



# Multi-Loop Amplitudes in the High-Energy Limit in $\mathcal{N} = 4$ SYM

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CP3 - UCLouvain

In collaboration with

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Goal: Analyze mathematical structure of scattering amplitudes.

Ideal playground for this is  $\mathcal{N} = 4$  SYM.

- Conformal symmetry
- Maximally transcendental

Polylogarithmic part of  $\ell$ -loop amplitude has weight  $2\ell$

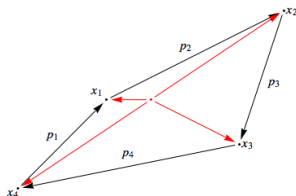
$$\mathcal{A}_4^{(1)} \sim \frac{1}{2} \log^2 \frac{s}{t} + \frac{2\pi^2}{3}$$

# The Planar Limit ('t Hooft limit)



Corresponds to  $N_c \rightarrow \infty$ ,  $a = \frac{N_c g^2}{8\pi^2}$  finite.

Define *dual coordinates*  $x_i$ , s.t.  $p_i = x_i - x_{i-1}$



## Dual conformal symmetry

- Fixes all 4&5 point results
- $A_N = A_N^{\text{BDS}} e^{R_N}$

[Drummond, Henn, Korchemsky, Sokatchev;  
Drummond, Henn, Smirnov, Sokatchev]

[Anastasiou, Bern, Dixon, Kosower;  
Bern, Dixon, Smirnov]

[Drummond, Henn, Korchemsky, Sokatchev]

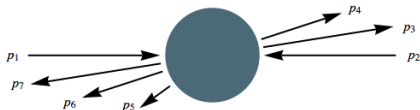
Forward scattering:

- $s \gg |t|$
- $\mathcal{A}_4^{(1)} \sim \frac{1}{2} \log^2 \frac{s}{t} + \frac{2\pi^2}{3}$



Multi Regge kinematics:

- Hierarchy in rapidity
- No hierarchy in transverse plane

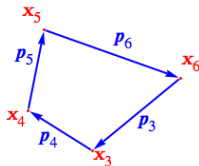
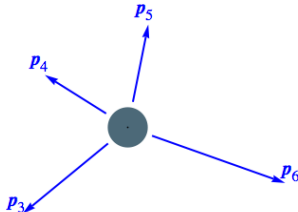
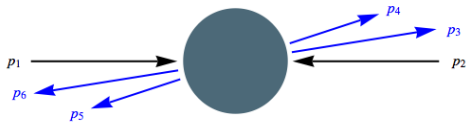


- $R_N \sim a^2 \left( \log \tau g_{N,LLA}^{(2)} + g_{N,NLLA}^{(2)} \right) + a^3 \left( \log^2 \tau g_{N,LLA}^{(3)} + \log \tau g_{N,NLLA}^{(3)} + g_{N,NNLLA}^{(3)} \right) + \dots$ ,  
with  $\log \tau \gg 1$

# Multi Regge Kinematics



Kinematical dependence encoded in  $N - 2$  transverse momenta.



$$R_N \sim a^2 \left( \log \tau g_{N,LLA}^{(2)}(\{\mathbf{x}_i\}) + g_{N,NLLA}^{(2)}(\{\mathbf{x}_i\}) \right) + \dots$$

# The Riemann Sphere With $n$ Marked Points

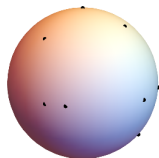


Riemann sphere  $\cong \mathbb{C}$  (stereographic projection)

Riemann spheres with  $n$  marked points

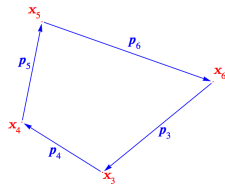


'Phase space' of  $N = n + 2$ -particle MRK amplitude.



Use  $SL(2, \mathbb{C})$  symmetry to fix three points to get  $N - 5$  cross ratios

$$z_i = \frac{(\mathbf{x}_1 - \mathbf{x}_{i+3})(\mathbf{x}_{i+2} - \mathbf{x}_{i+1})}{(\mathbf{x}_1 - \mathbf{x}_{i+1})(\mathbf{x}_{i+2} - \mathbf{x}_{i+3})}$$





Optical theorem tells us that branch cuts start at

$$x_{ij}^2 = (x_i - x_j)^2 = 0$$

In MRK, this corresponds to branch cuts at

$$\mathbf{x}_{ij}^2 = (\mathbf{x}_i - \mathbf{x}_j)^2 = 0$$

In the transverse part, however

$$\mathbf{x}_{ij}^2 \geq 0$$

⇒ Scattering amplitudes in planar  $\mathcal{N} = 4$  SYM in MRK are single-valued



Scattering amplitudes in planar  $\mathcal{N} = 4$  SYM in MRK are single-valued iterated integrals

→ Use *single-valued polylogarithms*  $\mathcal{G}$

[Brown]

Fulfil same differential equation as regular Polylogs

$$\partial_z G_{a_1, \dots, a_n}(z) = \frac{1}{z - a_1} G_{a_2, \dots, a_n}(z)$$

↓

$$\partial_z \mathcal{G}_{a_1, \dots, a_n}(z) = \frac{1}{z - a_1} \mathcal{G}_{a_2, \dots, a_n}(z)$$





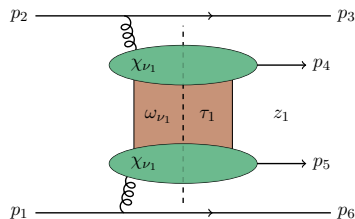
Single-valued polylogs are combinations of regular polylogs in  $z$  and  $\bar{z}$  such that all branch cuts cancel

$$\begin{aligned}\mathcal{G}_0(z) &= G_0(z) + G_0(\bar{z}) \\ &= \log z + \log \bar{z} = \log |z|^2\end{aligned}$$

Bloch-Wigner dilog:

$$\begin{aligned}D_2(z) &= \text{Im}(\text{Li}_2(z) + \log |z| \log(1 - z)) \\ &= \frac{1}{2i} \left( G_{0,1}(\bar{z}) - G_{0,1}(z) + \frac{1}{2}(G_0(z) + G_0(\bar{z}))(G_1(z) - G_1(\bar{z})) \right) \\ &= \frac{1}{4i} (\mathcal{G}_{1,0}(z) - \mathcal{G}_{0,1}(z))\end{aligned}$$

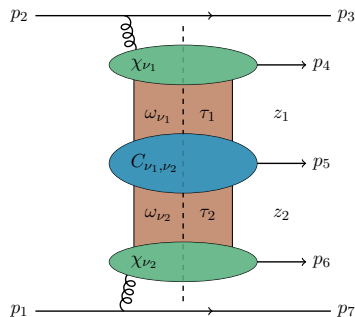
# The Remainder Function



[Bartels, Lipatov, Sabio-Vera; Lipatov, Prygarin]

$$\begin{aligned}
 &\sim \sum_{n_1} \int \frac{d\nu_1}{2\pi} \left( \frac{z_1}{\bar{z}_1} \right)^{\frac{n_1}{2}} |z_1|^{2i\nu_1} \chi_1^+ \tau_1^{-\omega_1} \chi_1^- \\
 &\equiv \mathcal{F}_1 \left[ \chi_1^+ \tau_1^{-\omega_1} \chi_1^- \right]
 \end{aligned}$$

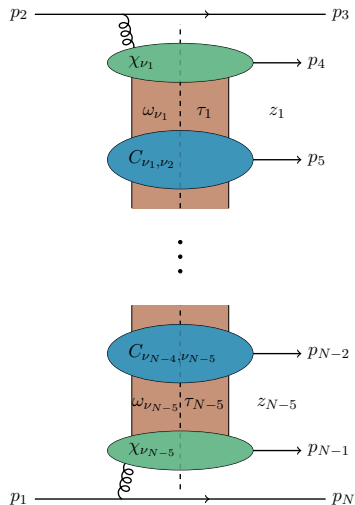
# The Remainder Function



[Bartels, Lipatov, Sabio-Vera; Bartels, Kormilitzin, Lipatov, Prygarin]

$$\begin{aligned} &\sim \mathcal{F}_2 \left[ \mathcal{F}_1 \left[ \chi_1^+ \tau_1^{-\omega_1} C_{12}^+ \tau_2^{-\omega_2} \chi_2^- \right] \right] \\ &\equiv \mathcal{F}_{12} \left[ \chi_1^+ \tau_1^{-\omega_1} C_{12}^+ \tau_2^{-\omega_2} \chi_2^- \right] \end{aligned}$$

# The Remainder Function



[Bartels, Lipatov, Sabio-Vera; Bartels, Kormilitzin, Lipatov, Prygarin]

# The Remainder Function



At LLA, replace all building blocks by their LO approximation

$$\chi_{0,1}^+ \tau_1^{aE_1} C_{0,12}^+ \tau_2^{aE_2} \chi_{0,2}^-$$

$$\tau_k^{aE_k} = a \log \tau_k E_k + \frac{a^2}{2} \log^2 \tau_k E_k^2 + \dots$$

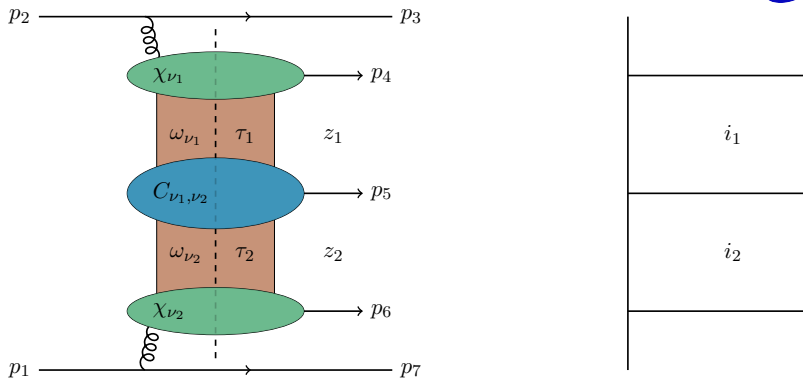
At LLA: 
$$\ell - 1 = \sum i_k = \#E_k$$

Define the *vacuum ladder*

$$\varpi_N = \chi_{0,1}^+ C_{0,12}^+ \cdots C_{0(N-6)(N-5)}^+ \chi_{0,N-5}^-$$

## An Example at 7 Points

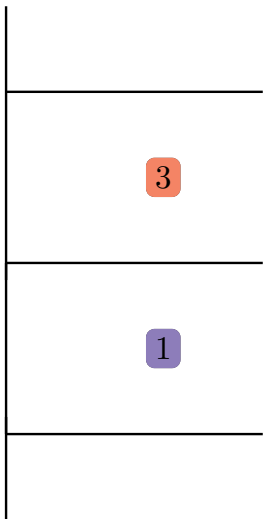
# A Graphical Representation - LLA



$$\mathcal{R}_{7,\text{LLA}}^{(\ell)} \sim \sum_{i_1+i_2=\ell-1} \frac{a^\ell}{i_1!i_2!} \log^{i_1} \tau_1 \log^{i_2} \tau_2 g^{(i_1,i_2)}$$



$$g^{(3,1)} = \mathcal{F}_{12} \left[ \varpi_7 E_1^3 E_2 \right] \equiv$$





The number of  $E_k$  in the integrand is related to the loop order of LLA amplitudes

→ Increase the loop order at a fixed logarithmic accuracy by inserting  $E_k$  into the integrand.

... but how?



The Fourier-Mellin transform

$$\mathcal{F}[F(\nu, n)] = \sum_{n=-\infty}^{\infty} \int \frac{d\nu}{2\pi} \left(\frac{z}{\bar{z}}\right)^{\frac{n}{2}} |z|^{2i\nu} F(\nu, n)$$

Products are mapped to convolutions

$$\mathcal{F}[F \cdot G] = \mathcal{F}[F] * \mathcal{F}[G] = \frac{1}{\pi} \int \frac{d^2w}{|w|^2} \mathcal{F}[F](w) \mathcal{F}[G]\left(\frac{z}{w}\right)$$

Use this to raise loop order

$$g^{(i_1, i_2)} = g^{(i_1-1, i_2)} * \mathcal{F}_1[E_1] = g^{(0,0)} * \mathcal{F}[E_1]^{*i_1} * \mathcal{F}_2[E_2]^{*i_2}$$



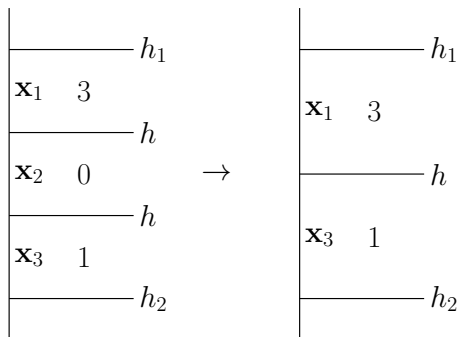
$$\mathcal{F}_k[E_k] = -\frac{z_k + \bar{z}_k}{2|1 - z_k|^2}$$

Single-valuedness allows us to solve the convolution integral by computing residues

[Schnetz]

$$\int \frac{d^2z}{\pi} f(z) = \text{Res}_{z=\infty} F(z) - \sum_i \text{Res}_{z=a_i} F(z) \quad \partial_{\bar{z}} F(z) = f(z)$$

→ Increasing loop order as simple as computing residues!



$$f(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = f(\mathbf{x}_1, \mathbf{x}_3)$$

[Del Duca, Druc, Drummond, Duhr, Dulat, RM, Papathanasiou, Verbeek]



$$\begin{aligned}
 R_{7,\text{LLA}}^{(3)} &\sim \log^2 \tau_1 \left| \begin{array}{c} \hline \mathbf{x}_1 \ 2 \\ \hline \mathbf{x}_2 \ 0 \\ \hline \end{array} \right. + \log^2 \tau_2 \left| \begin{array}{c} \hline \mathbf{x}_1 \ 0 \\ \hline \mathbf{x}_2 \ 2 \\ \hline \end{array} \right. + \log \tau_1 \log \tau_2 \left| \begin{array}{c} \hline \mathbf{x}_1 \ 1 \\ \hline \mathbf{x}_2 \ 1 \\ \hline \end{array} \right. \\
 &= \log^2 \tau_1 \left| \begin{array}{c} \hline \mathbf{x}_1 \ 2 \\ \hline \\ \hline \end{array} \right. + \log^2 \tau_2 \left| \begin{array}{c} \hline \mathbf{x}_2 \ 2 \\ \hline \\ \hline \end{array} \right. + \log \tau_1 \log \tau_2 \left| \begin{array}{c} \hline \mathbf{x}_1 \ 1 \\ \hline \mathbf{x}_2 \ 1 \\ \hline \end{array} \right.
 \end{aligned}$$



$$R_{8,LLA}^{(3)} \sim \log^2 \tau_1 \left| \begin{array}{c} \text{---} \\ \mathbf{x}_1 \ 2 \\ \text{---} \end{array} \right. + \log^2 \tau_2 \left| \begin{array}{c} \text{---} \\ \mathbf{x}_2 \ 2 \\ \text{---} \end{array} \right. + \log^2 \tau_3 \left| \begin{array}{c} \text{---} \\ \mathbf{x}_3 \ 2 \\ \text{---} \end{array} \right.$$

$$+ \log \tau_1 \log \tau_2 \left| \begin{array}{c} \text{---} \\ \mathbf{x}_1 \ 1 \\ \text{---} \\ \mathbf{x}_2 \ 1 \\ \text{---} \end{array} \right. + \log \tau_1 \log \tau_3 \left| \begin{array}{c} \text{---} \\ \mathbf{x}_1 \ 1 \\ \text{---} \\ \mathbf{x}_3 \ 1 \\ \text{---} \end{array} \right. + \log \tau_2 \log \tau_3 \left| \begin{array}{c} \text{---} \\ \mathbf{x}_2 \ 1 \\ \text{---} \\ \mathbf{x}_3 \ 1 \\ \text{---} \end{array} \right.$$



Reminder:

$$\text{LLA} \Leftrightarrow \sum i_k = \ell - 1$$

→ can only have limited number of 'insertions' at a given order

At MHV, a finite number of building blocks are enough to describe the  $\ell$ -loop amplitude for all numbers of particles



# Beyond MHV and LLA



Can flip helicities with convolutions, too!

$$\mathcal{F}_{12} [\chi_1^- C_{12}^+ \chi_2^-] = \mathcal{F}_{12} [\chi_1^+ C_{12}^+ \chi_2^-] * \mathcal{F}_1 \left[ \frac{\chi_1^-}{\chi_1^+} \right]$$
$$\mathcal{F}_1 \left[ \frac{\chi_1^-}{\chi_1^+} \right] = -\frac{z_1}{(1-z_1)^2}$$

Beyond MHV, replace  $e^{R_N} \rightarrow \mathcal{R}_N \equiv \frac{A_N}{A_N^{\text{BDS}}}$



Certain amplitudes will never factorize ...

- $\mathcal{R}_{N,+--+...}$  won't simplify

... but others still do

$$\mathcal{R}_{N,-+...+}^{(2)} \sim \log \tau_1 \left| \begin{array}{c} \text{---} - \\ \mathbf{x}_1 \quad 1 \\ \text{---} + \end{array} \right. + \sum_{k=2}^{N-5} \log \tau_k \left| \begin{array}{c} \text{---} - \\ \mathbf{x}_1 \quad 0 \\ \text{---} + \\ \mathbf{x}_k \quad 1 \\ \text{---} + \end{array} \right.$$



Formalism should still be applicable

- Has been applied at NLLA
- No conceptually new hurdles beyond NLLA

arXiv:1801.10605

Corrections to BFKL building blocks prohibit factorization in certain faces, but their number is also limited at a given logarithmic order

→ MHV amplitudes for all  $N$  still have finite number of building blocks



- Developed a framework that allows us to compute virtually any amplitude in MRK in  $\mathcal{N} = 4$  SYM at LLA and NLLA
- A finite set of building blocks encodes amplitudes for any number of particles.
- Computed the building blocks to yield:
  - ▶ All MHV 5-loop amplitudes at LLA
  - ▶ 8-point LLA amplitudes for any helicity configuration up to 4 loops
  - ▶ All MHV 3-loop amplitudes at NLLA (Not yet published)
  - ▶ 7-point NLLA NMHV up to 3 loops
- Determined the function space of MRK amplitudes at LLA and NLLA to all orders
- Computed many loops and legs in a special QFT