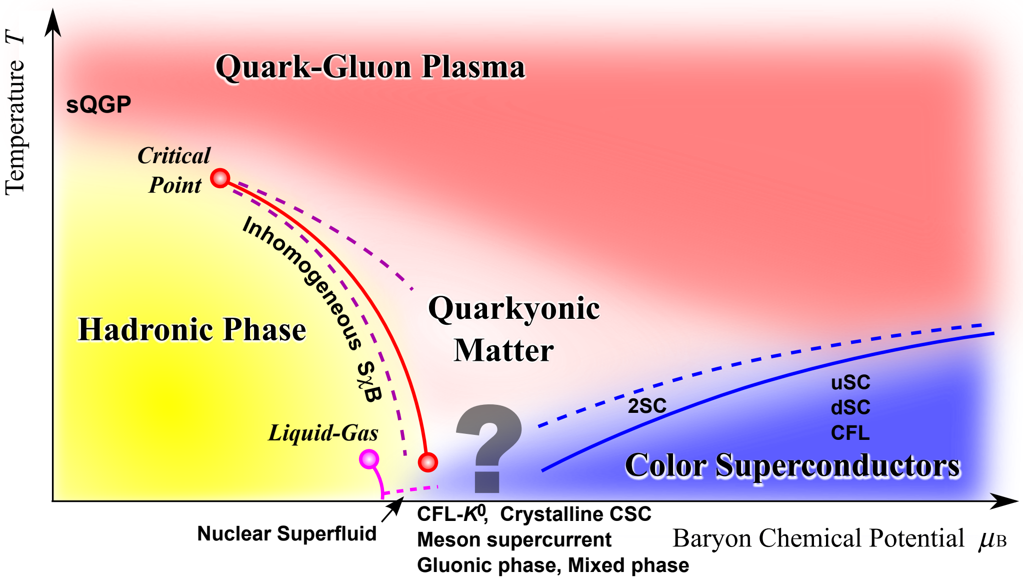
Motivation

check QCD in extreme conditions

- $E\uparrow$: collider physics



- $T\uparrow$, $\mu\uparrow$: equilibrium phase diagram



[Fukushima/Hatsuda]

- e.g. LEP, $e^+e^- \to X$
- check details of theory with jets
- nowadays: calc QCD background

- nature: early universe, n/qu stars
- $T_c \sim 170$ MeV $\sim 10\mu s$
- lab expt: SPS / RHIC / LHC HI / GSI

Motivation

Focus on equilibrium thermodynamics of QCD

- study confinement and chiral symmetry breaking
- phenomenologically relevant for astrophysics
- phenomenologically relevant for cosmology
- phenomenologically relevant for RHIC, LHC
- large T: theoretical limit tractable with analytic methods
  - ▷ goal: no models - stay within QCD!
  - ▷ goal: possibility of systematic improvements
  - ▷ parameters: T, μ_q, m_q, (N_c, N_f)

Interplay of methods

- QGP is strongly coupled system near $T_c \Rightarrow$ need e.g. LAT
- asymptotic freedom at high $T \Rightarrow$ weak-coupling approach in continuum
  - ▷ cave: strict loop expansion not well-defined
    IR divergences at higher orders [Linde 79; Gross/Pisarski/Yaffe 81]
- in general, one tries to use best of both; this talk: the weak-coupling side

Energy scales in hot QCD

Interactions make QCD a multi-scale system

- At asymptotically high T, $g \ll 1 \Rightarrow$ clean separation of 3 scales
- expansion parameter: $$g^2\, n_b(|k|) = \frac{g^2}{e^{|k|/T}-1} \stackrel{|k|\lesssim T}{\approx} \frac{g^2 T}{|k|}$$
- $|k| \sim \pi T / gT / g^2 T$
  aka hard/soft/ultrasoft scales
  are fully/barely/non- perturbative at high T
- no smaller momentum scales / larger length scales due to confinement

$\Rightarrow$ treatment of a multi-scale system: effective field theory !

Effective theory setup: QCD $\to$ EQCD

high T: QCD dynamics contained in 3d EQCD

- integrate out hard scales $|p| \gtrsim \pi T$: ψ, $A_\mu(n \neq 0)$

$$
\begin{aligned}
p_{\rm QCD}(T) &\equiv \frac{T}{V} \ln \int \mathcal{D}[A^a_\mu, \psi, \bar\psi] \exp\Big(-\int_0^{1/T} d\tau \int d^{3-2\epsilon}x\, \mathcal{L}^E_{\rm QCD}\Big) \\
&= p_{\rm E}(T) + \frac{T}{V} \ln \int \mathcal{D}[A^a_k, A^a_0] \exp\Big(-\int d^{3-2\epsilon}x\, \mathcal{L}_{\rm E}\Big)
\end{aligned}
$$

$$
\mathcal{L}_{\rm E} = \frac{1}{2} {\rm Tr}\, F^2_{kl} + {\rm Tr}\,[D_k, A_0]^2 + m^2_{\rm E} {\rm Tr}\, A^2_0 + \lambda^{(1)}_{\rm E} ({\rm Tr}\, A^2_0)^2 + \lambda^{(2)}_{\rm E} {\rm Tr}\, A^4_0 + \ldots
$$

- five matching coefficients [Braaten/Nieto 95, ..., Ghisoiu/Laine/Möller/YS 02-15]

$p_{\rm E} = T^4\,[\# + \# g^2 + \# g^4 + \# g^6 + \ldots]$, $m^2_{\rm E} = T^2\,[\# g^2 + \# g^4 + \# g^6 + \ldots]$,
$g^2_{\rm E} = T\,[g^2 + \# g^4 + \# g^6 + \# g^8 + \ldots]$, $\lambda^{(1),(2)}_{\rm E} = T\,[\# g^4 + \# g^6 + \ldots]$.

Effective theory setup: QCD $\rightarrow$ EQCD $\rightarrow$ MQCD

the IR of 3d EQCD is contained in 3d MQCD

- integrate out $|p| \gtrsim gT$: A_0

$$p_{\rm QCD}(T) \equiv p_{\rm E}(T) + p_{\rm M}(T) + \frac{T}{V}\ln\int \mathcal{D}[A_k^a]\exp\Big(-\int d^{3-2\epsilon}x\,\mathcal{L}_{\rm M}\Big)$$
$$\mathcal{L}_{\rm M} = \frac{1}{2}\,{\rm Tr}\,F_{kl}^2 + \ldots$$

- two matching coefficients [Kajantie et al. 03; P. Giovannangeli 04, Laine/YS 05]

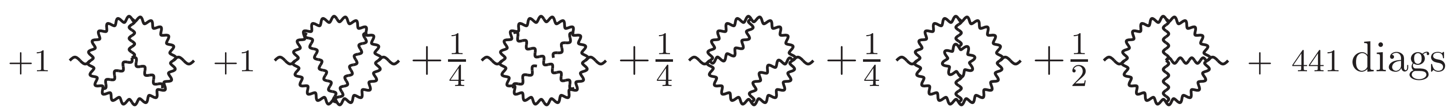
$$p_{\rm M} = Tm_{\rm E}^3\left[\# + \#\frac{g_{\rm E}^2}{m_{\rm E}} + \#\frac{g_{\rm E}^4}{m_{\rm E}^2} + \#\frac{g_{\rm E}^6}{m_{\rm E}^3} + \ldots\right],\; g_{\rm M}^2 = g_{\rm E}^2\left[1 + \#\frac{g_{\rm E}^2}{m_{\rm E}} + \#\frac{g_{\rm E}^4}{m_{\rm E}^2} + \ldots\right].$$

3-loop gauge coupling: hard contributions

determine the matching coefficients, with precision

latest: g_{E}^2 at 3 loops

- when reducing QCD to EQCD: determine e.g. $g_{\mathrm{E}}^2 = g^2(\mathcal{Z}_B + \delta\mathcal{Z}_B)^{-1}$
- bgf eff. action $\Gamma^{(2)}_{\mathrm{EQCD}}[B] = \frac{1}{2}B_i^a(p)B_j^a(q)\delta(p+q)(q^2\delta_{ij} - q_iq_j)(\mathcal{Z}_B + \delta\mathcal{Z}_B)$
- main ingredient: $\Pi'_T(0)$ at 3 loops
  - ▷ standard computer-algebraic methods
  - ▷ generation via QGRAF, $\sim$ 450 two-point diagrams



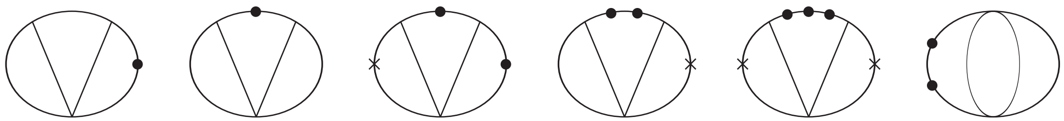
  - ▷ manipulation in FORM, $\sim 10^7$ vacuum sum-integrals
  - ▷ reduction via IBP/Laporta, $\sim 10^2$ masters, of which $\sim 10^1$ bosonic
  - ▷ basis transformation, $= 6$ non-trivial bosonic 3-loop master sum-integrals

3-loop gauge coupling: hard contributions

- calculate masters [3-loop pioneers: Arnold/Zhai 94]
  - ▷ compact (imag.) time interval → sum-integrals

$$\sum\!\!\!\!\!\!\!\int_P = T\sum_{n=-\infty}^{\infty}\int\frac{d^{3-2\epsilon}p}{(2\pi)^{3-2\epsilon}}\;;\;\; P^2 = P_0^2 + p^2 \text{ with } P_0 = 2\pi nT \text{ (bos)}$$

  - ▷ starting from 3 loops, these can be nasty objects
  - ▷ here, need 6 bosonic 3-loop master sum-integrals [Ghisoiu/YS 18, in prep]



  - ▷ beautiful methods, e.g. Tarasov at T
  - ▷ careful dissection: divergent pieces analytically
  - ▷ precision on finite/numerical piece? number content? [see example on next slide]

3-loop gauge coupling: hard contributions

- one of the six non-trivial 3-loop masters:

$$V_1 = \sum\!\!\!\!\!\!\!\int_P \sum\!\!\!\!\!\!\!\int_Q \sum\!\!\!\!\!\!\!\int_R \frac{1}{P^2\,[Q^2]^2\,(Q-P)^2\,R^2\,(R-P)^2}$$

$$= \frac{1}{(4\pi)^6}\left(\frac{e^{\gamma_{\rm E}}}{4\pi T^2}\right)^{3\epsilon}\frac{1}{6\,\epsilon^3}\Big[1+3\,\epsilon+\Big(13-3\zeta_3+\tfrac{9}{2}\,\zeta_2-6\big(\gamma_{\rm E}^2+2\gamma_1\big)\Big)\epsilon^2$$

$$+\Big(51-42(\gamma_{\rm E}^2+2\gamma_1)+24\zeta_2\big(\tfrac{19}{16}+\ln(2\pi)-12\ln G\big)+2\ln 2\big(12-12\gamma_{\rm E}^2-24\gamma_1-\zeta_3\big)$$

$$+6\gamma_{\rm E}\big(3\zeta_3-4-4\gamma_1\big)-36\gamma_2+\tfrac{25}{2}\,\zeta_3-16\zeta_3'+6\,c_1+6\,c_2+6\,c_3\Big)\epsilon^3+\mathcal{O}(\epsilon^4)\Big]$$

$$c_1 = \int_0^\infty {\rm d}x\,\frac{1}{3x}\left(\coth x-\frac{1}{x}\right)\Big[6x\big[\ln(1-e^{-2x})+x\big]^2-\pi^2\big[\ln(1-e^{-2x})+4x\big]-14x^3+$$

$$+6\ln(1-e^{-2x})\,{\rm Li}_2\left(e^{-2x}\right)+12x\big[{\rm Li}_2\left(1/(1-e^{-2x})\right)-i\pi\ln(1-e^{-2x})\big]-6\,{\rm Li}_3\left(1-e^{2x}\right)\Big]$$

$$\approx 0.68647205936406 18954(1)$$

$$c_2 = \sum_{n=1}^{\infty}\int_0^\infty {\rm d}x\,\frac{2\,e^{-x}}{n}\left[e^x{\rm Ei}(-x)+\gamma_{\rm E}+\ln\frac{x}{4n^2}\right]\times$$

$$\times\left[\psi(n+1)+e^x B(e^{-x/n},n{+}1,0)+e^x{\rm Ei}(-x)-\ln(1-e^{-x/n})+\ln\frac{x}{n^2}-\frac{x}{12n^2}\right]$$

$$\approx -3.2020672566(1)$$

$$c_3 = \int_0^\infty {\rm d}x\,\frac{4}{3x}\left(\coth x-\frac{1}{x}-1\right)\Big[4x^3-2\pi^2x+3\big[{\rm Li}_3\left(e^{2x}\right)+2\pi i\,x^2\big]-3\zeta(3)\Big]$$

$$\approx 10.33244698246374834(1)$$

3-loop gauge coupling: hard contributions

- arrive at renormalized NNLO result for effective gauge coupling [Ghisoiu/Möller/YS 14]

$$\text{recall:} \qquad g_{\rm E}^2 = g^2(\mathcal{Z}_B + \delta\mathcal{Z}_B)^{-1}$$

$$Z_B = 1 - \frac{g^2 N_c}{16\pi^2}\left[\frac{22}{3}L + \frac{1}{3}\right] - \left(\frac{g^2 N_c}{16\pi^2}\right)^2\left[\frac{68}{3}L + \frac{341}{18} - \frac{10\zeta_3}{9}\right]$$

$$- \left(\frac{g^2 N_c}{16\pi^2}\right)^3\left[\frac{748}{9}L^2 + \left(\frac{6608}{27} - \frac{10982\zeta_3}{135}\right)L + (\text{finite})\right] + \mathcal{O}(g^8)$$

$$\delta Z_B = \left(\frac{g^2 N_c}{16\pi^2}\right)^3 \frac{61\zeta_3}{5\epsilon} + \mathcal{O}(g^8)\ , \quad L = \ln\left(\frac{\bar{\mu}e^{\gamma_{\rm E}}}{4\pi T}\right)$$

- what is the IR divergence?

Dimension-six operators in EQCD

- examine higher-order operators in the effective theory [Chapman 94]

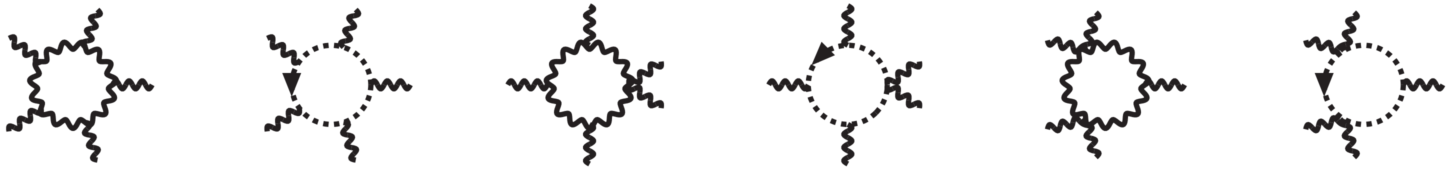
$$\begin{aligned}
\delta\mathcal{L}_\mathrm{E} &= \left(\sum\!\!\!\!\!\!\!\!\int_P{}' \frac{2g_\mathrm{E}^2}{P^6}\right) \mathrm{tr}\Big\{ c_1\,(D_\mu F_{\mu\nu})^2 + c_2\,(D_\mu F_{\mu 0})^2 \\
&+ i g_\mathrm{E}\big[c_3\,F_{\mu\nu}F_{\nu\rho}F_{\rho\mu} + c_4\,F_{0\mu}F_{\mu\nu}F_{\nu 0} + c_5\,A_0(D_\mu F_{\mu\nu})F_{0\nu}\big] \\
&+ g_\mathrm{E}^2\big[c_6\,A_0^2 F_{\mu\nu}^2 + c_7\,A_0 F_{\mu\nu} A_0 F_{\mu\nu} + c_8\,A_0^2 F_{0\mu}^2 + c_9\,A_0 F_{0\mu} A_0 F_{0\mu}\big] \\
&+ g_\mathrm{E}^4\big[c_{10} A_0^6\,\big]\Big\}
\end{aligned}$$

- ▷ color trace in adjoint rep; $\mathrm{tr}(AB) = A_{ab}B_{ba}$ with e.g. $(A_0)_{ab} = -i f^{abc} A_0^c$ etc.
- ▷ one lin rel between $c_4, \ldots, c_7 \Rightarrow$ keep redundancy for crosschecks
- ▷ the 1-loop sum-integral evaluates to Gamma and Zeta, and is finite

$$\sum\!\!\!\!\!\!\!\!\int_P{}' \frac{T^2}{P^6} = \frac{2\zeta_3}{(4\pi)^4}[1 + \mathcal{O}(\epsilon)]$$

Dimension-six operators in EQCD

- for doing loops with these operators, need c_i in d dimensions
  - ▷ eval e.g. 5pt function: 20 indep color/Lorentz structures



  - ▷ (done also 2-, 3-, 6pt fcts as checks)
  - ▷ LO results for dim6 operators [Laine/Schicho/YS 18]

$$c_1 = \frac{41-d}{120}\,,\quad c_2 = \frac{(d-1)(d-5)}{120}\,,\quad c_3 = \frac{1-d}{180}\,,\quad c_4 - 2c_7 = \frac{(41-d)(5-d)}{60}$$

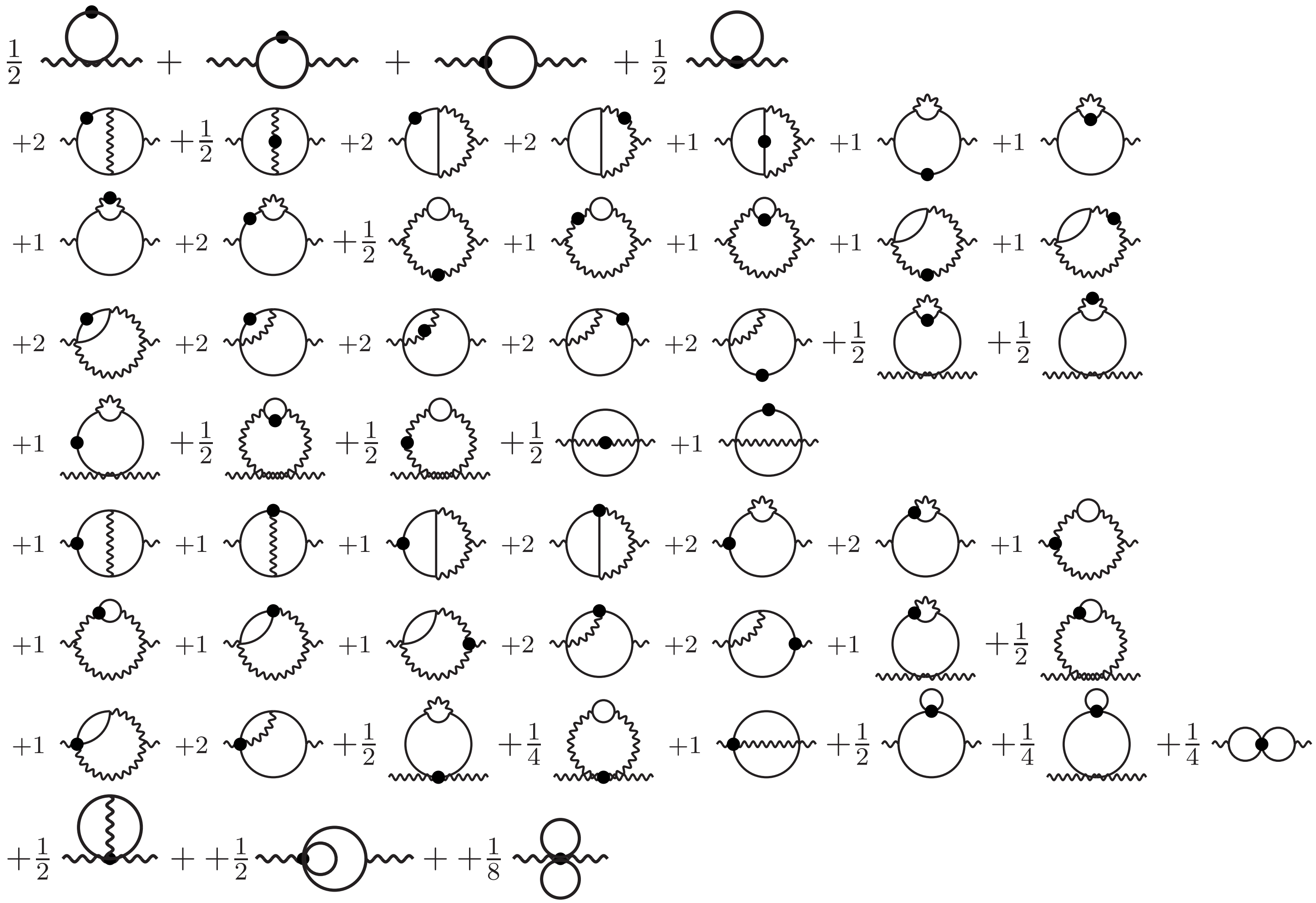
$$c_5 - 2c_7 = \frac{(21-d)(5-d)}{30}\,,\quad c_6 + c_7 = \frac{(d-25)(5-d)}{24}\,,\quad c_8 = \frac{(5-d)(3-d)(d-1)}{20}$$

$$c_9 = \frac{(5-d)(3-d)(d-1)}{30}\,,\quad c_{10} = \frac{(5-d)(3-d)(d-1)^2}{180}$$

  - ▷ $d = 3 - 2\epsilon \Rightarrow c_8$, c_9 and c_{10} couple to "evanescent" operators

Soft contributions

- effects of these dim6 operators?
  - ▷ when reducing EQCD to MQCD; determine $g_{\rm M}^2 = g_{\rm E}^2(Z_B + \delta Z_B)^{-1}$
  - ▷ from 2pt fct in bfg; need at least 2-loop for div (since 3d); blobs are new vertices



Soft contributions

- NLO results for 2pt correlator [Laine/Schicho/YS 18]

$$
\begin{aligned}
Z_B &= 1 + \left(\frac{g_{\rm ER}^2 N_c}{16\pi^2}\right)^2 \frac{m_{\rm ER}}{2\pi T}\left(\frac{875\zeta_3}{72}\right) \\
&- \left(\frac{g_{\rm ER}^2 N_c}{16\pi^2}\right)^3 \left(\frac{1097\zeta_3}{549}\right)\frac{61}{5}\left\{L + 2\ln\left(\frac{\bar{\mu}}{2m_{\rm ER}}\right) + \frac{\zeta_3'}{\zeta_3} - \gamma_{\rm E} + \frac{103771}{52656}\right\} \\
\delta Z_B &= \left(\frac{g_{\rm ER}^2 N_c}{16\pi^2}\right)^3 \left(-\frac{1097}{1098}\right)\frac{61\zeta_3}{5\epsilon}
\end{aligned}
$$

- note that this cancels $\frac{1097}{1098}$ of the IR divergence from the hard scales (!)
  - ▷ have integrated out hard $\sim T$ and soft $\sim gT \sim m_{\rm E}$ scales

Ultrasoft contributions

- remains to check contributions from ultrasoft $\sim g^2T$ scales [Laine/Schicho/YS 18]
  - ▷ same game: dim6 operators in MQCD

$$\delta\mathcal{L}_{\rm M} = \left(\sum\!\!\!\!\!\!\!\int_P{}' \frac{2g_{\rm M}^2}{P^6}\right) \,{\rm tr}\Big\{c_1\,(D_iF_{ij})^2 + ig_{\rm E}c_3\,F_{ij}F_{jk}F_{ki}\Big\}$$

  - ▷ evaluate UV div: screen IR by common mass (unphysical, but here ok)

- result (showing only the divergence here, coming from 2-loop diags)

$$\sum\!\!\!\!\!\!\!\int_P{}' \frac{g_{\rm M}^6 N_c^3}{P^6}\,\frac{T^2\,c_3}{32\pi^2\,\epsilon}[1+\mathcal{O}(\epsilon)] \ni \delta Z_B = \left(\frac{g^2N_c}{16\pi^2}\right)^3\left(-\frac{1}{1098}\right)\frac{61\zeta_3}{5\epsilon}$$

- adding up, IR div cancels perfectly
  - ▷ needed to include dim6 operators in EQCD and MQCD!

Summary

- thermodynamic quantities of QCD are relevant for cosmology and heavy ion collisons
- these quantities can be determined
  - ▷ numerically at $T \sim 200$ MeV; analytically at $T \gg 200$ MeV
  - ▷ multi-loop sports
  - ▷ eff. theories convenient
- 3d effective field theory opens up tremendous opportunities
  - ▷ analytic treatment of fermions (cf. LAT problems!)
  - ▷ universality
  - ▷ systematic improvement possible
- soft scale $m_{\rm E} \sim gT$ is formally larger than the ultrasoft scale $\sim g^2T$
  - ▷ however, soft scale plays an essential role in IR dynamics
  - ▷ in fact, it is 1097 times more important
    (in terms of an IR divergence in the 3-loop gauge coupling)